7 Free Logic

7.1 The Basics

Garson prefers to build his quantified modal systems on a kind of Free Logic instead. The core difference is that not all terms denote existing objects. One must then introduce notation for existence; Garson suggests that in at least most of the systems discussed, this can be introduced by definition:

(Def E) $\mathrm{E}t$ *abbreviates* $\exists x\, x = t$

(However, he also intimates that in some formulations, "E" is better taken as a primitive predicate letter.) The syntax is the same otherwise. Many other books use "E!t" instead of "Et" here.

The inference rules are reformulated as follows:

(∀Out) From $\forall x\, Ax$ infer $\mathrm{E}c \rightarrow Ac$.

(∀In) Where no premises or undischarged hypotheses of the subproof contain the constant c, from $\mathrm{E}c \rightarrow Ac$ infer $\forall x\, Ax$, provided that c does not occur in $\forall x\, Ax$ and c does not already occur in Ac within the scope of a quantifier binding the variable x.

System FL- consists of the inference rules of PL along with the above; system FL extends FL- by adding the (=In) and (=Out) of identity logic. (These are unchanged.)

The derived rules for the existential quantifier become:

(∃In) From $\mathrm{E}c\ \&\ Ac$ infer $\exists x\, Ax$.

(∃Out) The following proof scheme is derivable:

$\exists x\, Ax$

| $\mathrm{E}c\ \&\ Ac$ where c is a new constant

| ⋮

| B

B (∃Out)

Or if you prefer:

$\exists x\, Ax$

$\mathrm{E}c\ \&\ Ac$ (∃Out)   where c is new

⋮

B

(Establishing the above is homework.) The derived rules (∼∃Out), (∼∃In), (∼∀Out) and (∼∀In) carry over unchanged.

One of the chief differences is that in FL, unlike QL, it is not possible to prove that anything exists.

$\nvdash_{\mathrm{FL}} \exists \mathrm{x}\, \mathrm{Ex}$

$\nvdash_{\mathrm{FL}} \forall x\, Ax \rightarrow \exists x\, Ax$

On the other hand:

$\vdash_{\mathrm{QL}} \forall x\, Ax \rightarrow \exists x\, Ax$

It is necessary when doing formal semantics for classical quantified logic (QL) to assume that the domain of quantification for every model has at least one member. Not so with free logic.

Finally, Garson notes that QL can be viewed as an extension of FL, if we were to reformulate QL as FL plus the axiom schema:

(Q) $\mathrm{E}t$

(I.e., the assumption that every term represents something that exists.) The resulting system is equivalent to QL formulated above. I won't bother trying to show this in full, but it is worth showing briefly that (Q) is a theorem schema of QL as formulated earlier.

1.	$t = t$	(=In)
2.	$\exists x\, x = t$	1 (Q∃In)
3.	$\mathrm{E}t$	2 (Def E)

7.2 Modal Extensions of FL

As with QL, these are just got by adding the rules and axioms of one of our propositional modal logics to FL. Garson puts "f" before the name of the propositional system for the abbreviation, so we have:

System fK: The system obtained from FL by adding "□" to the syntax of the language, and the inference rules (□In) and (□Out).

System fT: The system obtained from fK by adding every instance of schema (M) as an axiom.

System fB: The system obtained from fT by adding every instance of schema (B) as an axiom.

System fS4: The system obtained from fT by adding every instance of schema (4) as an axiom.

System fS5: The system obtained from fT by adding every instance of schema (5) as an axiom.

Here's an example proof:

$\Box\,\forall x(Ax \to Bx), \Diamond\,\exists x\, Ax \vdash_{\mathrm{fK}} \Diamond\,\exists x\, Bx$

1.	$\Box\,\forall x(Ax \to Bx)$	
2.	$\Diamond\,\exists x\, Ax$	
3.	$\quad \Box, \exists x\, Ax$	
4.	$\quad\quad \mathrm{E}c\ \&\ Ac$	
5.	$\quad\quad \forall x(Ax \to Bx)$	1 ($\Box$Out)
6.	$\quad\quad \mathrm{E}c \to (Ac \to Bc)$	5 ($\forall$Out)
7.	$\quad\quad \mathrm{E}c\ \&\ Bc$	4,6 (PL)
8.	$\quad\quad \exists x\, Bx$	7 ($\exists$In)
9.	$\quad \exists x\, Bx$	3,4–8 ($\exists$Out)
10.	$\Diamond\,\exists x\, Bx$	2,3–9 ($\Diamond$Out)

(You'll notice above that I used "(PL)" as though it were an inference rule. I use this to shorthand any series of steps that can be done by propositional logic. I encourage you to do the same.)

7.3 Constant and Varying Domains

Garson's main reason for preferring the free logic formulations of modal logic is to allow (or more easily allow) for interpretations with *varying* domains, i.e., in which different things exist in different worlds.

That the classical systems push one into accepting constant domains can be seen in a variety of ways. We have alrady seen that (BF) and (CBF) are theorems of System qK. For example, (BF) seems to imply that being necessarily true of everything there actually is is enough to be necessarily true of everything in all accessible possible worlds, implying that those worlds don't contain anything not in the actual world.

Indeed, in qK we can prove certain results that seem to fairly straightforwardly require that all possible worlds contain the same objects.

($\forall\Box$E) $\vdash_{\mathrm{qK}} \forall x\,\Box \mathrm{E}x$ *Proof:*

1.	$\Box$	
2.	$\quad \mathrm{E}c$	(Q)
3.	$\Box \mathrm{E}c$	1–2 ($\Box$In)
4.	$\forall x\,\Box \mathrm{E}x$	3 (Q$\forall$In)

(ED) $\vdash_{\mathrm{qK}} \mathrm{E}c \to \Box \mathrm{E}c$ *Proof:*

1.	$\Box \mathrm{E}c$	($\forall\Box$E), (Q$\forall$Out)
2.	$\mathrm{E}c \to \Box \mathrm{E}c$	1 (PL)

(CD) $\vdash_{\mathrm{qK}} {\sim}\mathrm{E}c \to \Box{\sim}\mathrm{E}c$ *Proof:*

1.	$\mathrm{E}c$	(Q)
2.	${\sim}\mathrm{E}c \to \Box{\sim}\mathrm{E}c$	1 (PL)

None of these are provable in fK; nor are (BF) or (CBF). For the theorems just listed, this should be obvious. In the case of (BF) and (CBF), it is worth noting briefly how the proof is blocked. Earlier we saw the proof of (BF) in qK. For fK we'd only get this far:

1.	$\forall x\,\Box Ax$	
2.	$\mathrm{E}c \to \Box Ac$	1 ($\forall$Out)
3.	$\quad \Box$	
4.		

It is at this point in our qK proof that we used ($\Box$Out) to get Ac in the boxed subproof. However, since $\Box$ is not the main operator of line 2, this is not possible here. Something similar blocks the proof of (CBF).

The abbreviation (ED) means "expanding domains" since systems that have it require that the accessibility relation can only ever lead from one world to a world with the same or an expanded domain; i.e., if **wRv** then the domain of world **v** must be an expansion or identical to that of world **w**. In fK, (ED) is true if and only if (CBF) is true, and hence (CBF) can be taken as another guarantee that domains can only expand.

The abbreviation (CD) means "contracting domains" since systems that have it require that the accessibility relation only ever leads from one world to another world with the same or contracted domain; i.e., if **wRv** then the domain of **v** must be contracted from or identical to that of **w**. In fK, (CD) is true if and only if (BF) is true, and hence (BF) can be taken as another guarantee that domains can only contract.

Putting together entails that precisely the same objects are quantified over in all possible worlds accessible to a given world.

However, there are responses to these worries that a defender of classical quantified logic could give.

Possibilists might insist that Et, at least if defined as $\exists x\, x = t$, does not really mean that t *actually* exists. They regard quantifiers as ranging over both actual and possible objects. Hence, $\exists \mathrm{x}\,\mathrm{x} = \mathrm{g}$ should not be understood as saying that God (actually) exists, but only that there is some possible object which is God. To say that something actually exists, one would have to introduce another predicate, say "A" for *actuality*, which cannot be defined in terms of quantifiers. The rules of qK (or its extensions) do not entail that everything is actual, or that what is actual in one world is actual in others.

In response, Garson points out that this approach invites introducing restricted quantifiers of the form $\forall_{\mathrm{A}} x\, Bx$ and $\exists_{\mathrm{A}} x\, Bx$, where the former abbreviates $\forall x(\mathrm{A}x \rightarrow Bx)$, etc. The derived logical rules governing these restricted quantifiers would work effectively like the rules for FL, and hence, Garson thinks that even proponents of this approach should be interested in modal systems built on FL.

Actualists might give another response by insisting that Et does not capture the proper logical form of real existence statements (or at least not interesting existence statements). Here we'll find those that are likely to insist that there is no interesting sense in which "existence is a predicate".

Frege claimed that existence is best thought of a second-level concept, or concept applicable to concepts, not as a concept applicable to objects. (In contemporary terminology, we might prefer to call it a property of properties instead.) Frege regarded "God exists" as meaningless if "God" is thought of as a *name*, insisting instead that it is only meaningful if interpreted as "there is a God", i.e.:

$$\exists \mathrm{x}\, \mathrm{Gx}$$

When it comes to existence claims interpreted in this way, System qK (or even qS5) does not have the result that whatever exists, exists necessarily.

$$\nvdash_{\mathrm{qK}} \exists x\, Fx \rightarrow \Box \exists x\, Fx$$

Similarly, something like "Dragons do not exist" would be interpreted as follows:

$$\sim \exists \mathrm{x}\, \mathrm{Dx}$$

And again, there is nothing in qK forcing us to conclude that whatever doesn't exist in this sense necessarily doesn't:

$$\nvdash_{\mathrm{qK}} \sim \exists x\, Fx \rightarrow \Box \sim \exists x\, Fx$$

A slightly more sophisticated view along the same lines was endorsed by Russell. For Russell, an existence claim in the plural, e.g., "Tigers exist" or "Dragons exist", were interpreted as Frege suggesed. An existence claim in the singular, e.g., "Socrates exists" or "the presence King of France does not exist", were thought to be meaningful when, but only when, the "thing" claimed to exist was referred to by means of a description, or a name that can be interpreted as a description in disguise, i.e., something of the form $(\imath x)Ax$. In that case, the existence claim that $(\imath x)Ax$ exists would be analyzed as meaning:

$$\exists x\, \forall y(Ay \leftrightarrow x = y)$$

(I.e., that there is one and only one x that such that Ax.) The claim that $(\imath x)Ax$ does not exist as:

$$\sim \exists x\, \forall y(Ay \leftrightarrow x = y)$$

Here again, there is nothing in qK requiring that that which exists (or doesn't) in this sense necessarily exists (or doesn't).

Having a constant domain of quantification then, doesn't amount to the result that it's true in all possible worlds that "Socrates exists" or that "Dragons don't exist", since the truth or falsity of such claims don't boil down to what objects are in the domain of quantification, but rather what properties are instantiated by what and how many objects.

It is perhaps for these sorts of reasons that Wittgenstein wrote in the *Tractatus*:

> (2.022) It is obvious that an imagined world, however different it may be from the real one, must have *something*—a form—in common with it.
>
> (2.023) Objects are just what constitute this unalterable form.
>
> . . .
>
> (2.027) Objects, the unalterable, and the subsistent are one and the same.
>
> (2.0271) Objects are what is unalterable and subsistent, their configuration is what is changing and unstable.
>
> (2.0272) The configuration of objects produces a state of affairs.

The suggestion is that having the same objects in common between worlds does not mean that those worlds are in any way "alike" in what sorts of facts are true at them. They are alike only in terms of what sorts of questions may be asked of them, i.e., the truth of what sentences may be evaluated by their lights.

Garson does not give a very thorough response to this kind of approach. His main attitude seems to be that it is still worthwhile exploring systems allowing for varying domains "to keep our options open," and I agree that it is worth exploring them (if for no other reason than one needs to know what one is rejecting before rejecting it). Moreover, there are a number of philosophical challenges to the sort of view just presented (e.g., whether or not it forces on us a "combinatorial" view of modality, whether or not it requires thinking of entities as "bare particulars", whether it might run into trouble given the size of the actual universe, whether it has the right philosophy of language for names or existence statements, etc.)

It would be great if we could delve into all these philosophical issues, but for now we return to the details of the various logical systems.

7.4 Rigid Constants

From a rather different perspective in the philosophy of language, in his famous *Naming and Necessity*, Saul Kripke argued that proper names are "rigid designators", i.e., that they name the same thing in all possible worlds. (He contrasts them with "non-rigid designators," typically descriptions, such as "the first woman President," which might name a different object in different possible worlds. (It might name Hillary Clinton in one world—the actual world?—but Geraldine Ferraro in another, Condi Rice in another.)

From this, Kripke concluded that identity statements formed with *names* flanking "=" are *necessarily true* if true at all. There might have been worlds in which the heavenly body that shines brightest in the morning might not have been the heavenly body that shines brightest in the evening, but those are not worlds in which Hesperus is not Phosphorus. Samuel Clemens might not have written any books or have given himself a pseudonym, but those are not worlds in which Mark Twain is not Samuel Clemens.

If we use our simple constants "a", "b", "c", exclusively for proper names, Kripke view speaks in favor of accepting the following axiom schema:

(RC) $(b = c \rightarrow \Box b = c) \;\&\; (b \neq c \rightarrow \Box b \neq c)$

To avoid the absurd result that identity statements formed with descriptions would end up as necessarily true if true at all, one must either forgo treating descriptions as terms altogether (e.g., using Russell's theory of descriptions), or make a distinction between constants and other terms, and limit the above to the case where b and c are constants.

System rK: the obtained obtained from fK by adding every instance of (RC) as an axiom.

We also define rT, rB, rS4 and rS5 as you'd expect. For systems obtained from qK and its extensions by adding (RC), we use qrK, qrT, etc.

One interesting result about rK and its extensions is that we get substitutivity of identicals in all contexts, even within the scope of modal operators. Thus, treating "e", "f", "n" all as constants/names, even:

$$\mathrm{n = e}, \Box\mathrm{G(e,f)} \;\vdash_{\mathrm{rK}} \Box\mathrm{G(n,f)}$$

1.	n = e	
2.	□G(e, f)	
3.	n = e → □n = e	(RC), (PL)
4.	□n = e	1,3 (MP)
5.	□	
6.	G(e, f)	2 (□Out)
7.	n = e	4 (□Out)
8.	G(n, f)	6,7 (=Out)
9.	□G(n, f)	5–8 (□In)

To avoid seeing this as a problem, one must insist that "n" is an improper translation for "the number of planets," since that is a description (and non-rigid designator), not a name. Genuine names, according to Kripke, can always replace one another in modal contexts.

This seems to make rK and its extensions unsuitable as a basis for quantified epistemic or doxastic logical systems.

Garson claims that in order to avoid problems if systems based on rK were expanded to allow for terms other than constants, it would be necessary to add the following rule:

(∃i) Where c is a (rigid) constant that does not appear in any premises or undischarged hypotheses, and t is a nonrigid term from $t \neq c$ infer $\bot$.

System oK: the result of adding (∃i) as a rule to rK.

The exact motivation for this rule and this way of treating non-rigid terms can only become clear when we move to the semantics of quantified modal logic. Notice, however, that it does not say that $t \neq c$ leads to a contradiction when we obtain this result based on special knowledge about c, but rather, only when c is arbitrary (so that, in effect, t is not identical to anything, existent or nonexistent).