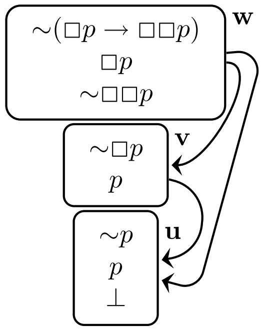
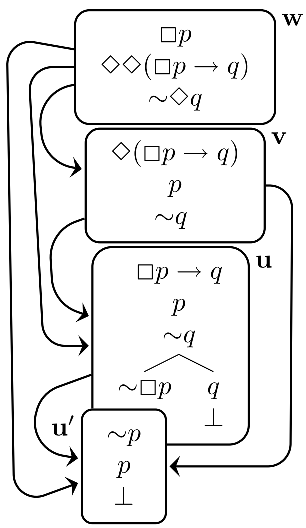


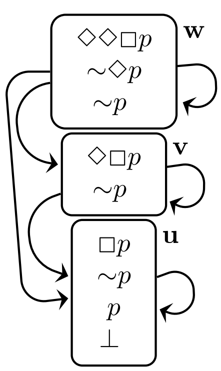
The outer arrow from **w** to **u** captures the transitivity of the accessibility relation in 4-models. Hence we get 'p' as true in **u**, generating the '$\bot$'.

Another example: $\Box p, \Diamond\Diamond(\Box p \to q) \vDash_{\text{K4}} \Diamond q$:



Now *that's* a tree.

Trees for checking S4-validity combine the properties of M-trees and 4-trees, e.g., we can show that $\Diamond\Diamond\Box p \vDash_{\text{S4}} \Diamond p$:

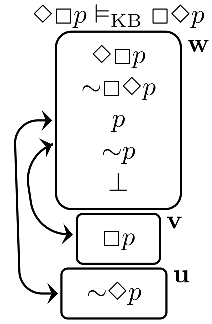


If it becomes tedious to add all the extra arrows in K4- and S4-trees, we can simply pretend that a sequence of arrows counts as a single arrow (or "accessibility path") for such trees when applying the tree rules. Just remember that the same does not hold for K-trees.

Of course, a tree can also be used to show that an argument or statement is K4- or S4-invalid.

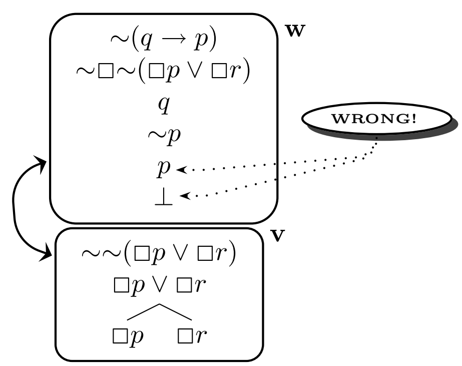
In KB-models the accessibility relation is symmetric: whenever it holds in one direction, it holds in the other. Trees for KB-models can be drawn by making every accessibility link go both ways. The result is that some of the rules are applied to a world box earlier on the same branch, which violates Garson's Placement Principle. However, this is harmless if there is no branching in between.

$\Diamond\Box p \vDash_{\text{KB}} \Box\Diamond p$



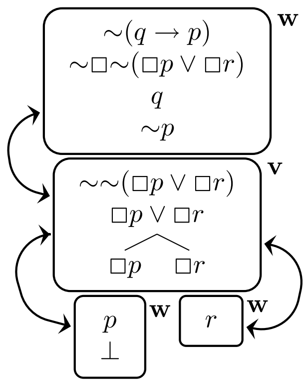
Here the 'p' in world **w** comes from world **v** below, and the "$\sim p$" comes from world **u**. The resulting $\bot$ in **w** closes the branch (and, in this case, the tree): the contradiction does not need to be at the bottom of the branch.

If there is branching in between, this method cannot be used. Here is a misuse of this method applied to the argument $\sim(q \to p)$ /∴ $\Box\sim(\Box p \lor \Box r)$:



On the left branch of the tree, we have "$\Box p$" in world **v**. Since the accessibility between **w** and **v** goes both ways, this does mean that *on that branch*, 'p' would have to be true in **w**. However, we cannot write 'p' at the bottom of the box for **w** we already have, since that would make it apply to both branches. Since 'p''s being true leads to $\bot$ in **w**, this makes it appear as if both branches are closed, whereas in fact only the left branch closes.

The correct method is to use a second box for **w** underneath, giving it the same label:



Since the box on the bottom left is a "continuation" of world **w**, 'p''s being true there leads to $\bot$ and closes that branch. However, the branch on the right remains open, and in fact, $\sim(q \to p) \nvDash_{\mathrm{KB}} \Box\sim(\Box p \vee \Box r)$.

Split world boxes like this are not needed often.

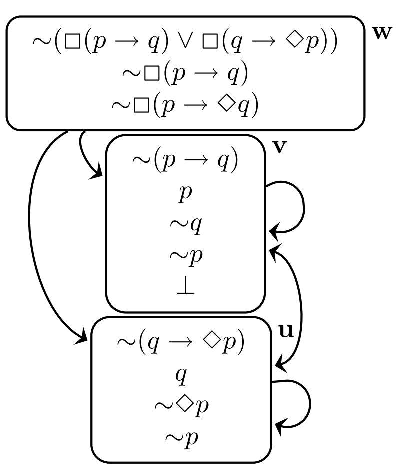
As you might guess, trees for checking B-validity combine the characteristics of KB-trees and M-trees, so that the accessibility relation would be both reflexive and symmetric.

In 5-models, the accessibility relation is Euclidian. This means that whenever **wRv** and **wRu** we also have that **vRu** for any worlds **w**, **v** and **u**.

For trees, this means that whenever, on a given branch, we have an arrow from **w** to one world and from **w** to another world, those two worlds are themselves connected in both ways.

Why in both ways? Well, suppose $\mathbf{wRw'}$ and $\mathbf{wRw''}$. Then $\mathbf{w'Rw''}$. But's also true that $\mathbf{wRw''}$ and $\mathbf{wRw'}$, and so $\mathbf{w''Rw'}$.

We automatically draw those lines in when constructing a K5-(or S5-)tree. Let us show that $\vDash_{\mathrm{K5}} \Box(p \to q) \vee \Box(q \to \Diamond p)$.

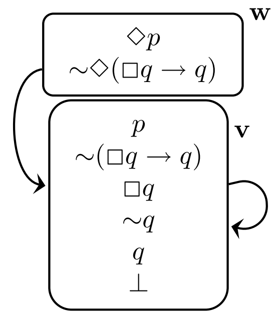


Here the "$\sim p$" in world **v** comes from the "$\sim\Diamond p$" in **u** below it. This closes the branch and tree.

You may have noticed that I gave worlds **u** and **v** here (but not **w**) reflexivity arrows. This is because whenever **R** is Euclidian it is also *shift reflexive*. Suppose **wRv**; then, trivially, **wRv** *and* **wRv**. It follows by the definition of a Euclidian relation that **vRv**.

Only the "root" world, **w**, need not be accessible to itself. In fact, in K5-trees, apart from the root world, every world on a given branch will be accessible to every other world on that branch, including itself.

An example where this is important is the tree for $\Diamond p \vDash_{\mathrm{K5}} \Diamond(\Box q \to q)$.



To obtain a tree where *every* world on a given branch is guaranteed to be accessible to/from every other world on the branch, one need only assume that its accessibility relation is not only Euclidian, but uniformly reflexive, so that even the root world of the tree will access itself. This is the case with S5-trees, since S5-models are defined to have both reflexive and Euclidian accessibility relations.

Further facts about **R** in S5-models:

- **R** is symmetric. Why? Suppose **wRv**. Since **R** is reflexive, **wRw**. Since **wRv** and **wRw**, **vRw**.
- **R** is transitive. Suppose **wRv** and **vRu**. Since **R** is symmetric, **vRw**. Since **R** is Euclidian and both **vRw** and **vRu**, it must be that **wRu**.

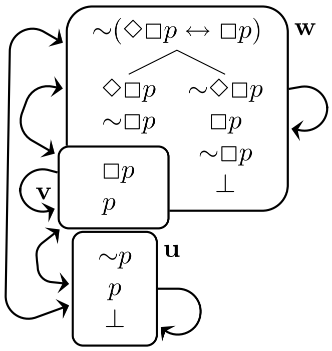
In other words, all S5-models are also S4-models and B-models.

When drawing a tree to check S5-validity, one may automatically insert arrows between any two worlds on the same branch.

It may be cleaner simply to leave off all accessibility arrows on an S5-tree, and simply apply the rules for true $\Box$- and false $\Diamond$-statements from within a given world to all worlds on the branch.

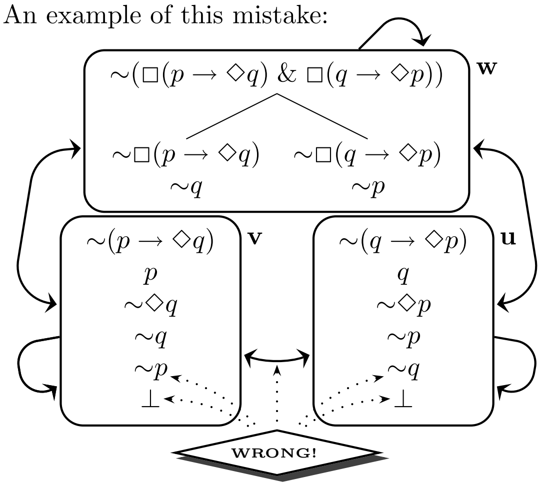
Just remember that this is only appropriate for S5-trees. (I'll continue to draw the arrows.)

Let us verify that $\vDash_{\mathrm{S5}} \Diamond\Box p \leftrightarrow \Box p$.

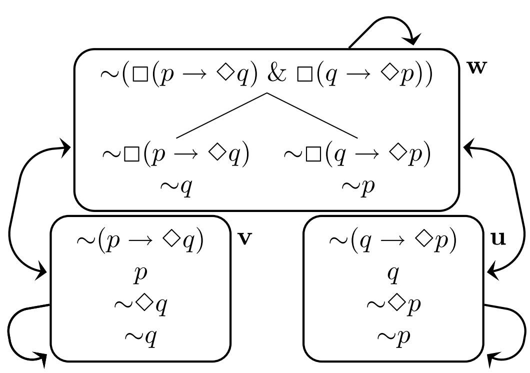


Although on S5-trees, any world on the same branch is accessible to every other world, one must be careful when drawing trees not to draw arrows between worlds on *different branches*. Each branch of a tree repesents a *different* potential modal model. As far as one branch is concerned (except for those on their common trunk) the worlds lying on other branches do not exist.

An example of this mistake:



The correct tree is instead:



As you can see, neither branch of this tree closes, as indeed $\nvDash_{S5} \Box(p \rightarrow \Diamond q)$ & $\Box(q \rightarrow \Diamond p)$. (This is also something to watch for when drawing K5 trees.)

Notice also in the above that the "$\sim q$" in **w** that comes from **v**, and the "$\sim p$" in **w** that comes from **u**, are placed on the appropriate branches, not in the "trunk".

5.6 Accessibility vs. Closeness

In philosophical discussions, S5 is the most widely accepted logic for absolute conceptions of modality like logical or metaphysical necessity. In S5, all possible worlds are equally accessible to one another. Hence, the accessibility relation must not be confused with the *ordering* relation among possible worlds (making certain worlds "nearer" or "more distant" from the actual world), often invoked in discussions of counterfactuals.

You may have seen reference to so-called subjunctive conditionals, usually written:

$$A \mathrel{\Box\!\!\rightarrow} B$$

Roughly, this means, "if it were that A, it would be that B."

The truth conditions of such subjunctive conditionals are controversial, but are often described somewhat like this (following Stalnaker):

$$\mathbf{a_w}(A \mathrel{\Box\!\!\rightarrow} B) = \begin{cases} \text{T} & \text{if } \mathbf{a_v}(B) = \text{T in all worlds} \\ & \quad \mathbf{v} \text{ such that } \mathbf{a_v}(A) = \text{T} \\ & \quad \text{and } \mathbf{v} \text{ is at least as} \\ & \quad \text{close to } \mathbf{w} \text{ as any other} \\ & \quad \text{world } \mathbf{u} \text{ such that} \\ & \quad \mathbf{a_u}(A) = T \\ \text{F} & \text{otherwise} \end{cases}$$

Unfortunately, $A \mathrel{\Box\!\!\rightarrow} B$ cannot be defined in terms of '$\Box$' and '$\rightarrow$'. Were we to add the sign '$\Box\!\!\rightarrow$' to the language of our propositional modal logic, in order to do its formal semantics, we would need our models to take the form $\langle \mathbf{W, R, C, a} \rangle$ where **C** would be a three-place relation between worlds (meaning, **v** is at least as close to **w** as **u** is).

Although both represent conditionals stronger than the material conditional $A \rightarrow B$, a subjunctive conditional $A \mathrel{\Box\!\!\rightarrow} B$ must not be confused with $A \dashv B$. Indeed, they have different properties:

$(A \dashv B)$ & $(B \dashv C) \vDash_{K+} A \dashv C$
$A \dashv B \vDash_{K+} \sim B \dashv \sim A$
$A \dashv B \vDash_{K+} (A$ & $C) \dashv B$

However,

$(A \mathrel{\Box\!\!\rightarrow} B)$ & $(B \mathrel{\Box\!\!\rightarrow} C) \nvDash A \mathrel{\Box\!\!\rightarrow} C$
$A \mathrel{\Box\!\!\rightarrow} B \nvDash \sim B \mathrel{\Box\!\!\rightarrow} \sim A$
$A \mathrel{\Box\!\!\rightarrow} B \nvDash (A$ & $C) \mathrel{\Box\!\!\rightarrow} B$

We will not be able to discuss formal languages containing '$\Box\!\!\rightarrow$' in detail in this course, however.