LECTURE III

Scattering: important tool to study the structure of matter
- study nature of interaction among particles
- produce new, unstable particles
- isolate specific, rare reactions

Elastic scattering \( \equiv a + b \rightarrow a + b \)
- species remains the same
- energy momentum ALWAYS conserved

Inelastic scattering \( a + b \rightarrow c + d + e \)

Inclusive scattering \( a + b \rightarrow a + x \)

Exclusive scattering \( a + b \rightarrow c + d + e \)
(track all particles)

\[
p - p' = k' - k
\]
\[
q^\mu = p^\mu - p'^\mu
\]

Can prove: \( q^2 \neq 0 \)
"Virtual" particle exchange i.e. \( E^2 - p^2 c^2 \neq m^2 c^4 \)

- No action at a distance
- In quantum theory: exchange of virtual quanta
- Quantum carries energy & momentum
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Energy-momentum non-conservation allowed for a time period consistent with the uncertainty principle $\Rightarrow \Delta t \leq \hbar / \Delta E$

Higher Energy $\Rightarrow$ shorter time $\Rightarrow$ smaller range

For a massless particle: small E, P possible $\Rightarrow$ long range force (e.g., electromagnetism)

For a massive quantum: Rest energy restricts range:

If $m = 0$: $V \sim 1/r$
If $m \neq 0$: $V \sim (1/r) e^{-r/R}$, $R = \hbar / mc$

$R = 10^{-15} m \Rightarrow m = 100 \text{ MeV} / c^2$ (Strong force)

$R = 10^{-18} m \Rightarrow m = 100 \text{ GeV} / c^2$ (Weak force)

To observe structure, look at deBroglie wavelength:

$\lambda = \frac{\hbar}{q}$

$10^{-14} m \Rightarrow 20 \text{ MeV}$ (nuclear size)
$10^{-15} m \Rightarrow 200 \text{ MeV}$ (nuclear structure)

Several GeV $\Rightarrow$ proton structure (quarks)

100 GeV $\Rightarrow$ Weak force quantum
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Calculation of $q$ for a scattering reaction:

\[ \begin{align*}
\text{Fixed' target reaction} & : \ k = k' \quad (\text{electron-proton scattering}) \\
4\text{-momenta} & : \ p, p', (\text{electron}), \ k, k', (\text{proton}) \\
\quad & : \ p - p' \equiv q \equiv 4\text{-momentum transferred} \\
(p-p')^2 & = p^2 + p'^2 - 2pp' = m_e^2c^2 + m_p^2c^2 - 2\left(\frac{EE'}{c^2}\right)
\end{align*} \]

Extreme Relativistic Limit (ERL)

\[ \begin{align*}
E, E' \gg m_e^2c^2, m_p^2c^2 & \implies E \approx |p|c \\
& \quad E' \approx |p'|c \\
\implies q^2 & \approx -2EE' (1-\cos\theta) < 0
\end{align*} \]

When $q^2 < 0$, \text{\textquoteleft}\text{space-like} \text{\textquoteright} reaction.

Calculation of $q$ for an annihilation reaction:

\[ \begin{align*}
q = p + k \implies q^2 & = p^2 + k^2 + 2pk \\
& \quad \text{If one particle at rest, then} \\
q^2 & \approx 2m_eE \\
q^2 & = m_{\text{new}}^2c^2 \implies m_{\text{new}} \approx \sqrt{2m_eE}
\end{align*} \]
For colliding beams:

\[ q^2 = p^2 + k^2 + 2 p \cdot k \leq 2E_{e^+}E_{e^-} \Rightarrow 2 \vec{p}_{e^+} \cdot \vec{p}_{e^-} \]

\[ \vec{p}_{e^+} \cdot \vec{p}_{e^-} = -|p_{e^+}|^2 \Rightarrow q^2 \approx 4E_{e^+}E_{e^-} \quad \text{(ERL)} \]

Thus \[ m_{\text{new}}c^2 \approx 2E_e \]

**Fixed Target:** \[ m_{\text{new}} \propto \sqrt{E} \]

**Colliders:** \[ m_{\text{new}} \propto E \]

- Colliding beams suitable for finding new particles.
- Fixed target better suited to study structure.

**Calculation of Geometric Cross-Section**

Particles 'a'

\[ N_b \text{ scattering centers: } N_b = \eta_b A d \]

\[ \sigma_b = \frac{\text{Cross-sectional area}}{\text{scattering center}} \]

\[ \eta_b = \text{density} \]
LECTURE III

Bombard target \( b \) with particles \( a \).
- Measure \( \frac{dN}{dt} = \text{# reactions / time} \)

Flux of \( a \):
\[
\Phi_a = \frac{N_a A_d}{A_t} = N_a \nu_a
\]

\[
\frac{dN}{dt} = \Phi_a N_b \sigma_b
\]

\( \Phi_a N_b \equiv L \) or luminosity.

\[
\text{# reactions / time} = \frac{\text{flux of } a \times \text{# of } b \times \text{area of } b}{\text{time}}
\]

\[
\Rightarrow \text{Area of } b \equiv \sigma_b = \frac{\text{# reactions / time}}{\text{# projectiles} \times \text{# scattering centers}}
\]

\[
\frac{\text{# projectiles}}{\text{time}} = \text{beam current}
\]

\[
\frac{\text{# scattering centers}}{\text{area}} \equiv \text{target thickness}
\]

Nucleus: \( 10^{-14} \text{ m} \) \( \Rightarrow \) 1 barn = \( 10^{-28} \text{ m}^2 \)

High Energy collisions:
\[
\sigma_{pp} (10 \text{ GeV}) \sim 10 \text{ mb}
\]

\[
\sigma_{np} (10 \text{ GeV}) \sim 100 \text{ fm}
\]
\[ \sigma_b = \frac{\text{# reactions/Time}}{\text{# projectiles} \times \text{# scattering centers} \times \text{Luminosity}} = \frac{\text{Rate}}{\text{time} \times \text{area}} \]

In a real experiment, hard to measure 'total' rate to obtain 'total' cross-section.

Differential count rate \( \Rightarrow \) differential X-section:

\[ \frac{dN}{dt} (E, \theta, \Delta \Omega) = 2 \frac{d\sigma (E, \theta) \Delta \Omega}{d\Omega} \]

Sometimes, you cannot measure all energies:

\[ \frac{d^2N}{d\Omega dE'} \Rightarrow \sigma_{\text{tot}} = \int_0^{E_{\text{max}}} \int d\Omega \frac{d^2N}{d\Omega dE'} \]

We now extract an expression for scattering of a spinless projectile scattering of a spinless nucleus using semi-classical methods.
LECTURE IV

Calculation of Rutherford Cross Section

Fermi's Second Golden Rule:

Reaction rate depends on the form of the potential. In Time-Dependent Perturbation theory, one defines a transition matrix element

\[ M_{fi} = \langle \Psi_f | H_{\text{int}} | \Psi_i \rangle \]

\[ W = \frac{\text{reaction rate}}{\# \text{beam} \times \# \text{target}} = \frac{2\pi}{\hbar} |M_{fi}|^2 S(E') \]

\[ S(E') = \frac{dn(E')}{dE'} \quad \text{reaction rate depends on} \quad \text{# of available final states} \]

If one particle is detected, must sum over all 4-momenta of other particles allowed by energy momentum conservation.

Quantization: Each particle occupies a volume \( h^3 \) in 6D \( E-P \) space.

In a volume \( V \) with a momentum range \( dp' \),

\[ dn(p') = \frac{V 4\pi p'^2 dp'}{(2\pi \hbar)^3} \]

\[ \frac{dn}{dE} \quad \text{calculation is straightforward & algorithmic} \]
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\[ W = \frac{dN/dt}{N_b \cdot N_a} = \Phi a \frac{N_d \delta_b}{N_b \cdot N_a} = \frac{N_a V a \delta_b}{N_a} \frac{d\delta_b V a}{V} \]

\[ \frac{d\sigma V a}{V} = \frac{2\pi}{\hbar} \left[ M_{fi} \right]^2 \frac{dW}{dE} \]

\[ \frac{d\sigma}{d\Omega} = \frac{V^2 E^{1/2}}{(2\pi)^2 \hbar c^4} \left[ M_{fi} \right]^2 \left( \frac{E \approx p c}{V a \approx c} \right) \]

Rutherford Scattering

No spin, no recoil, interaction is a small perturbation. (Born Approximation) \Rightarrow

Incoming & Outgoing particles are plane waves.

\[ \psi_i = \frac{1}{\sqrt{V}} e^{i p x / \hbar} \quad \psi_f = \frac{1}{\sqrt{V}} e^{i p' x / \hbar} \]

Via finite but large compared to interaction range

\[ H_{int} = -e \phi \quad \nabla^2 \phi = 8/\epsilon_0 \]

(Electromagnetism)

Define \( q^\mu = p^\mu - p'^\mu \) (4-momentum transfer)
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\[ M_{fi} = \int \gamma_f^* \mathrm{Hint} \gamma_i \, d^3x = \frac{\epsilon}{V} \int \phi \otimes e^{i q \cdot x / \hbar} \, d^3x \]

Define \[ f(x) = Z e^2 f(x) \]
Use \[ \int (\nabla^2 \nu - \nu \nabla^2 u) \, d^3x = 0 \]

\[ \Rightarrow M_{fi} = \frac{e^2 \hbar^2}{V |q|^2} \int \nabla^2 \phi \, e^{i q \cdot x / \hbar} \, d^3x \]

\[ = \frac{Z e^2 \hbar^2}{V |q|^2} \int f(x) \, e^{i q \cdot x} \, d^3x \]

\[ \Rightarrow F(q) = \text{Form Factor} \]

Define \[ \frac{e^2}{4 \pi \text{static}} = \frac{1}{137} \equiv \text{Fine Structure Constant} \]

For a point heavy charge, \[ f(x) = s^2 x \Rightarrow F(q) = 1 \]

\[ \Rightarrow \frac{d\sigma}{ds^2} = \frac{Z^2 e^2 E^2}{|q|^4} \]

To obtain Rutherford formula: use \[ E = E' \]

\[ |p| = |p'| \]

\[ | q | = 2 | p | \sin \theta / 2 \]

\[ \frac{d\sigma}{ds^2} = \frac{Z^2 e^2}{16 E^2 \beta^4 \sin^4 \theta / 2} \]

\[ (E = \gamma mc^2, \quad p = \gamma m \beta c, \quad \frac{E}{p} = \frac{1}{\beta}) \]
What happens with all the planck's c's?
Use natural units: \( \hbar = c = 1 \)

\[ \text{LENGTH} \leftrightarrow \text{TIME} \leftrightarrow \frac{1}{\text{MASS}} \]

\[ \text{MASS} \leftrightarrow \text{MOMENTUM} \leftrightarrow \text{ENERGY} \]

Need only 2 constants:
\[ \frac{\hbar c}{197 \text{ MeV} \cdot \text{fm}} \quad \frac{\hbar^2 c^2}{0.389 \text{ GeV}^2 \cdot \text{mb}} \]

For Rutherford Scattering, take 4 MeV \( \alpha \) particles
\( \Rightarrow \beta \ll 1 \Rightarrow E \approx m c^2 \)

\[ \Rightarrow \frac{\text{d} \sigma}{\text{d} \Omega} = \frac{Z^2 \alpha^2}{16 (m c^2)^2 \beta^4 \sin^4(0/2)} \left( \frac{\hbar^2 c^2}{\text{mb}} \right) \sim 10^{-23} \text{ m}^2 \]

* Fit very well with the hypothesis of a nucleus
  - With charge \( Z e \) smaller than \( 10^{14} \text{ m} \)
  - Few scatters at small angles \( (\sigma < 10^{-18} \text{ m}^2) \)
  - Significant scattering with \( \theta > 90^\circ \)
  \[ \Rightarrow F(q) \sim 1 \]

* Did not fit Thomson (plum pudding) model.
Rutherford formula is valid for $\beta \ll 1$ and spinless beam and a spinless target nucleus.

For a spin $\frac{1}{2}$ relativistic projectile, need to modify formula to incorporate conservation laws.

$$\left( \frac{d\sigma}{dS^2} \right)_R = \frac{2q^2E^2}{q^4} \quad \text{However } \frac{d\sigma}{dS^2} \bigg|_{0 \to 180^\circ} \to 0$$

for spin $\frac{1}{2}$ beam particles.

$$\Rightarrow \left( \frac{d\sigma}{dS^2} \right) = \left( \frac{d\sigma}{dS^2} \right)_{Mott} \cos^2 \theta / 2$$

Experimentally, Mott cross section was found to be systematically smaller $\Rightarrow F(\mathbf{q}) < 1$

$$\left( \frac{d\sigma}{dS^2} \right)_{\text{nucleus}} = \left( \frac{d\sigma}{dS^2} \right)_{M} |F(\mathbf{q})|^2 \quad (F(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}} f(x) \, d^3x)$$

$f(x)$ is the nuclear charge distribution.

Next lecture $\Rightarrow$ analyze electron scattering data.
\begin{align*}
\left( \frac{d\sigma}{d\Omega} \right)_{\text{Nucleus}} &= \left( \frac{d\sigma}{d\Omega} \right)_{\text{M}} |F(q)|^2 \\
F(q) &= \int e^{i q \cdot r} \, f(r) \, d^3 r
\end{align*}

As \( q \) increases, one samples only a fraction of charge distribution since de Broglie wavelength becomes smaller than nuclear size.

For a uniform sphere of charge, \( f(r) \rightarrow f(r) = \Theta(r) \)
\[ \Theta = \text{constant for } r < R \quad \Theta = 0 \text{ for } r > R \]

This leads to diffraction patterns (see slide)
First minumum at \( |q| R / \hbar \approx 4.5 \)

Inspect slide: For \( ^{12}C \), first minumum at \( 26^\circ = \theta / 2 \)
Since \( E = 420 \text{ MeV} \), \( q \approx 2E \sin \theta / 2 \)
\[ \Rightarrow \quad R = 2.4 \text{ fm} \]

- Different forms of \( \langle \rho \rangle \) produce different \( F(q) \)
- Detailed measurements show that nuclei are diffuse spheres of charge.

One can get a good estimate of a nuclear size with one single \( d\sigma / d\Omega \) measurement at small, non-zero \( q \) value. (no detailed knowledge of \( F(q) \) required) \( \Rightarrow \)
If \( q \cdot R / \hbar \ll 1 \), then

\[
F(q^2) \approx 4\pi \int_0^\infty f(r) r^2 \, dr - \frac{1}{6} \frac{q^2}{\hbar^2} 4\pi \int_0^\infty f(r) r^4 \, dr \ldots
\]

Since \( 4\pi \int_0^\infty r^2 f(r) \, dr = 1 \) and \( \langle r^2 \rangle = \int_0^\infty r^2 f(r) \, r^2 \, dr \),

\[
\Rightarrow F(q^2) = 1 - \frac{1}{6} \frac{q^2}{\hbar^2} \langle r^2 \rangle \Rightarrow \langle r^2 \rangle = -6 \frac{\hbar^2}{q^2} \frac{dF(q^2)}{dq^2}
\]

---

- Analysis of precision spectra of \(^{12}\text{C}\) electron scattering

\[
\begin{array}{c}
\text{e}^- \rightarrow p' \\
\text{K}^- \rightarrow \text{K}^+
\end{array}
\]

\[ E = 4.95 \text{ MeV} \quad \theta = 65.4^\circ \]

For elastic scattering, \( p^2 = p'^2 + 2pK \)

\[
p + K = p' + K' \Rightarrow p^2 + k^2 + 2pK = p'^2 + K'^2 + 2p'K' \]

What is peak at 482 MeV?

\[ p + K = p' + K' \quad \Rightarrow \quad p^2 + k^2 + 2pK = p'^2 + K'^2 + 2p'K' \]

\[
\Rightarrow pK = pp' + p'K - p'^2
\]

\[ p = (E, p) \quad p' = (E', p') \quad K = (M, 0) \]

If \( m_e \rightarrow 0 \), then \( E = |p| \quad \Rightarrow \quad E' = |p'| \)

\[
\Rightarrow pK = EM_N = EE' - |p||p'|\cos \theta + EM_N - m_e^2
\]

\[
\Rightarrow E' = E \left[ \frac{1}{1 + \frac{E}{M_N} (1 - \cos \theta)} \right]
\]

For \( M_N = 12M_p \), \( E' = 482.6 \text{ MeV} \Rightarrow \) elastic peak

i.e. \(^{12}\text{C}\) nucleus remains intact.
What are the other peaks?

\[ e^- \rightarrow e^- + ^{12}C \]

\[ ^{12}C \rightarrow ^{12}C^* \]

\[ E_{\text{elastic}} - E_{\text{inelastic}} = \text{energy of excited state} \]

\[ \Rightarrow \text{study nuclear structure} \]

Large resonance at roughly \( 463 \text{ MeV/c} \) is called the "Giant dipole resonance".

- Rising continuum at lower \( E' \)?
- Sometimes a proton or neutron is knocked out
  \[ \Rightarrow \text{quasi-elastic scattering} \]

- Must first give up 6 to 7 MeV to break up nucleus.

- \( E' = 378 \text{ MeV} \) if \( M_N = M_p \) (mass of proton)

- Peak is very broad. Why?

Protons & neutrons are confined to a space of roughly a sphere of radius 2.5 fm.

\[ \Delta p \approx \frac{\hbar}{\Delta x} \Rightarrow \Delta p = 100 \text{ MeV for } \Delta x \sim 10^{-15} \text{ m} \]

- If energy of incident electron is greater than a few GeV, then one can break up the proton and infer its structure.

- This was done for the first time in the mid-1960s
  \[ \Rightarrow \text{Deep Inelastic Scattering} \]
LECURE V

\[ p + k = p' + k' \]

\[ k'^2 = W^2 = (p + k - p')^2 = (q + k)^2 \]

\[ \frac{d\sigma}{d\Omega} \text{ is now a function of } q^2 \text{ and } W^2 \]

The cross section is seen to rise as \( q^2 > 1 \text{ GeV}^2 \)

\[ \Rightarrow \text{Contrary to expectation since } F(q) \text{ is getting smaller!} \]

Parametrize \[ \left( \frac{d\sigma}{d\Omega} \right)_{DIS} = \left( \frac{d\sigma}{d\Omega} \right)_{M} W_2(x, \frac{q^2}{2kq}) \]

\[ x = \frac{-q^2}{2kq} = \frac{-q^2}{2M(E-E')} \Rightarrow W_2(x) \text{ independent of } q^2! \]

\[ \Rightarrow x \text{ is the fraction of proton momentum } \]

\[ \Rightarrow \text{F(q)} = 1 \text{ again!} \]

\[ \Rightarrow \text{Conclusion: proton have several hard objects with no structure that are} \]

\[ \Rightarrow \text{travelling at very high speeds.} \]
UNSTABLE PARTICLES: How does one describe them?

Consider an assembly of identical particles each with decay probability \( \lambda \).

\[
\begin{align*}
\frac{dN}{dt} &= -\lambda N dt \\
N &= N_0 e^{-\lambda t}
\end{align*}
\]

\[
\tau = \frac{1}{\lambda} \equiv \text{lifetime i.e. time interval for } N_0 \text{ to reduce to } N_0/e
\]

Unstable particle does not have definite energy.

\[
|\psi(t)|^2 = |\psi(0)|^2 \Rightarrow \text{disagrees with unstable model.}
\]

Introduce \( E = E_0 - \frac{i\Gamma}{2} \Rightarrow |\psi(t)|^2 = |\psi(0)|^2 e^{-\frac{\Gamma t}{\hbar}} \)

\[
\Rightarrow P = \frac{h}{\tau} = \frac{\hbar}{\tau}
\]

What does it mean?

Introduce: \( \omega = \frac{E}{\hbar} \) & the Fourier transform

\[
\psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{g}(\omega) e^{-i\omega t} d\omega
\]

\[
P(E) \approx g \star \tilde{g} \Rightarrow P(E) = \frac{1}{2\pi} \frac{\Gamma}{(E-E_0)^2 + (\Gamma/2)^2}
\]

\[
\Gamma = \text{width} \quad \tau = \frac{\hbar}{\Gamma}
\]

\[
\Delta E \Delta t \gg \hbar \Rightarrow \Delta E \sim \Gamma > \frac{\hbar}{2}
\]

Small lifetime \( \Rightarrow \) Large Width.