NUCLEI: made up of protons & neutrons

$Z$ protons & electrons defines atomic number $A$ defines atomic mass

$N$ neutrons defines neutron number $A = Z + N$

$A \times Z \times N$ defines $A$

$^3_2H, ^2_1H, ^3_1H$ are isotopes, also $^3_2He, ^4_2He$

Same $A \Rightarrow$ isotopes  
Same $N \Rightarrow$ isotones

Binding energy $B = M_N - ZM_p - ZM_n - ZM_e$

- $B$ roughly constant $\approx 7-8$ MeV/nucleon
  for heavy nuclei

Higher $B$, lower energy state $B/A$

$B/A$ constant $\Rightarrow$ nuclear density constant

LIQUID DROP MODEL: $E_V = \alpha_1 A + A$

Connections:
- Surface nucleons: $E_s = -\alpha_3 A^{2/3}$
- Coulomb energy: $E_c = -\alpha_2 Z^2 A^{-1/3}$
- Pauli's principle: $E_p = -\alpha_4 (Z-N)^2 / A$
Liquid Drop Model Limitations: Mean-free path comparable to size $\Rightarrow$ quantum liquid.

Other Extreme: Nucleons move independently of each other except for obeying the exclusion principle. $\Rightarrow$ degenerate Fermi gas

$\#$ of spatial states in volume $V$ $\&$ between $p_f$ and $dp$:

$$dN = \frac{4\pi p^2 dp}{h^3} V \quad (2$ nucleons/state)$$

$\Rightarrow A = Z + N = 4 \int_0^{p_f} \frac{4\pi p^2 dp}{h^3} V = \frac{16\pi}{3h^3} p_f^2 V$

$V = \frac{4}{3} \pi r_0^3 A$ $\&$ $r_0 = 1.21 \text{ fm}$

$\Rightarrow p_f \approx 250 \text{ MeV} \quad \& \quad E_F \approx \frac{p_f^2}{2M} \approx 33 \text{ MeV}$

Electron scattering (quasi-elastic) validates this.

Picture of potential:

Total $E_F$ minimizes for $N/Z$

Leading term $\approx (N-Z)^2/A$

Fermi Gas Model usually works for large $N$

$\Rightarrow$ electrons in solid $\Rightarrow$ quantization neglected.

System of nucleons $\Rightarrow$ small $N$ $\Rightarrow$ definite angular momentum.
Empirical observations:
- N or Z = 2, 8, 20, 28, 50, 82, or 126 are exceptionally stable nuclei with large B.
- Large # of stable isotopes
- First excited state has large E
- Isotopic abundances

Shell Model: Each nucleon assigned to a specific energy level \( \Rightarrow \) analog with atomic levels

Central Potential with Spherical Symmetry
\[ V = R_n_l(r) Y_m(l, \theta) \]
\[ n = 1, 2, 3, 4 \ldots \] # nodes + 1
\[ l = s, p, d, f, g \]
N levels are \( 2(2l+1) \) degenerate; \( P = (l)^l \)

For low Z nuclei, \( V \) is that of 3-D harmonic oscillator.
\[ E = (N+3/2) \hbar \Omega \]
\[ N = 2(n-1) + l \]

For heavy nuclei:
\[ V(r) = -V_0 / 4 + e^{-(r-R)/a} \]

Breaks degeneracy and defines energy levels

This works well for \( A \leq 20 \) but for large A, spin-orbit coupling plays a big role.
\[ V(r) = V_c(r) + V_{ls}(r) \langle \ell s \rangle / r^2 \]

\[ \Delta E_{ls} = \frac{2l+1}{2} \langle V_{ls}(r) \rangle \]

\[ \langle V_{ls} \rangle < 0 \quad \Rightarrow \quad \langle V_{ls} \rangle \sim E_N - \frac{k}{2} E_{N-1} \]

- Changes heigancy of single particle energy levels
- Explains magic numbers

J^p of nuclei can be studied in experiments to verify predictions of shell model.

Closed shells: \( J^p = 0^+ \)
\[ N, Z = 2, 8, 20, \ldots \]

Examples: \[ ^{16}O, \quad ^{40}Ca, \quad ^{48}Ca, \quad ^{50}Ni \]

One particle or one hole states: Last nucleon or "hole" defines \( J^p \)

- Example: \( ^{15}N \quad ^{16}O \) "minor" nuclei
- 8 nucleons \( \Rightarrow \) closed shell
- 7 nucleons \( \Rightarrow \) "hole" in \( 1P_{\frac{1}{2}} \) \( \Rightarrow \) \( J^p = \frac{1}{2}^- \)

First excited state: nucleon in \( 1d_{\frac{5}{2}} \) or \( 2s_{\frac{1}{2}} \) \( \Rightarrow \) large binding energy

Example: \( ^{17}O, \quad ^{19}F \) "minor" nuclei
- 1 nucleon in \( 1d_{\frac{5}{2}} \) \( \Rightarrow \) \( 5/2^- \)
- 1 excited state: \( \frac{1}{2}^+ \), small binding energy
Nuclear Stability:

- **α Decay**: In a heavy nucleus, some probability for 2 neutrons & 2 protons to bond into 4He.
  - 4He is very stable: $B = 28.3\text{ MeV}$
  - $\Delta B(A,2) \approx B(A-4,2) + 28.3\text{ MeV}$

Rate of decay depends sensitively on $B$.
  - Stable nuclei until $^{207}$Bi
  - Larger nuclei: U & Th isotopes have long $T$.

- **Fission**: Heavy nucleus splits into 2 roughly equal pieces spontaneously.
  - Energetically favorable.

- **Induced Fission**: Depends on binding energy:
  - $^{235}\text{U}$: 0 energy neutron induces fission
  - $^{238}\text{U}$: 1.4 MeV of energy needed

- Fission produces more neutrons: $^{235}\text{U}
- Chain reaction: large fission probability + critical
- Controlled chain reaction: $^{238}\text{U}$ + small $^{235}\text{U}$ + moderator

Bomb vs nuclear reactor.
Beta-Decay: Nuclei of equal mass number but different Z have a minimum with a quadratic dependence for M(A, Z)

If \( M(A, Z) > M(A, Z+1) \) then \( \beta^- \) decay occurs:
\[
\beta^- \rightarrow n + e^- + \nu_e \quad (^{101}_{43} \text{Tc} \rightarrow ^{101}_{44} \text{Ru} + e^- + \nu_e)
\]

If \( M(A, Z) > M(A, Z-1) \) then \( \beta^+ \) decay occurs:
\[
\beta^+ \rightarrow n + e^+ + \nu_e \quad (^{101}_{43} \text{Rh} \rightarrow ^{101}_{44} \text{Ru} + e^+ + \bar{\nu_e})
\]

In this case another possibility:

\[
M(A, Z) - M(A, Z-1) < 2 MeV \quad \text{then, Electron Capture: } \beta^+ e^- \rightarrow n + \nu_e
\]

Heavy nucleus: nucleus radius large
- electronic orbits small.
- K-shell electrons have maximum at nuclear center.

Lifetime of unstable nuclei depend on energy released (phase space)
\( J^P \) of mother & daughter nuclei.

\[ \text{eg: } ^{40}_{18} \text{K has long half-life.} \]
\[ 89\% \beta^- \text{ to } ^{40}_{16} \text{Ca} \]
\[ 11\% K \text{ capture to } ^2 \text{excited state of } ^{40}_{18} \text{Ar} \]
\[ 10^{-3} \beta^+ \text{ decay to } ^0 \text{G.S. of } ^{40}_{18} \text{Ar} \]
Magic Numbers

30 November, 2004

Lecture 22: Nuclei
Nuclear $\beta$-Decay: $\beta^-$ decay
- $\beta^+$ decay
- electron capture.

- Recall neutron decay: like $\mu$ decay except $\Gamma$ was modified $1 \rightarrow 1.24$

- Nuclear decay: 3 main differences
  * $M_{fi}$ contains overlap of initial $i$ and final wave $f$.
  * Phase space ($\beta^+$ or $\beta^-$) determined by binding energies.
  * Collisions correct $\Gamma$ as $\beta$s exit the nucleus.

$$d\Gamma = \frac{2\pi}{\hbar} |M_{fi}|^2 \frac{d\Omega}{\Delta E} \quad E_0 = E_0 + E_V + E_R = E_0 + E_V$$

In experiments, typically the $\beta$s are detected.

$$d\Gamma = \frac{2\pi}{\hbar} |M_{fi}|^2 (4\pi)^3 (E_0 - E_e)^2 P_e dP_e$$

$$\frac{d\Gamma}{\sqrt{P_e} dP_e} \propto |M_{fi}|^2 (E_0 - E_e) \frac{dN}{dP_e dP_e}$$

Endpoint would be distorted by small neutrino mass.

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Define $f \equiv \int dP_e F_\pm (2P_e P_e)(E_0 - E_e)^2$

$$F_\pm \equiv \text{Fermi function}$$

$$P = \frac{1}{\tau} = \frac{1}{2\pi^3} |M_{fi}|^2 f \Rightarrow \tau = \frac{2\pi^3}{|M_{fi}|^2}$$

E"
$f_T$ is a measure of $|M_{fi}|$

$\begin{array}{l}
\text{N} \rightarrow \text{P} & 10 \text{ minutes} & 1100 \\
\text{bHe} \rightarrow \text{bLi} & 0^8 \text{s} & 810 \\
\text{He} \rightarrow \text{He} & 1 \text{ minute} & 3076 \\
\text{Li} \rightarrow \text{Li} & 6 \times 10^9 \text{ years} & 7 \times 10^9 \\
\end{array}$

Non-relativistic QM: $\mu \propto M^2$

e.g., e-p scattering: $\Phi_e = -e^\lambda (\gamma_e \gamma_\nu) \gamma_\nu \gamma_\nu$

$M_{fi} = \int \Phi_e \Phi_p (d^3 r)$

$\Phi_\nu^N = \int \frac{G_F}{\sqrt{2}} \gamma_\nu^\dagger \gamma^\nu d^3 r$

$M_{fi} = \frac{G_F}{\sqrt{2}} \int \gamma_e^\dagger \gamma_e \gamma_p^\dagger \gamma_p d^3 r$ \text{FERMI TRANSITION}

But weak interaction is V-A

$M_{fi} = \frac{G_F}{\sqrt{2}} \int \gamma_e^\dagger \gamma_p^\dagger \gamma_p \gamma_e d^3 r$ \text{GAMOW-TELLER TRANSITION}

$\gamma_e^\dagger \gamma_e (r) \sim \exp \left( i \vec{q} \cdot \vec{r} \right) = 1 - \frac{\vec{q} \cdot \vec{r}}{\vec{q}^2} + \frac{(\vec{q} \cdot \vec{r})^2}{2! \vec{q}^2} + \ldots$

$\vec{l} = \vec{r} \times \vec{p}$ so expansion is with increasing $l$. 

$2/12/04$ 

Lecture XXIII
9 \text{few MeV/c} \quad \nu \text{few fm} \Rightarrow |qR|/h \sim 10^{-2}

Each increase in \( l \) reduces \( M_{fi} \) by \( 10^4 \) to \( 10^3 \)

Fermi Decays - E2 in spin 0 state
- Nucleon spin unaffected.
\( \Delta P = 0, \Delta J = 0 \)

Selection rule:

\( 0^+ \leftrightarrow 0^+ \) Super allowed, \( M_{fi} \sim 1 \).
\( \Rightarrow \) Complete Wavefunction overlap.

Gamow-Teller Decays - E1 in spin 1 state
\( \Delta P = 0, \Delta J = 0, \pm 1 \).

\( \Delta l = 0 \) : Allowed
\( \Delta l = 1 \) : "Once forbidden"
\( \Delta l = 2 \) : "Twice forbidden"

Examples: Super Allowed: \( ^{14}\text{O} \rightarrow ^{14}\text{N} \quad 0^+ \rightarrow 0^+ \)

\( \text{If isospin symmetry} \Rightarrow \text{no change of} \quad \text{wave function} \)

\( ^{14}\text{N} \rightarrow ^{14}\text{N}, \quad ^{14}\text{N} \)

\( \Delta l = 4 \) !
FUSION \( B/A \) rises with \( A \) for \( A < 20 \)
\[ \Rightarrow \text{fuse 2 light nuclei} \Rightarrow \text{release energy} \]
- Must overcome Coulomb barrier \( \sim \) few MeV
- Scattering? No efficient
- Heating? Need \( 10^{10} \) K!

However tails of Maxwellian velocity distribution provide good probability for \( \sim 10^7 \) K \( \Rightarrow \) stars.

\[ \text{SUN} \quad M \sim 10^{30} \text{kg} \Rightarrow 5 \times 10^{56} \text{ H atoms} \]

\[ ^1\text{H} + ^1\text{H} \rightarrow ^2\text{H} + e^+ + \nu_e + 0.42 \text{ MeV} \]
Weak interaction! Otherwise no stars!

\[ ^1\text{H} + ^3\text{H} \rightarrow ^3\text{He} + \text{more energy} \]

\[ ^3\text{He} + ^3\text{He} \rightarrow ^6\text{He} + 2^1\text{H} + 12.86 \text{ MeV} \]

\[ \Rightarrow \quad 4( ^1\text{H}) \rightarrow ^4\text{He} + 2e^+ + 2\nu_e + \text{energy} \]

Solar luminosity \( \sim 4 \times 10^{38} \text{ W} \Rightarrow 5 \times 10^{55} \text{He atoms} \]
\( \Rightarrow 10\% \) of mass burned so far!

\[ \text{CNO Cycle} \]

\[ Z( ^4\text{He}) \rightarrow ^{12}\text{C} + \text{energy} \]

\[ ^{12}\text{C} \rightarrow \text{N} \rightarrow 0 \rightarrow ^{12}\text{C} + ^4\text{He} \]

\[ \Rightarrow \quad ^{12}\text{C} + 4( ^1\text{H}) \rightarrow ^{13}\text{C} + ^9\text{He} + 2e^+ + 2\nu_e + \text{energy} \]
Also: $^3$He + $^4$He $\rightarrow$ $^7$Be + T

\[ \text{downward} \]

$^7$Be + e$^-$ $\rightarrow$ $^7$Li + νe  \quad $^7$Be + $^1$H $\rightarrow$ $^8$B

$\delta$B $\rightarrow$ $^8$Be + e$^+$ + νe

Net result: Clean solar neutrino spectrum with thresholds for different reactions.

5 MeV $\nu$e easy to measure,
0.5 MeV $\nu$e very hard.

Pioneering Solar $\nu$ Experiment: $^{37}$Cl Homestake

* Birth of Underground Physics
  * Large tank of oil (C_2 Cl_4)
  * $\nu$e capture: $\nu$e + $^{37}$Cl $\rightarrow$ $^{37}$Ar + e$^-$
  * Cosmic ray interactions minimized.
  * $^{37}$Ar has $\text{t}_1/2 = 36$ days: $^{37}$Ar electron capture

* Removal of $^{38}$Ar: - Add $^{38}$Ar as carrier
  - Remove Ar by circulating He and condensing.
  - Ar transferred to an apparatus for counting.

* Counting $^{39}$Ar: - $^{37}$Cl emits X-ray & electrons.
- Backgrounds: - Cosmic rays \( \Rightarrow \) measured at shallow depth
- Neutrons from \( ^{238}\text{U} \) in rock \( \Rightarrow \) shielded by water

From 1970 to 1990: 339 events!!
0.5 atoms/day!!

\[ \Rightarrow \] 3 times lower than predicted
\[ \Rightarrow \] Solar neutrino problem.