2 Modal System K

Modal logic is not the name of a single logical system; there are a number of different logical systems that make use of the signs '$\Box$' and '$\Diamond$', each with its own set of rules.

Each system reflects a different understanding or conception of '$\Box$' and '$\Diamond$', or the logical properties of necessity and possibility.

We begin with System K, which forms the basis for many other systems. (I.e., those systems are formed by adding additional rules to K.) System K is named after the logician-philosopher Saul Kripke, who was among the first to systematize formal semantics for modal logic.

The rules of K are consistent with many different understandings of '$\Box$' and '$\Diamond$' (including, among others, deontic ones).

It should perhaps be noted that Garson's method for formalizing K is different from the usual way (or Kripke's way), but it results in the same theorems and derivability results.

2.1 Syntax

The logical language of System K is derived from that of PL by adding a single primitive monadic propositional operator, '$\Box$'.

It behaves *syntactically* precisely as '$\sim$' does. Its scope is only what immediately comes after it. "$\sim p \to q$" means "if not p then q,", whereas "$\sim(p \to q)$" means "it's not true that if p then q." Similarly, "$\Box p \to q$" means "if, necessarily, p, then q,", whereas "$\sim(p \to q)$" means "necessarily, if p then q."

The sign for possibility is not taken as primitive, but defined in terms of '$\Box$'; we'll also define signs for so-called *strict implication* and *strict equivalence*.

(Def $\Diamond$)	$\Diamond A$	*abbreviates*	$\sim\Box\sim A$
(Def $\strictif$)	$(A \strictif B)$	*abbreviates*	$\Box(A \to B)$
(Def $\strictiff$)	$(A \strictiff B)$	*abbreviates*	$(A \strictif B)$ & $(B \strictif A)$

The sign '$\strictif$' was introduced by C.I. Lewis in his pioneering work on modal logic, much of which was motivated by the desire to define a logical notion of "if... then..." that did not share the oddities of the material conditional. "$(A \strictif B)$" can be read, "A strictly implies B," or "A entails B."

2.2 Boxed Subproofs

The inference rules of K involve the notion of a boxed subproof, which is a subproof that is headed not with a hypothesis, but rather simply with the sign '$\Box$'.

Informally, the $\Box$-header indicates that *every line* in the subproof is shown to be necessarily true.

We cannot bring statements into boxed subproofs using the (Reit) rule—that would result in our regarding every true or assumed-true statement as necessarily true. Instead we have another rule, ($\Box$Out).

($\Box$Out) — if a line of the form $\Box A$ is available in the subproof previous to a given boxed subproof, one may infer A within the boxed subproof.

The purpose of a boxed subproof is to be able to prove statements of the form $\Box A$. Hence we have:

($\Box$In) — if a statement A is derived within a boxed subproof, the boxed subproof may be closed and one may infer $\Box A$ outside of that subproof.

System K consists only of the two rules above along with the rules of PL.

As an example proof in K, below we prove that every instance of the *Distribution schema* is a theorem of K:

(Dist) $\vdash_K \Box(A \to B) \to (\Box A \to \Box B)$
Proof:

1.	$\mid\ \Box(A \to B)$	
2.	$\mid\ \mid\ \Box A$	
3.	$\mid\ \mid\ \Box(A \to B)$	1 (Reit)
4.	$\mid\ \mid\ \mid\ \Box$	
5.	$\mid\ \mid\ \mid\ A$	2 ($\Box$Out)
6.	$\mid\ \mid\ \mid\ A \to B$	3 ($\Box$Out)
7.	$\mid\ \mid\ \mid\ B$	5,6 (MP)
8.	$\mid\ \mid\ \Box B$	4–7 ($\Box$In)
9.	$\mid\ \Box A \to \Box B$	2–8 (CP)
10.	$\Box(A \to B) \to (\Box A \to \Box B)$	1–9 (CP)

Using boxed subproofs and PL rules, it is possible to derive $\Box A$ in K for any truth-table tautology A. For example, here is a proof of $\Box(p \to (q \to p))$:

1.	$\mid\ \Box$	
2.	$\mid\ \mid\ p$	
3.	$\mid\ \mid\ \mid\ q$	
4.	$\mid\ \mid\ \mid\ p$	2 (Reit)
5.	$\mid\ \mid\ q \rightarrow p$	3–4 (CP)
6.	$\mid\ p \rightarrow (q \rightarrow p)$	2–5 (CP)
7.	$\Box(p \rightarrow (q \rightarrow p))$	1–6 ($\Box$In)

Indeed, any theorem of K (something that can be proven without use of premises or undischarged hypotheses) can be proven to be necessary in K.

The Rule of Necessitation (Nec): if $\vdash_K A$ then $\vdash_K \Box A$.

Indeed, the usual formulation of K is to take (Nec) as a primitive *inference rule* along with every instance of (Dist) as an *axiom* (i.e., something that can be written into any line of a proof or subproof with no other line as justification); these two together are equivalent to having ($\Box$In) and ($\Box$Out).

The rule of necessitation coheres with almost all conceptions of necessity: if something can be proven using the rules of logic alone, then it must be necessary.

Do not confuse this rule with the untrue claim that $\vdash_K A \rightarrow \Box A$.

Indeed, those conceptions of necessity that are consistent with the rules of System K can roughly be characterized by two features: (i) what can be proven true without assumptions is necessary (i.e, (Nec)), and (ii) that which necessarily follows from necessary truths is also necessary (i.e, (Dist)).

You'll notice that from here on out, I'll almost never give proofs involving lowercase statement letters like 'p', 'q' and 'r', but rather schematic letters like 'A', 'B' and 'C'. This makes the results proven general so that they can be applied in other contexts. Moreover, if all instances of a certain schema are shown to be theorems, we shall allow ourselves to introduce one into any proof at any time, rather than duplicate the steps of the proof already given.

2.3 Inferences Involving Possibility

Becuase '$\Diamond$' is defined in terms of '$\Box$', no special basic rules are needed for '$\Diamond$'.

Here are some easy derived rules:

($\sim\Diamond$) From $\sim\Diamond A$ infer $\Box\sim A$. (Really just (DN)).

($\sim\Box$) From $\sim\Box A$ infer $\Diamond\sim A$. *Proof:*

1.	$\sim\Box A$	
2.	$\mid\ \Box\sim\sim A$	
3.	$\mid\ \sim\Box A$	1 (Reit)
4.	$\mid\ \mid\ \Box$	
5.	$\mid\ \mid\ \sim\sim A$	2 ($\Box$Out)
6.	$\mid\ \mid\ A$	5 (DN)
7.	$\mid\ \Box A$	4–6 ($\Box$In)
8.	$\mid\ \bot$	3, 7 ($\bot$In)
9.	$\sim\Box\sim\sim A$	2–8 (IP)
10.	$\Diamond\sim A$	9 (Def $\Diamond$)

Because such statements are actually negations of statements with '$\Box$', if '$\Diamond$' occurs on both sides of a conditional, it may be easier to prove the contrapositive of the result you're after.

(Dist$\Diamond$) $\vdash_K \Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$. *Proof:*

1.	$\mid\ \Box(A \rightarrow B)$	
2.	$\mid\ \mid\ \Box\sim B$	
3.	$\mid\ \mid\ \Box(A \rightarrow B)$	1 (Reit)
4.	$\mid\ \mid\ \mid\ \Box$	
5.	$\mid\ \mid\ \mid\ \sim B$	2 ($\Box$Out)
6.	$\mid\ \mid\ \mid\ A \rightarrow B$	3 ($\Box$Out)
7.	$\mid\ \mid\ \mid\ \sim A$	5,6 (MT)
8.	$\mid\ \mid\ \Box\sim A$	4–7 ($\Box$In)
9.	$\mid\ \Box\sim B \rightarrow \Box\sim A$	2–8 (CP)
10.	$\mid\ \sim\Box\sim A \rightarrow \sim\Box\sim B$	9 (CN)
11.	$\mid\ \Diamond A \rightarrow \Diamond B$	10 (Def $\Diamond$)
12.	$\Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$	1–11 (CP)

The above theorem schema is also important for obtaining the following derived rule, represented schematically:

$$
\begin{array}{lll}
\Diamond A & & \\
\quad \mid \Box & & \\
\quad \mid \quad \mid A & & \\
\quad \mid \quad \mid \vdots & & \\
\quad \mid \quad \mid B & & \\
\Diamond B & & \textbf{(}\Diamond\textbf{Out)}
\end{array}
$$

In short, if we know that A is possible, and within a boxed subproof, can prove that B would be true if we assumed A were true, we are allowed to conclude that B is possible. This is because we cold fill in the remainder of this proof schema as follows:

$$
\begin{array}{ll}
\Diamond A & \\
\quad \mid \Box & \\
\quad \mid \quad \mid A & \\
\quad \mid \quad \mid \vdots & \\
\quad \mid \quad \mid B & \\
\quad \mid A \to B & \text{(CP)} \\
\Box(A \to B) & (\Box\text{In}) \\
\Box(A \to B) \to (\Diamond A \to \Diamond B) & (\text{Dist}\Diamond) \\
\Diamond A \to \Diamond B & \text{(MP)} \\
\Diamond B & \text{(MP)}
\end{array}
$$

Proofs of this form are quite common. So common, in fact, that it is worthwhile to introduce a further abbreviated notation for representing the two subproofs involved in an application of ($\Diamond$Out) as though it were a single subproof. Garson does this as follows:

$$
\begin{array}{l}
\mid \Box, A \\
\mid \vdots
\end{array}
$$

This kind of subproof is called a *world-subproof*. If we understand necessity as "truth in all possible worlds", a normal boxed subproof represents what we can prove about all possible worlds absolutely. A world-subproof represents what we can prove about all those possible worlds in which a certain hypothesis holds—in this case, all those where A is true.

We shall generalize this style of abbreviation to other contexts in which we have subproofs within subproofs, so that, for example:

$$
\begin{array}{ll}
\mid A, B, C & \\
\mid \vdots & \\
\mid D & \\
A \to (B \to (C \to D)) & \text{(CP)}
\end{array}
$$

can be used as a convenient shorthand for three embedded conditional proofs of the form:

$$
\begin{array}{ll}
\mid A & \\
\mid \quad \mid B & \\
\mid \quad \mid \quad \mid C & \\
\mid \quad \mid \quad \mid \vdots & \\
\mid \quad \mid \quad \mid D & \\
\mid \quad \mid C \to D & \text{(CP)} \\
\mid B \to (C \to D) & \text{(CP)} \\
A \to (B \to (C \to D)) & \text{(CP)}
\end{array}
$$

One must simply be careful with regard to (Reit) and ($\Box$Out) for subproofs with multiple headers. (Reit) cannot be used with any header list containing $\ldots, \Box, \ldots$, and ($\Box$Out) can only traverse past one such $\Box$, unless it is applied to something of the form $\Box\Box A$ or $\Box\Box\Box A$, etc.

If it is convenient to have the multiple headers on separate lines (so that they can be cited separately, for example), we also use the abbreviated notation:

$$
\begin{array}{l}
\mid A \\
\mid B \\
\mid C \\
\mid \vdots \\
\mid D
\end{array}
$$

to the same effect.

Here's ($\Diamond$Out) in action. Let's show:

(Dist$\Diamond$&) $\vdash_K \Diamond(A \,\&\, B) \to (\Diamond A \,\&\, \Diamond B)$

1.	$\Diamond(A \,\&\, B)$	
2.	$\quad \Box, A \,\&\, B$	
3.	$\quad A$	3 (&Out)
4.	$\Diamond A$	1,2–3 ($\Diamond$Out)
5.	$\quad \Box, A \,\&\, B$	
6.	$\quad B$	5 (&Out)
7.	$\Diamond B$	1,5–6 ($\Diamond$Out)
8.	$\Diamond A \,\&\, \Diamond B$	4,7 (&In)
9.	$\Diamond(A \,\&\, B) \to (\Diamond A \,\&\, \Diamond B)$	1–8 (CP)

The converse of the above, however, does not hold, but we do have:

$$\vdash_K (\Box A \,\&\, \Diamond B) \to \Diamond(A \,\&\, B)$$

1.	$\Box A \,\&\, \Diamond B$	
2.	$\Box A$	1 (&Out)
3.	$\Diamond B$	1 (&Out)
4.	$\quad \Box, B$	
5.	$\quad A$	2 ($\Box$Out), (Reit)
6.	$\quad A \,\&\, B$	4,5 (&In)
7.	$\Diamond(A \,\&\, B)$	3,4–6 ($\Diamond$Out)
8.	$(\Box A \,\&\, \Diamond B) \to \Diamond(A \,\&\, B)$	1–7 (CP)

3 Extensions of K (T, B, S4, S5)

Logical systems capturing stronger or at least more specific conceptions of necessity can be obtained by adding additional inference rules or axioms to K.

Consider the following schemata:

(M): $\Box A \to A$

(B): $A \to \Box\Diamond A$

(4): $\Box A \to \Box\Box A$

(5): $\Diamond A \to \Box\Diamond A$

According to (M), *whatever is necessary is true*. This axiom should hold for any system codifying a conception of *alethic* modality, where necessity is understood in terms of necessary truth. An obvious derived rule, which we'll also abbreviate as (M), allows us to conclude A from $\Box A$ within the same subproof. (Notice that rule ($\Box$Out) of K does not allow this.) 'M' stands for *modality*, where this word is understood in the narrow sense.

According to (B), *what is actually true is necessarily possible*. This is named after L. E. J. Brouwer.

According to (4), *if something is necessary, it is necessarily necessary*. This is one of the original axioms used by C. I. Lewis in codifying modal logic, and retains the number used in his numbering of axioms.

According to (5), *if something is possible, it is necessarily possible*. This, the strongest of these principles, coheres with the understanding of necessity as truth in *all* possible worlds absolutely. For something to be possible is for it to be true at some world. If it is true at some world, it is true at all worlds that it is true at some world, which is what it is for it to be necessarily possible. Again, the numbering comes from Lewis.

We can define a number of systems by adding the instances of one or more of the above to K:

System T: the system obtained from K by allowing every instance of (M) as an an axiom.

System M: another name for System T.

System B: the system obtained from T by allowing every instance of (B) as an axiom.

System S4: the system obtained from T by allowing every instance of (4) as an axiom.

System S5: the system obtained from T by allowing every instance of (5) as an axiom.

Note that these systems add every instance of certain axiom schemata, not individual axioms. Axiom (M) covers not only

$\Box p \to p$

but everything of that form, including

$\Box{\sim}p \to {\sim}p$
$\Box\Box p \to \Box p$
$\Box({\sim}p \to \Diamond q) \to ({\sim}p \to \Diamond q)$, etc.

In the first proof scheme below, we shall write "$\Box{\sim}A \to {\sim}A$", and cite (M). This is not confusing a statement with its negation but rather making use of a more specific schema all of whose instances are also instances of a more general schema.