

# Philosophy 511: Modal Logic

## Course Handouts

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### What is Modal Logic?

Modal logic might be defined in one of three ways, each broader than the previous definition.

The *narrowest* definition would be this:

**Modal Logic** is the study of the logic of necessity and possibility.

Generally this involves the study of two operators,  $\Box$  and  $\Diamond$ , meaning ‘necessarily’ and ‘possibly’ (or “it is necessary that...” and “it is possible that...”), respectively. Nevertheless, even under this narrow definition, we are free to regard these symbols as standing for different conceptions of necessity (or possibility):

1. **Logical Necessity**, i.e., true in virtue of logic alone. (E.g., If P then P.)
2. **Metaphysical Necessity**, i.e., true in virtue of the structure of reality. (E.g., nothing can be both blue all over and red all over simultaneously.)
3. **Lawful/Nomological/Causal/Scientific/Physical Necessity**, i.e., true in virtue of the laws of nature. (E.g., water boils at 212 °F.)

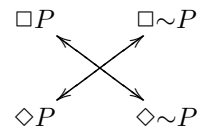
These all describe different ways in which it could be impossible for something not to be true. For each conception of necessity, there would be a corresponding conception of possibility. This narrowest conception of modal logic is sometimes called *alethic modal logic*.

A broader definition would be this:

**Modal Logic** is the study of pairs of operator that obey the *modal square of opposition*.

What does that mean? Notice that  $\Box$  and  $\Diamond$  are related to another under their normal interpretation so that ‘ $\sim\Box\sim$ ’ is equivalent to ‘ $\Diamond$ ’, and ‘ $\sim\Diamond\sim$ ’ is equivalent to ‘ $\Box$ ’. (Something is possible just in case

it is not necessarily not the case; something is necessary just in case it is not possibly not the case.) Because of this, the true logical opposite of saying that something is necessary is saying that it is possibly not the case, and the opposite of saying it is possible is saying that it is necessarily not the case. We can draw a chart like this:



This is the **modal square of opposition**. The arrows indicate which forms are the logical contradictions of one another.

But notice that there are other pairs of operators related to each other in just this way. Consider the pairs ‘it is (morally) obligatory that...’ and ‘it is (morally) permissible that...’, and interpret the box and diamond in the modal square of opposition as meaning these. One *might* even say that we have here a different conception of necessity and possibility where necessity does not entail truth. Since these pairs of operators obey the modal square of opposition, the study of the logic of these operators (usually called *Deontic logic*) would count as a species of modal logic for the broader definition.

The broadest definition for our purposes could be this:

**Modal Logic** is the study of logical operators that generate intensional contexts.

So what’s an intensional context? In terminology dating back to the Enlightenment, but updated and popularized more recently by the philosopher and logician Rudolf Carnap, the *extension* of a sentence is its truth-value, the extension of a predicate is the class of things to which it applies, and the extension of a name/description is its bearer. However, the *intension* of a sentence is the abstract thought or

proposition it expresses, the intension of a predicate is the property, concept or idea it represents, and the intension of a name is its ‘meaning’, ‘sense’ or the ‘individual concept’ it uses to pick out its bearer.

In the logical systems you’ve likely studied so far, starting from a given statement, it is always possible to infer what results from replacing a subexpression in that statement with one with the same extension.

- In propositional logic, for example, from ...  $P$  ... and  $P \leftrightarrow Q$  it is always possible to infer ...  $Q$  ...
- In first-order predicate logic, from ...  $F$  ... and  $\forall x(Fx \leftrightarrow Gx)$  one can always infer ...  $G$  ...
- In first-order predicate logic with identity, from ...  $a$  ... and  $a = b$  one can always infer ...  $b$  ...

As a result of this, these kinds of logical systems are called ‘*extensional*’. Notice, however, that in the scope of an operator like ‘necessarily’, replacing an expression with one with the same extension can change a true statement to a false one. Consider:

- $\square(2 + 2 = 4)$  (TRUE)
- $\square(\text{Kevin teaches philosophy.})$  (FALSE)
- $\square(\text{All cordates have hearts.})$  (TRUE)
- $\square(\text{All cordates have kidneys.})$  (FALSE)
- $\square(8 > 7)$  (TRUE)
- $\square(\text{The number of planets} > 7)$  (FALSE)

Contexts in which one cannot validly replace an expression with another with the same extension are known as *intensional contexts*, since in these contexts, intension matters, not just extension. Operators that generate such contexts are known as *intensional contexts*. The  $\square$  and  $\diamond$  of alethic modal logic are obviously intensional operators. However, others exist, including those involving belief, knowledge and other so-called *propositional attitudes*. (These are studied in *Doxastic logic* and *Epistemic logic*, among others.) Consider, e.g.:

Newton knew that  $2 + 2 = 4$ . (TRUE)  
 Newton knew that Kevin teaches philosophy. (FALSE)

No one doubts that all cordates have hearts. (TRUE)  
 No one doubts that all cordates have kidneys. (FALSE)

Hegel believed that  $8 > 7$ . (TRUE?)  
 Hegel believed that the number of planets  $> 7$ . (FALSE)

This course focuses mostly on alethic modal logic, which falls under all three definitions, though we shall sometimes briefly consider other forms of modal logic (in the broader senses).

# 1 Propositional Logic Revisited

We begin our discussion of modal logic by looking at systems that add modal operators to *propositional logic*, also known as *sentential logic*.

## 1.1 Notation

Following Garson, we use lowercase letters ( $p, q, r, p', p'', \text{etc.}$ ) for simple or atomic statements. (You may be used to using uppercase letters.)

Garson uses uppercase letters ( $A, B, C, \text{etc.}$ ) schematically for any statements, simple or complex. You may be used to using fancy script letters, lowercase letters or Greek letters for this purpose.

	Garson	Alternatives
Negation	$\sim$	$\neg, -$
Conjunction	$\&$	$\wedge, \bullet$
Disjunction	$\vee$	$+$
Conditional	$\rightarrow$	$\supset$
Biconditional	$\leftrightarrow$	$\equiv$
Absurdity	$\perp$	$\times, f$

Garson uses  $\perp$  as a sign for a logically absurd proposition or contradiction. It is not only used at the end of indirect proofs (AKA reductio proofs), but can occur as part of a larger statement.

It is useful in the setting of more advanced logical systems to take some notation as *primitive*, and nother notation as defined. Garson takes  $\rightarrow$  and  $\perp$  as primitive. *Officially*, the language of Garson’s propositional logic only makes use of these two symbols (along with lowercase letters). However, the others are used informally as *abbreviations* for equivalent statement forms making use of only these.

(Def $\sim$ )	$\sim A$	<i>abbreviates</i>	$(A \rightarrow \perp)$
(Def $\vee$ )	$(A \vee B)$	<i>abbreviates</i>	$(\sim A \rightarrow B)$
(Def $\&$ )	$(A \& B)$	<i>abbreviates</i>	$\sim(A \rightarrow \sim B)$
(Def $\leftrightarrow$ )	$(A \leftrightarrow B)$	<i>abbreviates</i>	$(A \rightarrow B) \& (B \rightarrow A)$

## 1.2 Deductive system

With the above notational simplifications, we can reduce all of standard propositional logic to five rules:

*Hypothesis* (Hyp): a hypothesis or assumption can be made at any time, provided it begins a new subproof.

*Modus ponens* (MP): from  $A \rightarrow B$  and  $A$  infer  $B$  within a given subproof.

*Double negation* (DN): from  $\sim\sim A$  infer  $A$  within a given subproof.

*Conditional proof* (CP): if  $B$  is derived in a subproof that begins with hypothesis  $A$ , the subproof in question can be closed, and one may infer  $A \rightarrow B$ .

*Reiteration* (Reit): an available statement  $A$  can be copied from the previous subproof into a given subproof that begins with a hypothesis.

(We also allow ourselves to replace a defined expression with what it is defined as, and vice-versa, though strictly speaking, this is not a new rule of our system, but just rewriting a line we already have with different abbreviations.)

Example:

1.	$p \rightarrow (q \rightarrow r)$	(Premise)
2.	$p \rightarrow q$	
3.	$p \rightarrow (q \rightarrow r)$	1 (Reit)
4.	$p$	
5.	$p \rightarrow (q \rightarrow r)$	3 (Reit)
6.	$q \rightarrow r$	4,5 (MP)
7.	$p \rightarrow q$	2 (Reit)
8.	$q$	4,7 (MP)
9.	$r$	6,8 (MP)
10.	$p \rightarrow r$	4–9 (CP)
11.	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	2–10 (CP)

Garson uses Fitch-style notation for tracking subproofs, which does not use ‘show’ lines, and marks hypotheses with horizontal lines and the scope of their applicability with vertical lines. You may be more

used to another style, but hopefully it won’t be difficult to adjust.

Apart from (Reit) we can only appeal to lines in the current subproof. This restriction may seem pointless, but its necessity will become obvious when we move to the modal systems. In the previous example, we cannot skip lines like 3, 5 or 7.

Because the other signs are all defined, we do not need any additional primitive rules. All other rules of propositional logic can be considered *derived rules*. Notice, for example, that we only introduced one form of (DN) above. What about the inference from  $A$  to  $\sim\sim A$ ? Such a rule is not necessary, since we can do it using (MP) and (CP):

$A$	
$\sim A$	
$A \rightarrow \perp$	(Def $\sim$ )
$A$	(Reit)
$\perp$	(MP)
$\sim A \rightarrow \perp$	(CP)
$\sim\sim A$	(Def $\sim$ )

Once we have shown that an inference pattern is derivable, we can introduce an abbreviation for it and use it in later proofs as if it were a primitive rule. (We’ll use (DN) for the above as well.) For example, we can use (MT) for the rule derivable as follows:

$A \rightarrow B$	
$\sim B$	
$A$	
$A \rightarrow B$	(Reit)
$\sim B$	(Reit)
$B$	(MP)
$B \rightarrow \perp$	(Def $\sim$ )
$\perp$	(MP)
$A \rightarrow \perp$	(CP)
$\sim A$	(Def $\sim$ )

One startling fact is that given the definition of  $\sim$ , Indirect Proof or *reductio ad absurdum* is the same thing as (CP). Any contradiction of the form  $B$  and  $\sim B$  easily yields  $\perp$ —by (MP)! Assuming  $A$  and yield-

ing a contradiction  $\perp$  yields  $A \rightarrow \perp$  by (CP), which simply is  $\sim A$ .

Here's a demonstration for the derivability of one of the two forms of  $\&$ -Elimination (or simplification):

1.	$A \& B$	
2.	$\sim(A \rightarrow \sim B)$	1 (Def $\&$ )
3.	$\sim B$	
4.	$A$	
5.	$\sim B$	3 (Reit)
6.	$A \rightarrow \sim B$	4-5 (CP)
7.	$\sim(A \rightarrow \sim B)$	2 (Reit)
8.	$(A \rightarrow \sim B) \rightarrow \perp$	7 (Def $\sim$ )
9.	$\perp$	6,8 (MP)
10.	$\sim B \rightarrow \perp$	3-9 (CP)
11.	$\sim\sim B$	10 (Def $\sim$ )
12.	$B$	11 (DN)

The abbreviations Garson uses for propositional logic are listed on pages 35-36 of his book. I am not fussy about abbreviations. If you would like to use different abbreviations for the same rules (and additional abbreviations for further derived rules, etc.), you may use whatever you like provided I'll be able to understand them when I read your homework.

It is perhaps worth noting that Garson uses the abbreviation ( $\vee$ -Out) for the "proof by cases" strategy, not for disjunctive syllogism, which conflicts with Hardegree's usage of ( $\vee$ O).

Garson's calls his logical system PL. (Feel free to use 'SL' for "sentential logic" instead if you are used to that.)

**A theorem** of PL is a statement that can be proven outside of any subproofs without using any premises or undischarged hypotheses. (A statement you can prove using the logical rules alone.)

When  $A$  is a theorem of PL we write  $\vdash_{PL} A$  (or simply  $\vdash A$  if context makes it clear we are talking about PL).

If a statement  $B$  can be proven in PL from the statements  $A_1, A_2, \dots, A_n$ , we write  $A_1, A_2, \dots, A_n \vdash_{PL} B$ .

Note that *unlike* ' $\perp$ ', the sign ' $\vdash$ ' is not itself part of the logical symbols included in PL. It is only part of the language we use when discussing PL and other

logical languages (the so-called '*metalanguage*'). It cannot occur embedded *inside* a PL statement (e.g.,  $A \rightarrow (B \vdash A)$ , etc.)

### Some important facts about PL:

- PL is *sound*, i.e., if  $\vdash_{PL} A$  then  $A$  is a truth-table tautology, and if  $A_1, A_2, \dots, A_n \vdash_{PL} B$  then  $A_1, A_2, \dots, A_n, \therefore B$  is valid according to truth tables.
- PL is *complete*, i.e., if  $A$  is a truth-table tautology, then  $\vdash_{PL} A$ , and if  $A_1, A_2, \dots, A_n, \therefore B$  is valid according to truth tables, then  $A_1, A_2, \dots, A_n \vdash_{PL} B$ .

We shall not attempt to prove these facts here. If you are interested, we cover Soundness and Completeness proofs for a system very similar to PL in the Mathematical Logic I course (Phil 513).

Garson's PL isn't even the most austere way to formulate the laws of propositional logic. Indeed, it is possible to get by with just a single logical primitive, the *Sheffer stroke*, where  $(A|B)$  means "not both  $A$  and  $B$ ", and define all the others in terms of it. Using this primitive one can get by with only two rules:

*Nicod's axiom* (NA): any statement of the form  $(B|(C|D))|((E|(E|E))|((F|C)|((B|F)|(B|F))))$  may be written into any line of a proof.

*Nicod's rule* (NR): from  $B|(C|D)$  and  $B$  one may infer  $D$ .

However simple, unfortunately, this system is far too alien to our ordinary ways of thinking to be practical for most purposes.