An argument $A_1, A_2, \ldots A_n$ /∴ B is said to have a **K-counterexample** iff there exists a K-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$ and world $\mathbf{w}$ in $\mathbf{W}$, for which $\mathbf{a_w}(A_1)$ = T and $\mathbf{a_w}(A_2)$ = T and …and $\mathbf{a_w}(A_n)$ = T but $\mathbf{a_w}(B)$ = F.

A statement A is a **K-valid** iff for *every* K-model, $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$, and world $\mathbf{w}$ in $\mathbf{W}$, $\mathbf{a_w}(A)$ = T.

Notions of **K-satisfiability**, **K-equivalence**, etc., can be defined similarly.

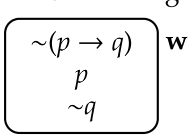
We write "$A_1, A_2, \ldots A_n \vDash_K B$" to mean that the argument $A_1, A_2, \ldots A_n$ /∴ B is K-valid, or simply "$\vDash_K A$" to assert that the statement A is K-valid (or K-logically true).

We later attempt to show that if $A_1, A_2, \ldots A_n \vdash_K B$ then $A_1, A_2, \ldots A_n \vDash_K B$ (soundness of K), and also that if $A_1, A_2, \ldots A_n \vDash_K B$ then $A_1, A_2, \ldots A_n \vdash_K B$ (completeness of K). Similarly, (or as a special case of the above), we will want to show that $\vDash_K A$ iff $\vdash_K A$.

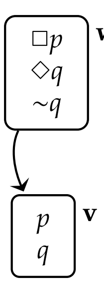
5.4 Trees for K

To expand the truth-tree method for determining the validity of an argument of statement in propositional modal logic, we need two things. First, we enclose certain parts of trees within boxes to represent different possible worlds (and add arrows between boxes to represent the accessibility relation). Second, we add tree formation rules dealing with the operators '□' and '◇'.

A world box, with a label such as '**w**', containing statements represents that those statements are true at world **w**. E.g.:



represents a world **w** where "$p \rightarrow q$" is false, and hence where 'p' is true and 'q' false.

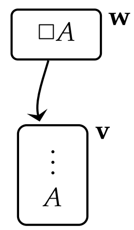


represents two worlds, **w** and **v**: in **w**, "□p", "◇q" are true but 'q' is false, and in **v**, 'p' and 'q' are both true. It also holds that **wRv** (**v** is accessible from **w**). 'q' being true in one world and false in another does not yield ⊥.

To close a branch we need A and $\sim A$ in the *same* world box.

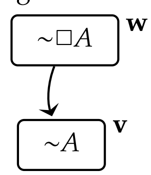
We add rules governing the modal operators. As before, for each operator, we have two rules: one for statements with it as main operator and one for their negations.

First, for true statements of the form □A:



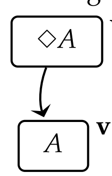
for *every* world **v** on the current tree branch accessible to **w**.

A statement of the form □A is false when there is at least one world **v** such that **wRv** and $\mathbf{a_v}(A)$=F, i.e., $\mathbf{a_v}(\sim A)$=T. This gives us the following tree rule for negations of □-statements:



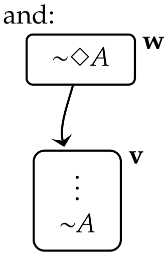
where **v** is a *new* world not already on the tree.

Given the definition of '◇' in terms of '□', we get the following rules for '◇'.



where **v** is a *new* world not already on the tree.

and:



for *every* world **v** on the current tree branch accessible to **w**.

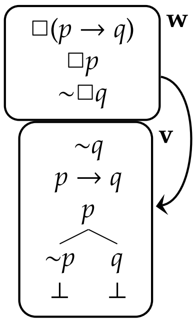
Notice that the second and third rules resemble quantifier rules for ∃-instantiation/elimination/out, where you introduce *new* names.

The first and fourth resemble the rules for ∀-instantiation/elimination/out, where these can be done to existing names.

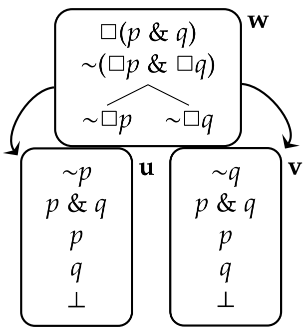
(There is a slight difference, however, in that one is not allowed to apply the modal tree rules in the latter group to create new worlds; the worlds *must* already exist.)

The proper application of these rules is best seen by example.

Here's a tree showing the K-validity of $\Box(p \to q)$, $\Box p \;/\!\therefore\; \Box q$.



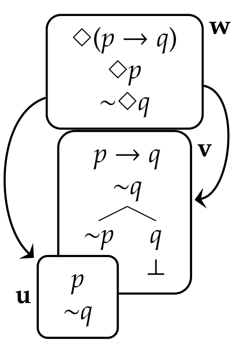
Here's one for $\Box(p \,\&\, q) \;/\!\therefore\; \Box p \,\&\, \Box q$.



Note that branches can span worlds, and worlds can exist 'on' a certain branch but not another. In the above, **u** is on the left branch, and **v** on the right (only).

Construction and proper placement of worlds on trees with multiple branches is a complicated affair, which is difficult to cover fully on the handouts. Be sure to read chapter 4 of Garson's book carefully.

Trees also help us find counterexaples to K-invalid arguments, such as $\Diamond(p \to q), \Diamond p \;/\!\therefore\; \Diamond q$.



A couple things to note here.

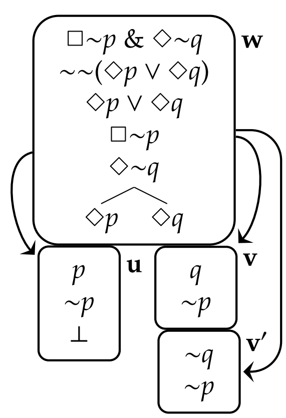
World **u** is on left branch that begins inside world **v**. However, notice that it is *accessible* from world **w**. This is because it is the result of applying the tree rule to "$\Diamond p$", which is found in **w**, not **v**.

More importantly, one of the branches of this tree did not close. ('p' being true in world **u** does not conflict with it being false in **v**.)

This allows us to construct a K-counterexample to the argument, $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$, where the premises are true and the conclusion false at some world.

- The worlds making up the set **W** of the model are those found on the open branch of the tree. In this case $\mathbf{W} = \{\mathbf{w}, \mathbf{v}, \mathbf{u}\}$.
- The accessibility relation **R** of this model is given by the arrows between the worlds on the branch of the tree. If no arrow is drawn we can assume the accessibility relation does not hold. In this case $\mathbf{R} = \{\langle \mathbf{w}, \mathbf{v} \rangle, \langle \mathbf{w}, \mathbf{u} \rangle\}$.
- The valuation function **a** can be assumed to assign a given statement the truth value T for a world if it is found on the open branch inside the box for that world, and assign it a F for that world if its negation (or neither) is found there. In this case, $\mathbf{a}_{\mathbf{w}}(p)$=F, $\mathbf{a}_{\mathbf{w}}(q)$=F, $\mathbf{a}_{\mathbf{v}}(p)$=F, $\mathbf{a}_{\mathbf{v}}(q)$=F, $\mathbf{a}_{\mathbf{u}}(p)$=T and $\mathbf{a}_{\mathbf{u}}(q)$=F.
- In this model, "$\Diamond(p \to q)$" is true at **w**, since "$p \to q$" is true in at least one world (viz. **v**) accessible from **w**.
- "$\Diamond p$" is also true at **w**, since 'p' is true in at least one world (viz. **u**) accessible from **w**.
- However, "$\Diamond q$" is false at **w**, since there is *no* world accessible from **w** where 'q' is true.
- Hence $\Diamond(p \to q), \Diamond p \nvDash_{\mathrm{K}} \Diamond q$.

Another example:



Here, the branch containing world **u** closes. In constructing our counterexample, we include only those worlds on our *open* branch. So here $\mathbf{W} = \{\mathbf{w}, \mathbf{v}, \mathbf{v}'\}$.

We can then see that "$\Box{\sim}p$" is true at $\mathbf{w}$, since 'p' is false at every world in our model accessible from $\mathbf{w}$. "$\Diamond{\sim}q$" is true at $\mathbf{w}$, since 'q' is false in one world accessible from $\mathbf{w}$, but so is "$\Diamond q$", since 'q' is true at another. Hence, "${\sim}(\Diamond p \vee \Diamond q)$" is false.

Some tips for K-trees:

- Observe and obey:

  The Placement Principle The results of applying a tree rule are placed on every open branch *below* the statement to which it is applied (and nowhere else).

- A tree is finished when either all branches close or all possible rules are applied to all the statements on a branch but the brach remains open.

- Checkmarks (✓) can be helpful to keep track of which statements have had rules applied to them. *However,* care must be taken with true $\Box$-statements and false $\Diamond$-statements, since these apply to every world introduced by applying the rules to false $\Box$-statements and true $\Diamond$-statements, and so may need to be applied multiple times.

- Rules for propositional connectives should always be applied *before* rules for modal operators, to avoid applying the rules in the wrong world box.

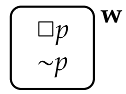
- Rules that create new worlds should be applied before rules for true $\Box$-statements and false $\Diamond$-statements. The latter two rules can never create new worlds.

(Some additional examples may be covered on the board.)

5.5 Semantics and Trees for Extensions of K

The definition given earlier for a K-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$ places absolutely no restrictions on the accessibility relation $\mathbf{R}$. Indeed, there is no guarantee in a K-model that any given world is accessible to itself. Hence, in order for $\mathbf{a}_\mathbf{w}(\Box A) = \mathrm{T}$ it need not necessarily be the case that $\mathbf{a}_\mathbf{w}(A) = \mathrm{T}$.

You'll recall that in System K one cannot deduce A from $\Box A$ within the same subproof. This is as it should be. The inference $\Box p$ /∴ p is K-invalid, as can be seen from this (rather simple) tree.



This tree is complete as is. We cannot apply the tree rule to $\Box p$ since there are no worlds accessible from $\mathbf{w}$ to which we could apply it. (Recall that this rule does not allow us to create a new world.)

To capture the proper semantics for alethic intepretations of '$\Box$', it behooves us to define a special category of model:

An M-model is a K-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$ in which the accessibility relation $\mathbf{R}$ is *reflexive*, i.e., for all $\mathbf{w}$ in $\mathbf{W}$, it holds that $\mathbf{wRw}$.

(M-models may also be called T-models.)

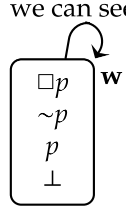
An argument $A_1, A_2, \ldots A_n$ /∴ B is **M-valid**, or $A_1, A_2, \ldots A_n \vDash_\mathrm{M} B$, iff there is no M-model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$ and world $\mathbf{w}$ in $\mathbf{W}$ for which $\mathbf{a}_\mathbf{w}(A_1) = \mathrm{T}$ and $\mathbf{a}_\mathbf{w}(A_2) = \mathrm{T}$ and ... and $\mathbf{a}_\mathbf{w}(A_n) = \mathrm{T}$ but $\mathbf{a}_\mathbf{w}(B) = \mathrm{F}$.

The notions of M-counterexamples, M-satisfiability, M-equivalence, etc., are defined similarly.

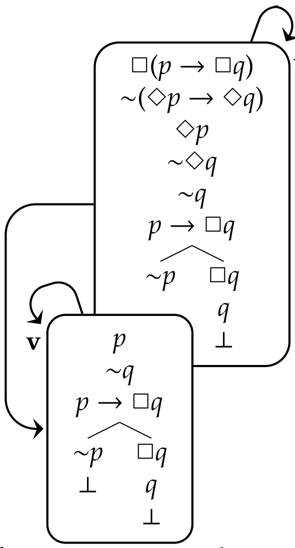
Every M-model is a K-model but not vice-versa. Hence, an argument may have a K-counterexample without having an M-counterexample, but not vice-versa. It follows that an argument or statement may be M-valid without being K-valid.

Here's another way of putting the same point. There are more K-models than M-models; hence it is *easier* for a statement to be M-valid (true in all M-models) than for it to be K-valid (true in all K-models).

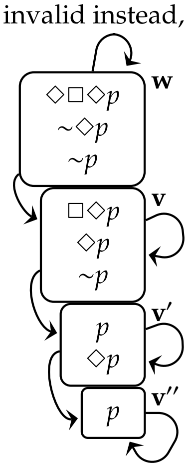
Trees for testing M-validity can be effected by immediately placing an arrow from each world to *itself* as soon a world is introduced. This means that the tree rules for true $\Box$-statements and false $\Diamond$-statements are applied *within* a given box. Here we can see that the argument $\Box p$ /∴ p is M-valid.



On the reverse, we give a tree showing $\Box(p \rightarrow \Box q) \vDash_\mathrm{M} \Diamond p \rightarrow \Diamond q$:



Of course, a tree might reveal an argument to be M-invalid instead, e.g., $\Diamond\Box\Diamond p \nvDash_{M} \Diamond p$:



If you find it tedious to add arrows from every world box to itself, you can leave them out, provided that you remember that every world is accessible to itself when applying the tree rules for M-trees, but that this is not the case for K-trees.

The example just given shows that even M-validity does not capture the appropriate sense of validity when '□' and '◇' are interpreted in their stronger senses. We can capture these stronger senses of validity by defining additional species of K-models that put different sorts of restrictions on the accessibility relation.

A 4-model (or K4-model) is a K-model ⟨**W**, **R**, **a**⟩ in which the accessibility relation **R** is *transitive*, i.e., for all worlds **w**, **v** and **u** in **W**, if **wRv** and **vRu** then **wRu**.

A KB-model is a K-model ⟨**W**, **R**, **a**⟩ in which the accessibility relation **R** is *symmetric*, i.e., for all worlds **w** and **v** in **W**, if **wRv** then **vRw**.

A 5-model (or K5-model) is a K-model ⟨**W**, **R**, **a**⟩ in which the accessibility relation **R** is *Euclidian*, i.e., for all worlds **w**, **v** and **u** in **W**, if **wRv** and **wRu** then **vRu**.

A B-model is a KB-model that is also an M-model.

An S4-model (or M4-model) is 4-model that is also an M-model.

An S5-model (or M5-model) is a 5-model that is also an M-model.

Each of these categories of models gives rise to corresponding definitions of validity, counterexamples, satisfiability, equivalence, etc. For example, a statement *A* is S4-valid iff it is true in every world in every model whose accessibility relation is both reflexive and transitive.

Contemporary research into different possible restrictions on the accessibility relation for modal models, the resulting conceptions of validity, etc., and the axiomatization of deductive systems capturing these notions of validity, have identified a number of other conditions.

The labels of these conceptions are given in the chart below. Some of these can be mixed and matched, some are stronger or weaker restrictions than others, etc.

	axiom	R-condition	description
D	$\Box A \rightarrow \Diamond A$	Serial	wRv for some v
CD	$\Diamond A \rightarrow \Box A$	Unique	if wRv and wRu then v = u
□M	$\Box(\Box A \rightarrow A)$	Shift Reflexive	if wRv then vRv
L	$\Box(\Box A \rightarrow B) \vee ((B \,\&\, \Box B) \rightarrow A)$	Connected	if wRv and wRu then vRu or uRv or v = u
C4	$\Box\Box A \rightarrow \Box A$	Dense	if wRv then vRu or uRv for some u
C	$\Diamond\Box A \rightarrow \Box\Diamond A$	Convergent	if wRv and wRu then vRt or uRt for some t

Some of these are important for non-alethic understandings of '□' and '◇'; others have been studied from a purely formal point of view.

However, we shall focus on KB- B-, 4-, 5-, S4- and S5-models and the corresponding conceptions of validty. We can expand the tree method to cover these by adopting the appropriate rules for drawing arrows between worlds.

In 4-trees, for example, whenever an arrow connects a first world box with a second, and another connects that second box with a third, we can draw an arrow from the first to the third.

Let us show, for example that a paradigm instance of the axiom schema (4) is 4-valid, i.e., $\vDash_{K4} \Box p \rightarrow \Box\Box p$: