

1. Constants are terms.
2. “ $\perp$ ” is a sentence.
3. If  $t_1, \dots, t_n$  are terms and  $P$  is a predicate letter then  $P(t_1, \dots, t_n)$  is a sentence.
4. If  $A$  is a sentence, then  $\Box A$  is a sentence.
5. If  $A$  and  $B$  are sentences, then  $(A \rightarrow B)$  is a sentence.
6. If  $At$  is a sentence containing term  $t$ , and  $x$  is a variable, then  $\imath x Ax$  is a term.
7. If  $At$  is a sentence containing term  $t$ , and  $x$  is a variable, then  $\forall x Ax$  is a sentence.

## 11.5 Deductive System

Garson builds his deductive system for Modal Description Theory on system oK (see page 47), which is an extension of rK, the system with rigid constants. Thus we have, among other rules, (RC):

**(RC)**  $(b = c \rightarrow \Box b = c) \ \& \ (b \neq c \rightarrow \Box b \neq c)$

Note in the above,  $b$  and  $c$  are constants, not other terms, and hence, not description terms. As you proved in homework, we in effect have a strong rule of substitution of identicals in all contexts:

**(R=Out)** From  $b = c$  and  $Ab$  infer  $Ac$ , regardless of the complexity of  $Ab$ .

However, here too, use of the above is restricted to when  $b$  and  $c$  are both constants (since it relies on (RC)). We cannot substitute descriptive terms for identicals in all contexts.

In effect, we treat constants as our rigid terms, and descriptions as non-rigid terms. Hence, the other rule of oK can be stated as follows:

**( $\exists$ i)** Where  $c$  is a constant that does not appear in any premises or undischarged hypotheses or in  $\imath x Ax$ , from  $\imath x Ax \neq c$  infer  $\perp$ .

We also expand oK by adding the following axiom schema:

(1)  $\forall y[(Ay \ \& \ \forall x(Ax \rightarrow x = y)) \leftrightarrow y = \imath x Ax]$

(We must here require that  $x$  and  $y$  be distinct variables.) Or, allowing us to use Garson’s shorthand:

(1)  $\forall y(1Ay \leftrightarrow y = \imath x Ax)$

We may then define:

**System  $\imath K$**  is System oK + (1)

**System  $\imath T$**  is System oT + (1), and so on for  $\imath B$ ,  $\imath S4$ ,  $\imath S5$ , etc.

It should be noted that we still do not allow ourselves to instantiate quantifiers directly to descriptive terms. The ( $\forall$ In) and ( $\forall$ Out) rules are formulated only for constants.

We get the following derived rules:

**( $\imath$ Out)** From  $c = \imath x Ax$  and  $Ec$  infer  $Ac \ \& \ \forall x(Ax \rightarrow x = c)$ . *Proof:*

$$\begin{array}{ll} c = \imath x Ax & \\ Ec & \\ \forall y[(Ay \ \& \ \forall x(Ax \rightarrow x = y)) \leftrightarrow y = \imath x Ax] & (1) \\ Ec \rightarrow [(Ac \ \& \ \forall x(Ax \rightarrow x = c)) \leftrightarrow c = \imath x Ax] & (\forall\text{Out}) \\ (Ac \ \& \ \forall x(Ax \rightarrow x = c)) \leftrightarrow c = \imath x Ax & (\text{MP}) \\ c = \imath x Ax \rightarrow (Ac \ \& \ \forall x(Ax \rightarrow x = c)) & (\leftrightarrow\text{Out}) \\ Ac \ \& \ \forall x(Ax \rightarrow x = c) & (\text{MP}) \end{array}$$

**( $\imath$ In)** From  $Ac \ \& \ \forall x(Ax \rightarrow x = c)$  and  $Ec$  infer  $c = \imath x Ax$ . (*Proof* as above, but with first premise and conclusion swapped.)

**( $E\imath\exists$ )**  $\vdash_{\imath K} E \imath x Ax \leftrightarrow \exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y))$   
*Proof:*

1.	E $\imath x Ax$	
2.	$\exists y y = \imath x Ax$	1 (Def E)
3.	$Ec \ \& \ c = \imath x Ax$	
4.	Ec	3 (&Out)
5.	$c = \imath x Ax$	3 (&Out)
6.	$Ac \ \& \ \forall x(Ax \rightarrow x = c)$	4,5 ( $\imath$ Out)
7.	$Ec \ \& \ (Ac \ \& \ \forall x(Ax \rightarrow x = c))$	4,6 (&In)
8.	$\exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y))$	7 ( $\exists$ In)
9.	$\exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y))$	2,3-8 ( $\exists$ Out)
10.	$E \imath x Ax \rightarrow \exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y))$	1-9 (CP)
11.	$\exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y))$	
12.	$Ed \ \& \ (Ad \ \& \ \forall x(Ax \rightarrow x = d))$	
13.	Ed	12 (&Out)
14.	$Ad \ \& \ \forall x(Ax \rightarrow x = d)$	12 (&Out)
15.	$d = \imath x Ax$	13,14 ( $\imath$ In)
16.	$Ed \ \& \ d = \imath x Ax$	13,15 (&In)
17.	$\exists y y = \imath x Ax$	16 ( $\exists$ In)
18.	E $\imath x Ax$	17 (Def E)
19.	E $\imath x Ax$	11,12-18 ( $\exists$ Out)
20.	$\exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y)) \rightarrow E \imath x Ax$	11-19 (CP)
21.	$E \imath x Ax \leftrightarrow \exists y(Ay \ \& \ \forall x(Ax \rightarrow x = y))$	10,20 ( $\leftrightarrow$ In)

You may if you wish also cite (E $\exists$ ) for the derived rule for going from one side of an instance the above biconditional to the other.

As noted above, the quantifier rules for  $\mathcal{IK}$  are formulated to deal directly only with constants. Consider the rule of ( $\forall$ Out), *from*  $\forall x Ax$  *infer*  $Ec \rightarrow Ac$ ; this gives us the derived rule ( $\exists$ In), i.e., *from*  $Ec \ \& \ Ac$  *infer*  $\exists x Ax$ .

Even in a Free logic, we would not want this rule to hold generally for all terms, i.e., *from*  $\forall x Ax$  *infer*  $Et \rightarrow At$ , or *from*  $Et \ \& \ At$  *infer*  $\exists x Ax$  where  $t$  may be a descriptive term. For example, let “ $Sx$ ” mean “ $x$  is a spy”, and “ $Txy$ ” mean  $x$  is taller than  $y$ . Then the description:

$$\mathcal{IK}[Sx \ \& \ \forall y((Sy \ \& \ y \neq x) \rightarrow Tyx)]$$

means, basically, “the shortest spy.” It seems clear that we want to allow that the shortest spy could exist and it be necessary that the shortest spy is a spy, i.e.:

$$\begin{aligned} & E \mathcal{IK}[Sx \ \& \ \forall y((Sy \ \& \ y \neq x) \rightarrow Tyx)] \ \& \\ & \square S \mathcal{IK}[Sx \ \& \ \forall y((Sy \ \& \ y \neq x) \rightarrow Tyx)] \end{aligned}$$

could be true (which is of the form  $Et \ \& \ At$ ), without concluding that something is necessarily a spy:

$$\exists x \square Sx$$

We do, however, have the following weaker form of ( $\forall$ Out) for descriptive terms, i.e., that *from*  $\forall x Px$  *infer*  $E \mathcal{IK} Ax \rightarrow P \mathcal{IK} Ax$ , where  $P$  is a simple predicate letter. *Proof*:

- |     |   |                        |
|-----|---|------------------------|
| 1.  | $\forall x Px$                                    |                        |
| 2.  | $E \mathcal{IK} Ax$                               |                        |
| 3.  | $\exists y y = \mathcal{IK} Ax$                   | 2 (Def E)              |
| 4.  | $Ec \ \& \ c = \mathcal{IK} Ax$                   |                        |
| 5.  | $Ec$  | 4 (&Out)               |
| 6.  | $c = \mathcal{IK} Ax$                             | 4 (&Out)               |
| 7.  | $Ec \rightarrow Pc$                               | 1 ( $\forall$ Out)     |
| 8.  | $Pc$  | 5,7 (MP)               |
| 9.  | $P \mathcal{IK} Ax$                               | 6,8 (=Out)             |
| 10. | $P \mathcal{IK} Ax$                               | 3,4–9 ( $\exists$ Out) |
| 11. | $E \mathcal{IK} Ax \rightarrow P \mathcal{IK} Ax$ | 2–10 (CP)              |

If  $P$  were something complex, not a simple predicate, the above argument would be blocked at line 9. While

the inference rules for identity logic are formulated to make use of all terms, not just constants, allowing the inference in the above case, as is, (=Out) only allows substitutions of identicals in atomic statements, as above. While we do get the stronger (R=Out) for systems, such as this one, built on  $\mathcal{IK}$ , the inductive argument for it requires (RC), which only applies when the identity statement is formed using constants only. Such is not the case with line 6 above.

Another interesting result is the following, that if the  $P$  exists, then the  $P$  is  $P$  (where  $P$  is a simple predicate):

- |   |   |                        |
|---|---|------------------------|
| $\vdash_{\mathcal{IK}} E \mathcal{IK} Px \rightarrow P \mathcal{IK} Px$ | <i>Proof</i> :  |                        |
| 1.  | $E \mathcal{IK} Px$                                     |                        |
| 2.  | $\exists y(Py \ \& \ \forall x(Px \rightarrow x = y))$  | 1 (E $\exists$ )       |
| 3.  | $Ec \ \& \ (Pc \ \& \ \forall x(Px \rightarrow x = c))$ |                        |
| 4.  | $Ec$  | 3 (&Out)               |
| 5.  | $Pc \ \& \ \forall x(Px \rightarrow x = c)$             | 3 (&Out)               |
| 6.  | $Pc$  | 5 (&Out)               |
| 7.  | $c = \mathcal{IK} Px$                                   | 4,5 ( $\exists$ In)    |
| 8.  | $P \mathcal{IK} Px$                                     | 6,7 (=Out)             |
| 9.  | $P \mathcal{IK} Px$                                     | 2,3–8 ( $\exists$ Out) |
| 10.   | $E \mathcal{IK} Px \rightarrow P \mathcal{IK} Px$       | 1–9 (CP)               |

Notice, however that it does not hold generally that  $A \mathcal{IK} A$  or even  $P \mathcal{IK} Px$ . If it did, we would get  $(\mathcal{IK} x \neq x) \neq (\mathcal{IK} x \neq x)$ , which would contradict (=In), and make the system inconsistent.

It also shows that this sort of description logic could not be formulated outside of a free logic, in a classical system in which  $Et$  holds for all terms  $t$ , or else we would get  $E \mathcal{IK} x \neq x$ , and the above contradiction. Similar considerations apply to such descriptions as “ $\mathcal{IK}(Fx \ \& \ \sim Fx)$ ”.

## 11.6 Semantics and Metatheory for Modal Description Logic

Garson only discusses how to do objectual semantics for modal description logic, which is somewhat natural since he builds his system upon  $\mathcal{OK}$ .

Moreover, some natural ways of doing substitutional semantics won’t work when terms can contain sentences. For example, it is tempting to think that whether or not  $\forall x Ax$  is true ought to depend on the truth value of  $At$  all terms  $t$  such that  $Et$  is true.

But notice that those terms may include  $\forall x Ax$  itself (e.g.,  $\iota y(Py \ \& \ \forall x Ax)$ ), and whether or not they refer to something that exists may depend on the truth of  $\forall x Ax$ , getting us stuck in an infinite regress. There are ways around this, but it complicates things greatly.

For objectual semantics this problem is avoided since what matters for the truth-value of a quantified statement is not the truth of other statements containing (possibly more complex) terms, but rather the truth of hybrids.

In particular, Garson defines an  $\iota K$  model as an  $\circ K$  model  $\langle \mathbf{W}, \mathbf{R}, \mathbf{D}, \mathbf{a} \rangle$  in which it holds that:

- (1) For every term of the form  $\iota x Ax$  and every  $\mathbf{w}$  in  $\mathbf{W}$ , if there is one and only one object  $\mathbf{o}$  in  $\mathbf{Dw}$  for which the hybrid is such that  $\mathbf{a}_{\mathbf{w}}(\mathbf{Ao}) = \mathbf{T}$ , then  $\mathbf{a}_{\mathbf{w}}(\iota x Ax) = \mathbf{o}$ , and otherwise,  $\mathbf{a}_{\mathbf{w}}(\iota x Ax)$  is some object  $\mathbf{o}'$  which is a member of  $\mathbf{D}$  but not a member of  $\mathbf{Dw}$ .

On this approach, every term, descriptive or otherwise, is given an entity in the domain as its semantic value; if the description in question is satisfied uniquely in a given world, then the semantic value of the description will be that thing. Otherwise, we need to pick some object in the overall domain (that does not exist at a given world, but may exist at others); this object may not actually satisfy the description.

Notice moreover that since there are always descriptive terms formed from descriptions that cannot be true of one and only one thing, e.g., " $\iota x(x \neq x)$ ", every world  $\mathbf{w}$  in every model needs to be able to assign semantic values to such terms outside of  $\mathbf{Dw}$ . Hence,  $\mathbf{Dw}$  can never be exhaustive of the entire domain  $\mathbf{D}$ . Here we see how this approach is incompatible with classical quantifier rules, which yield constant domains so that  $\mathbf{Dw} = \mathbf{D}$ .

We can define  $\iota K$  validity in terms of truth in all  $\iota K$  models, and the following results are obtainable. (I didn't think it worth our time to examine the details.)

**Soundness:** if  $A_1, \dots, A_n \vdash_{\iota K} B$  then  
 $A_1, \dots, A_n \vDash_{\iota K} B$ .

**Completeness:** if  $A_1, \dots, A_n \vDash_{\iota K} B$  then  
 $A_1, \dots, A_n \vdash_{\iota K} B$ .

We could define  $\iota B$ ,  $\iota S4$  and  $\iota S5$  models, etc., and corresponding notions of validity, and achieve analogous results for them.

## 11.7 The Puzzles Revisited

How does Garson's modal description theory fare with regard to the puzzles Russell used to motivate his theory? It does have some advantages:

- Since it is built on a Free Logic, many existence claims are obviously contingent, and sentences containing unsatisfied descriptions can be meaningful and even true.
- If we treat the  $\mathbf{K}$  operator for "George IV wished to know whether ..." as a modal operator,  $s = \iota x Ax, \mathbf{K}s = \iota x Ax \not\vdash \mathbf{K}s = s$ , because substitutions of identicals cannot be done inside the scope of modal operators when one of the two halves is a description.
- We do not have a proof for thinking that the round square is round, or that it is square, since we have only  $\vdash_{\iota K} E \iota x(Rx \ \& \ Sx) \rightarrow (R(\iota x(Rx \ \& \ Sx)) \ \& \ S(\iota x(Rx \ \& \ Sx)))$ , which given the assumption that nothing is both round and square, only proves that  $\sim E \iota x(Rx \ \& \ Sx)$ . Garson says that this show that "the round square is not round, and it is not square. In fact, it is not." This is misleading; we cannot prove either that it is round or that it is not. Same goes with whether it is a square (or even both).

Yet Garson neglects to mention the flip side.

- It doesn't tell us whether or not the King of France is bald, but still insists that this does have a truth value; and for Garson there is no ambiguity. For him, it depends on whether or not some non-actual object is in the extension of the baldness predicate in this world, which seems rather odd. Indeed, it is compatible with his view that the King of France is a grape, etc.
- It also doesn't solve those belief puzzles with no descriptions involved. It does no better with referential uses of descriptions.
- It enforces existence/uniqueness conditions every bit as stringent as Russell's for descriptions such that refer to *existing* things.
- It blocks inferences descriptions were introduced to capture (e.g., from Aristotle is the teacher of Alexander to Aristotle taught Alexander.)
- So far, it has no way at all of resolving scope ambiguities. (We'll see his approach to this in the next chapter.)