LECTURE X

Complex Scalar Field

Suppose you had 2 scalar fields of the same mass:

\[ L = \frac{1}{2} \left[ \partial_{\mu} \phi_1 \partial^{\mu} \phi_1 - m^2 \phi_1^2 \right] + \frac{1}{2} \left[ \partial_{\mu} \phi_2 \partial^{\mu} \phi_2 - m^2 \phi_2^2 \right] \]

Define \( \phi = \frac{\phi_1 + i \phi_2}{\sqrt{2}} \), \( \phi^* = \frac{\phi_1 - i \phi_2}{\sqrt{2}} \)

\[ \Rightarrow L = \partial_{\mu} \phi^* \partial^{\mu} \phi - m^2 \phi^* \phi \]

Observation: Nothing fixes the direction of \( \phi_1 \& \phi_2 \)

i.e. Rotation in \( \phi_1 - \phi_2 \) space leaves \( L \) invariant.

\[ \phi'_1 = \phi_1 \cos \alpha + \phi_2 \sin \alpha \]
\[ \phi'_2 = -\phi_1 \sin \alpha + \phi_2 \cos \alpha \]

\[ \Rightarrow \phi' = e^{-i\alpha} \phi \quad \phi'^* = e^{i\alpha} \phi^* \]

Invariance of \( L \) has important consequences:

(Noether's Theorem)

Consider an infinitesimal transformation

\[ \phi' = (1-i\epsilon \phi) \Rightarrow S \phi = -i\epsilon \phi \quad \& \quad S \phi^* = +i\epsilon \phi^* \]

\[ \delta L = \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial^{\mu} \phi)} \partial^{\mu} \delta \phi + (\phi \rightarrow \phi^*) \]

Demand \( \delta L = 0 \Rightarrow \partial^{\mu} S \mu = 0 \) with

\[ S \mu = i \left( \phi \partial_{\mu} \phi^* - \phi^* \partial_{\mu} \phi \right) \]

\[ \partial^{\mu} S \mu = \frac{\partial S^0}{\partial t} + \nabla \cdot S \]

(Conservation of "current")
LECTURE X

E&M analogy: \[ \nabla \cdot \mathbf{J} + \frac{\partial \varphi}{\partial t} = 0 \] (continuity Eqn)

\[ \int S d^3x = \Phi = \text{constant (charge)}; \int \nabla \cdot \mathbf{J} d^3x = \int J \cdot nds \]

\[ \Rightarrow \text{change in "charge" density} = \text{flow of "current"} \]

* \[ \int S d^3x \] is a generic charge
* Interchange of \( \varphi \) and \( \varphi^* \) \[ \Rightarrow \text{change in sign of } S^k \]
* \( e^{-i} \) \[ \Rightarrow \text{Global gauge transform} = \text{Invariance Law} \]

Example: Conservation of probability from Schrödinger Eqn:

\[ \frac{\partial \psi}{\partial t} = \frac{i}{2m} \nabla^2 \psi \Rightarrow \frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{J} = 0 \]

\[ S = \psi^* \psi \]

"Probability" \( \quad \) \[ J = \frac{-i}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \]

"Current"

Dirac Equation: Quest for a relativistically covariant eqn for massive particles. Motivation: Conservation of probability \( \Rightarrow \) linear in time derivative \( \Rightarrow \) Lorentz invariance \( \Rightarrow \) linear in space derivative

General form: \[ \frac{i}{\hbar} \frac{\partial \psi}{\partial t} = \mathcal{H} \psi \] with \( \mathcal{H} = -i \left[ \mathbf{p} + \beta m \right] \)

Must obey \[ E^2 = p^2 + m^2 \Rightarrow -\frac{\partial^2 \psi}{\partial t^2} = \left(-\nabla^2 + m^2\right) \psi \]
LECTURE X

\[ \Rightarrow \alpha_i \beta_j + \beta_j \alpha_i = 0 \ (i \neq j) \quad \& \quad \alpha_i \beta + \beta \alpha_i = 0 \]

If \( m = 0 \), then \( \beta \) is absent \( \Rightarrow \alpha \alpha_j + \alpha_j \alpha = 2 \delta_{ij} \)

One solution are the \( 2 \times 2 \) pauli matrices:

\[ \Rightarrow \ i \frac{\text{d} \psi}{\text{d} t} = (\overrightarrow{\mathbf{p}} \cdot \overrightarrow{\mathbf{p}}) \psi \]

Here \( \psi \) is a \( 1 \times 2 \) spinor.

For \( m \neq 0 \), no \( 2 \times 2 \) soln \( \Rightarrow \) must go to \( 4 \times 4 \)

One set of solns:

\[ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ 0 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

Introduce the \( \gamma \) matrices

\[ \gamma^i = \beta \alpha^i \quad \frac{\text{d} \psi^0}{\text{d} t} = \beta \overrightarrow{\mathbf{p}} \]

\[ \text{Satisfy} \quad \{ \gamma^i, \gamma^j \} = 2 \delta_{ij} \]

\[ \overrightarrow{\mathbf{p}} \cdot \nabla + \beta m \]

\[ \Rightarrow \ i \frac{\text{d} \psi}{\text{d} t} = (-\overrightarrow{\mathbf{p}} \cdot \nabla + \beta m) \psi \]

\[ i \beta \frac{\text{d} \psi}{\text{d} t} = [(-i \beta \overrightarrow{\mathbf{p}} \cdot \nabla) + m] \psi \]

\[ \Rightarrow \ (i \gamma^\mu \partial_{\mu} - m) \psi = 0 \]

\( \psi \) is a 4 element column matrix.

\( \psi^* \psi \) is a number but not Lorentz invariant.

Introduce \( \overline{\psi} = \psi^+ \gamma^0 \Rightarrow \overline{\psi} \psi \) is a Lorentz scalar.
LECTURE X

\( \bar{\psi} \) is called the adjoint spinor. \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \)

\( \bar{\psi} \gamma^\mu \psi \) is a Lorentz 4-vector.

Another important \( \gamma \) matrix is \( \gamma^5 = i \sigma^2 \gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\( \bar{\psi} \gamma^5 \psi \) is a pseudoscalar.

\( \bar{\psi} \gamma^\mu \gamma^5 \psi \) is a pseudo-vector or axial-vector.

Consider the Dirac Eqn in restframe \( \Rightarrow \bar{\psi} = 0 \)

\( \Rightarrow i \gamma^0 \partial \psi - m \psi = 0 \)

\( \Rightarrow \partial \psi = -i m \gamma^A \psi \quad \& \quad \partial \psi = +i m \gamma^B \psi \)

\( \psi_A, \psi_B \) are 2 component spinors \( \Rightarrow \) 4 total solutions:

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Note: \( \gamma^B = i m c^2 \bar{\gamma} \)

Compare with Schrödinger Eqn Soln: \( \psi = e^{-iEt/m} \psi(0) \)

\( \psi_A \) has \( E = mc^2 \) but \( \psi_B \) has \( E = -mc^2 \) ???

\( \Rightarrow \) Reinterpret as \( E = +mc^2 \) with \( t \rightarrow -t \)

\( \Rightarrow \) Every particle has anti-particle.

2 fold degeneracy \( \Rightarrow \) \( \psi \) commutes with new operator.

So there exist simultaneous eigenfns.
LECTURE X

Commuting operator is \( \vec{\Sigma} \equiv \begin{pmatrix} \sigma_x & \sigma_y \\ \sigma_y & -\sigma_x \end{pmatrix} \)

\[ [\vec{H}, \vec{L}] = -i (\vec{L} \times \vec{P}) \quad [\vec{H}, \vec{\Sigma}] = +2i (\vec{L} \times \vec{P}) \]

\[ \implies [\vec{H}, \vec{J}] = 0 \quad \text{with} \quad \vec{J} = \vec{L} + \frac{1}{2} \vec{\Sigma} \]

Not conserved but \( \vec{J} \) is conserved \( \implies \text{spin } \frac{1}{2} \)

\( \bar{\psi} \) satisfies a different equation \( \implies \)
Start with \( i \frac{\partial \bar{\psi}}{\partial t} + i \sigma^k \frac{\partial \bar{\psi}}{\partial x^k} - m \bar{\psi} = 0 \)

Conjugate Eqn: \( -i \frac{\partial \psi^+}{\partial t} + i \sigma^k \frac{\partial \psi^+}{\partial x^k} - m \psi^+ = 0 \)

Since \( \gamma = \gamma^+ \gamma^0 \): \( -i \frac{\partial \psi^+}{\partial t} - i \gamma^0 \gamma^k \frac{\partial \psi^+}{\partial x^k} - m \psi^+ = 0 \)

Thus: \( i \gamma^m \partial_m \psi - m \psi = 0 \) (1)
\( i \partial_m \bar{\psi} \gamma^m + m \bar{\psi} = 0 \) (2)

\( \bar{\psi} \gamma \gamma^0 + (\bar{\psi} + 2) \gamma \psi \Rightarrow \bar{\psi} \gamma^m \partial_m \psi + \partial_m \bar{\psi} \sigma^m \psi = 0 \)

\[ \implies \partial_m (\bar{\psi} \sigma^m \psi) = 0 \]

Introduce \( j^m = \bar{\psi} \gamma^m \psi \) is a conserved current.
\( \implies \) due to invariance of Dirac Eqn
(Equality of particle & antiparticle)
LECTURE X

Electromagnetic "current": \( j^\mu = -e F^\mu_\nu \)

For a free particle: \( \psi = A e^{-i p^\mu x_\mu} \)

\[ \Rightarrow \quad j^\mu = -e p^\mu \]

If we replace \( E \rightarrow -E \) and \( p \rightarrow -p \), then \( j^\mu = +e (-p^\mu) \)

Thus, positrons have opposite charge.

Electrons propagating forwards \( \leftrightarrow \) positrons propagating backwards.

Next steps: Maxwell's Equations in covariant form,
- Lagrangian for spin \( \frac{1}{2} \) fermions
- Lagrangian for free photons
- Interaction between electrons & photons by demanding "local" gauge invariance
Lecture XI

Gauge Invariance in Classical EM
\[ \nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \]

- \( \mathbf{A} \) is the vector potential function \( \nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \)
- \( \mathbf{B} \) is unchanged if \( \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \)
  \( (\Lambda \) is an arbitrary scalar function) \]

However, \[ \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0. \]

This suggests \[ \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Lambda \] if thus

- if \( \mathbf{A} \rightarrow \mathbf{A} + \nabla \Lambda \), then \( \mathbf{V} \rightarrow \mathbf{V} - \partial \Lambda / \partial t \)
- or \( A_\mu \rightarrow A_\mu - \partial \Lambda / \partial t \) (\( A_\mu = (V, A) \))

Covariant formulation of Maxwell's Eqns:

\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = \mathbf{J} \]

\[
\begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{pmatrix}
\]

F\(^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \)

Form guarantees \( \nabla \cdot \mathbf{B} = 0 \)
\( \nabla \times \mathbf{E} - \partial \mathbf{B} / \partial t = 0 \)

Define \( \mathbf{J}_\mu = (\mathbf{J}, \mathbf{J}) \)

Then: \( \partial \mu F^{\mu\nu} = \mathbf{J}^{\nu} \)

It can be shown that \( \partial \mu \mathbf{J}^{\mu} = 0 \Rightarrow \) conserved current

\[ L = -\frac{i}{4} F^{\mu\nu} F_{\mu\nu} - J_\mu A^\mu \] leads to \( \mathbf{E} - \mathbf{L} \mathbf{E} = \partial F^{\mu\nu} = \mathbf{J}^{\nu} \)
LECTURE XI

Phase Invariance in QM:
Consider an observable \( \langle 0 \rangle = \int \psi^* 0 \psi \, d^3 x \)

→ Unchanged by \( \gamma(\alpha) \rightarrow e^{i\alpha} \gamma(\alpha) \)

But what if \( \alpha \) is a function of \( x \)?
Laws of physics should allow different phases.

Consider \(-i \frac{\nabla^2 \psi}{2m} = \frac{\partial \psi}{\partial t} \) if \( \psi \rightarrow \psi' = e^{i\xi(\mathbf{r},t)} \psi \)

Then \(-i \frac{\nabla^2 \psi'}{2m} \neq \frac{\partial \psi'}{\partial t} \) unless,

\[ \nabla \rightarrow \nabla - i q \mathbf{A} \equiv \mathcal{D} \mathbf{f} \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + i q \mathbf{V} = \mathcal{D} \]

Also \( \mathbf{A} \rightarrow \mathbf{A} + \nabla (\frac{\xi}{q}) \) if \( \mathbf{V} \rightarrow \mathbf{V} - \frac{\partial}{\partial t} (\frac{\xi}{q}) \)
with \( \frac{\xi}{q} = \mathbf{A} \)

Now \[ \left\{ \frac{(-i \nabla + q \mathbf{A})^2 + q \mathbf{V} \psi^2 \psi}{2m} \right\}_{\mathcal{H}} \]

This Hamiltonian leads to \( \mathbf{F} = q \mathbf{E} + \mathbf{V} \times \mathbf{B} \)

Thus, \( \mathbf{A} \) is the vector potential of E&M

Demanding \( \text{LOCAL} \) phase invariance
produces the \( \mathbf{E} \) & \( \mathbf{B} \) fields.
LECTURE XI

Covariant formulation: \( \phi(x, \mu) \rightarrow e^{iQx(\mu)} \phi(x, \mu) \)

\[ \partial \mu \phi \rightarrow e^{iQx} [\partial \mu \phi + iQ(\partial \mu x) \phi] \]

Introduce \( D \mu = \partial \mu + iQ \Lambda \mu \)

Then \( D \mu \phi \rightarrow e^{iQx} [\partial \mu \phi + iQ(\partial \mu x) \phi + iQ \Lambda \mu \phi] \)

Thus, \( D \mu \rightarrow e^{iQx} D \mu \) provided \( \Lambda \mu \rightarrow \Lambda \mu - \partial \mu x \)

\( \Rightarrow \Lambda \mu \) has the properties of \( \Lambda \mu \) of the Maxwell's Eqns.

Recipe for Quantum Electrodynamics

Dirac Eqn:\n\begin{align*}
(\gamma^{\mu} \partial_{\mu} - m) \psi &= 0 \quad (1) \\
\left(\gamma^{\mu} (\partial_{\mu} \gamma^{\mu} + m \gamma^{0}) \right) &= 0 \quad (2)
\end{align*}

\[ L_{\text{Dirac}} = \bar{\psi} \left( i \gamma^{\mu} \partial_{\mu} - m \right) \psi \]

\[ \frac{\delta L}{\delta (\partial_{\mu} \psi)} = 0 \quad \frac{\delta L}{\delta \psi} = (i \gamma^{\mu} \partial_{\mu} - m) \psi = 0 \Rightarrow (1) \]

\[ \frac{\delta L}{\delta \partial \mu \psi} = \bar{\psi} i \gamma^{\mu} \Rightarrow \frac{\delta L}{\delta \psi} = - \bar{\psi} m \]

\[ \partial \mu \left[ \frac{\partial \psi}{\partial \partial \mu \psi} \right] = i \partial \mu \bar{\psi} \gamma^{\mu} = \frac{\delta L}{\delta \psi} = - m \bar{\psi} \]

\[ \Rightarrow i \partial \mu \bar{\psi} \gamma^{\mu} + m \bar{\psi} = 0 \Rightarrow (2) \]
LECTURE XI

Start with $L_{free} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi$

Replace $\partial_\mu$ with $D_\mu \equiv \partial_\mu - i e A_\mu$

$\Rightarrow L = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - e A_\mu \bar{\psi} \gamma^\mu \psi$

$\bar{\psi} \gamma^\mu \psi$ is the conserved electromagnetic current.
$A_\mu$ is the vector potential of E&M.

We now have kinetic energy term for $e^+ e^-$, mass term for $e^+ e^-$, $A$ interaction term between photons and electrons.

We need the kinetic energy term for photons:
$L = \frac{1}{4} (\partial_\nu A_\mu - \partial_\mu A_\nu) (\partial^\nu A^\mu - \partial^\mu A^\nu) = \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$

$\frac{1}{2} (E^2 + B^2)$ is invariant under gauge tranfos.

$L_{QED} = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi - e A_\mu \bar{\psi} \gamma^\mu \psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}$

Note: $m^2 A_\mu A^\mu$ terms break invariance $\Rightarrow$ photons must be massless!

RECAP: - Formulated Dirac Eqn $\Rightarrow$
- Eqn for motion of massive spin $\frac{1}{2}$ particles and antiparticles.
- Global phase tranfos reveals conserved current.
- Local phase invariance reveals new vector field.
- $A$ interaction term with conserved current.
- Finally add photon kinetic energy term.
LECTURE XI

Why is \( \mathcal{V} \delta^\mu \gamma A_\mu = j^\mu A_\mu \) an interaction term?

Consider a scalar field and a static charge density:

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) - \mathcal{G}
\]

\[
\Rightarrow \partial_\mu \partial^\mu \phi - m^2 \phi = \mathcal{G}
\]

Consider a point source \( S = g \delta(x) \)

Then \( (\nabla^2 + m^2) \phi = g \delta(x) \)

Introduce \( \Phi(R) = \frac{1}{(2\pi)^{3/2}} \int d^3 x \ e^{-iR \cdot x} \Phi(x) \)

\[
\Rightarrow \Phi(R) = \frac{g}{(2\pi)^{3/2}} \frac{1}{k^2 + m^2}
\]

\( \frac{1}{k^2 + m^2} \) is the propagator which becomes \( \frac{1}{k^2} \) for \( m = 0 \)

\[
\phi(x) = \frac{g}{4\pi} e^{-m|x|} \Rightarrow \text{infinite range \& potential for } m = 0
\]

In 1934, Yukawa proposed that \( \phi \) is a meson field produced by source nucleons.

RECAP: In a quantum field theory:
- Interaction term \( \mathcal{G} \) or \( j^\mu A_\mu \) denotes the emission or absorption of force quanta.
- This gives rise to the potential term.
- This leads to a force or interaction.
  Thus: Local phase invariance \( \Rightarrow \) Force.
LECTURE XII

Having motivated \( S \& E \) we now derive Feynman rules (which allow us to carry out scattering calculations).

Example from non-relativistic perturbation theory:

Let \( H_0 \phi_n = E_n \phi_n \) with \( \int \phi_n^* \phi_m \, d^3x = \delta_{mn} \)

Add a time dependent potential:

\[
H_0 + V(\vec{x}, t) = i \frac{d \psi}{dt} = \sum_n \frac{a_n(t)}{n} \phi_n(\vec{x}) e^{-i E_n t}
\]

One obtains:

\[
\frac{d a_f}{dt} = -i \sum_n \frac{a_n(t)}{n} \int \phi_n^* V_{fi} \phi_i \, d^3x \, e^{-i(E_f-E_n)t}.
\]

This is a recursive relation.

\( a_f \) for a given final state is related \( M_{fi} \) in Fermi's Golden Rule.

**FIRST ORDER**

\[
\alpha_f = T_{fi} = i \int d^3x \phi_i^* V \phi_i
\]

**SECOND ORDER**

\[
\alpha_f = \sum_{n \neq i} V_{fi} V_{ni} \delta(E_f-E_i) \frac{d(E_f-E_i)}{E_i-E_n}
\]
XII LECTURE

Need free particle solutions to Dirac Hamiltonian.

\[ y = \alpha \ u(p) \ e^{i \chi \cdot \ p} \]

\[ p = (E, \vec{p}) \quad \chi \cdot \ p = \chi \mu \ p^\mu \]

\[ \partial_\mu \ y = -i \ p^\mu \ \alpha \ e^{i \chi \cdot \ p} \ u \]

\[ \Rightarrow \ (\gamma^\mu \ p^\mu - m) \ u = 0 \quad \text{(Momentum-Space Dirac Eqn)} \]

Let \( u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \Rightarrow \gamma^\mu \ p^\mu - m = \begin{pmatrix} E - m & -\vec{p} \cdot \vec{\sigma} \\ \vec{p} \cdot \vec{\sigma} & -E - m \end{pmatrix} \)

\[ \Rightarrow \ u_A = \frac{\vec{p} \cdot \vec{\sigma}}{E - m} \ u_B \quad u_B = \frac{\vec{p} \cdot \vec{\sigma}}{E + m} \ u_A \]

One set of solutions:

\[ u^1 = \begin{pmatrix} 1 \\ 0 \\ \frac{P_3}{E + m} \\ \frac{P_3 + i P_y}{E + m} \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ \frac{P_3}{E + m} \\ \frac{P_3 + i P_y}{E + m} \end{pmatrix} \quad u^3 = \begin{pmatrix} \frac{P_3}{E + m} \\ -\frac{P_3}{E + m} \\ 1 \\ 0 \end{pmatrix} \quad u^4 \]

\( u^i \) are not eigenstates of \( E \) unless \( \vec{p} \) is along \( z \)-axis.

Then: \( u^1 \) is \( e^- \) with spin "up" if \( u^2 \) is \( e^- \) with spin "down".

Rewrite negative energy states:

\[ u^{3,4}(-E, -\vec{p}) = u^{2,1}(E, \vec{p}) \]

\[ (\gamma^\mu \ p^\mu - m) \ u = 0 \]

\[ (\gamma^\mu \ p^\mu + m) \ u = 0 \]
LECTURE XII

Now we introduce \( V = -e \delta \mu A^\mu \)

\[
\frac{1}{2} \epsilon_{fi} = -i \int \bar{\psi}_f \gamma^\nu \gamma_i d^4x = -i e \int \bar{\psi}_f A^\mu \gamma_i d^4x = i \int \bar{\psi}_f A^\mu d^4x
\]

\[
\bar{\psi}_f \gamma_\mu \psi_i = -i e \bar{\psi}_f \gamma_\mu \psi_i e^{-(P_f - P_i) \cdot x}
\]

\( A^\mu \) electron emitting a photon.

\( \bar{\psi}_f \gamma_\mu \psi_i \) electron absorbing a photon.

\( \bar{\psi}_i \gamma_\mu \psi_f \) positron absorbing a photon.

These processes do not satisfy energy-momentum conservation.

Now, let \( A^\mu \) be produced by a second charged fermion.

\[
\Rightarrow \partial_\mu F^{\mu\nu} = j^{(2)}_{\nu} \quad \text{if } \text{since } A^\mu \text{ can go to } A^\mu + \partial_\mu \lambda,
\]

choose \( \partial_\mu A^\mu = 0 \) \( \Rightarrow \partial_\mu A^{\mu \nu} = j^{(2)}_{\nu} \)

\[
\bar{\psi}_f \gamma_\mu \psi_i e^{i q \cdot x} = \frac{U(B) \chi^{\mu} U(D)}{-i (P_D - P_B) \cdot x}
\]

\[
P_D - P_B \equiv q
\]

\[
\text{Since } \partial_\mu A^{\mu \nu} e^{i q \cdot x} = -q^2 e^{i q \cdot x},
\]

\[
A^\mu = -\frac{1}{q^2} j^{(2)}_{\mu}
\]
LECTURE XII

Thus, \( T_{fi} = -i \int d^4x \left( \frac{1}{q^2} \right) J^{\mu}(x) d^4x \).

Use Dirac \( \mathcal{D} \) function:

\[
\mathcal{D}(x) = \begin{cases} 
  0 & \text{if } x \neq 0 \\
  \infty & \text{if } x = 0 
\end{cases} \quad \int_{-\infty}^{\infty} \mathcal{D}(x) dx = 1.
\]

\[
\Rightarrow \int_{-\infty}^{\infty} f(x) \mathcal{D}(x) dx = f(0)
\]

If \( \phi(x) = C \int d^4p e^{-i(p \cdot x)} \phi(p) \) and \( \bar{\phi}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{i(p \cdot x)} \phi(x) \)

If \( \bar{\phi}(p) = \mathcal{D}(p) \) then \( \phi(x) = \frac{1}{\sqrt{2\pi}} \int dx e^{i(p \cdot x)} \)

Thus \( \mathcal{D}(p) = \frac{1}{2\pi} \int d^4x e^{i(p \cdot x)} \)

\[
\Rightarrow \int d^4x e^{i(P_a - P_c + P_b - P_d \cdot x)} = (2\pi)^4 \delta^4(P_a - P_c + P_b - P_d)
\]

\[
\Rightarrow T_{fi} = \left[ -i (-e \bar{u}_c \gamma_\mu u_A) \left( \frac{-1}{q^2} \right) (-e \bar{u}_d \gamma^\mu u_B) \right] X 
\]

\[
X = (2\pi)^4 \delta^4(P_a - P_c + P_b - P_d)
\]

The quantity in [ ] is \( M_{fi} \).
LECTURE XII

Feynman Rules

I. Draw all possible diagrams taking initial state to final state:

\[ e^- \mu^- \rightarrow e^- \mu^- \]

\[ e^+ e^- \rightarrow \mu^+ \mu^- \]

II. Label all incoming and outgoing 4-momenta:
- \( P_1, P_2, \ldots, P_n \) and corresponding spins \( s_1, s_2, \ldots, s_n \).
- Internal 4-momenta \( q_1, q_2, \ldots, q_m \).
- Assign arrows: external fermions go "forward",
- external anti-fermions go "backward",
- internal fermions go in direction of flow,
- external photons go "forward",
- Internal photon lines are arbitrary.

III. External lines contribute following factors:
- Fermions \( \{ \text{Incoming} \rightarrow \text{Outgoing} \} \)
- Antifermions \( \{ \text{Incoming} \leftarrow \text{Outgoing} \} \)
- Photons \( \{ \text{Incoming} \rightleftharpoons \text{Outgoing} \} \)

\[ A_\mu = \gamma_\mu p_\mu \epsilon_\alpha^{(s)} \]

\[ \epsilon_\mu, p_\mu = 0 \quad \epsilon_\mu^{(s)} p_\mu = 0 \]

\[ s = 1, 2 \]
LECTURE XII

Vertex factors:

\[ i g e \gamma_\mu \]

\[ g e = e \sqrt{\frac{4\pi}{\hbar c}} = \sqrt{4\pi} \alpha \]

Propagator:

photons: \[-i \frac{g_{\mu\nu}}{\beta^2}\]

electrons: \[ i \frac{(\gamma^\nu \gamma^\mu + m)}{\gamma^2 - m^2} \]

For each vertex, add a factor

\[ (2\pi)^4 \delta^4 (k_1 + k_2 + k_3) \]

\[ k_1, k_2, k_3 \text{ are 4-vectors; Sign convention: outward = -ve} \]

(NOTE: This is actual direction; opposite for antifermions!)

For each internal momentum \( q_i \), write

\[ d^4 q_i \] & integrate over \( d^4 x \).

\[ (2\pi)^4 \]

Cancel the \( S \)-function:

The result will include a factor

\[ (2\pi)^4 \delta \left( p_1 + p_2 + \cdots - p_n \right) \]

reflecting overall energy-momentum conservation.

\[ \Rightarrow \text{Remove this factor & what remains is} - i M \]
LECTURE XIII

Full calculation of $e^+ \rightarrow e^- \mu^-$

$\mu^-$

Proceed backward along each fermion line

$\int \left[ \bar{u}_{s_3}^-(p_3) (i g \gamma^\mu) u_{s_4}^+(p_1) \right] - \frac{i g_{\mu\nu}}{q^2} \left[ \bar{u}_{s_4}^+(p_4) (i g \gamma^\nu) u_{s_2}^-(p_2) \right] \frac{d^4q}{(2\pi)^4} \delta(p_1 - p_3 - q) \frac{1}{(p_2 + q - p_4)^2}$

$\frac{i g e^2}{(p_1 - p_3)^2} \left[ \bar{u}_{s_3}^+(p_3) \gamma^\mu u_{s_4}^+(p_1) \right] \left[ \bar{u}_{s_4}^+ \gamma^\mu u_{s_2}^-(p_2) \right]$

$M = \frac{-g e^2}{(p_1 - p_3)^2} \left[ U(3) \gamma^\mu U(1) \right] \left[ U(4) \gamma^\mu U(2) \right]$

$M$ is just a number (1x1 matrix)

Golden Rule for Scattering: $1+2 \rightarrow 3+4 \cdots n$

$\sigma = |M|^2 \frac{S}{4\sqrt{(p_1 \cdot p_3)^2 - (m_1 m_2)^2}} \left[ \frac{d^3p_3}{(2\pi)^3 2E_3} \cdots \frac{d^3p_n}{(2\pi)^3 2E_n} \right]$}

$S = 1/j!$ for each group of $j$ identical particles.
- Simplify form of $d\sigma$ for the special case of $1+2 \rightarrow 3+4$
  - Integrating over $|p_3| \& \ p_4$ gives the probability of observing particle $p_3$ with a solid angle $d\Omega$
  \[ \frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{S |M|^2 |p_4|}{(E_1+E_2)^2 |p_i|} \]
  in the center-of-mass frame.

- Calculation of $M_{fi}$ can be simplified by summing over initial and final state polarizations.
  - If one starts with unpolarized particles $f$ do not observe polarization in the final state; then one must average over initial spins $f$
  - Sum over final spins

$\Rightarrow$ Specific Algorithm for summing spins

Start with $M = -\frac{\alpha}{2} \sqrt{(u(3)\gamma^\mu u(1)) [\tilde{u}(4)\gamma_\mu u(2)]}$

$|M|^2 = MM^* = \frac{\alpha^4}{4} \sqrt{[\tilde{u}(3)\gamma^\mu u(1)] [\tilde{u}(4)\gamma_\mu u(2)] [\tilde{u}(3)\gamma^\mu u(1)] [\tilde{u}(4)\gamma_\mu u(2)]}$

Need to compute $[\tilde{u}(a)\gamma_1 u(b)] [\tilde{u}(a)\gamma_2 u(b)]^*$
LECTURE XIII

\[ [\bar{u}(a) \gamma_2 u(b)]^k = [u(a)^+ \gamma^0 \gamma_2 u(b)]^+ = u(b)^+ \gamma_2^+ \gamma^0 + u(a) \]

where \( \gamma_2 = \gamma^0 \gamma_2 + \gamma^0 \).

Let \( G = [\bar{u}(a) \gamma_2 u(b)] [\bar{u}(b) \gamma_2 u(a)] \)

Then \( G = \bar{u}(a) \Gamma_2 \{ \sum_{s_b} U_{sb}(p_b) \bar{u}_{s_b}(p_b) \} \gamma_2 u(a) \)

We use \( \sum_{s_1 s_2} U_{s_1}(p) \bar{U}_{s_2}(p) = 4 \gamma^\mu p_\mu + m \)

Then \( G = \text{Tr} [\gamma_1 (\gamma^\mu p_\mu + m) \gamma_2 (\gamma^\nu p_\nu + m)] \)

For \( e^- e^+ \) scattering, \( \gamma_2 = \gamma^\nu \Rightarrow \gamma_2 = \gamma^0 \gamma^\nu + \gamma^\nu = \gamma^\nu \)

Thus, \( \langle |M|^2 \rangle = \sum_{\text{spin}} \frac{1}{M^2} \)

\[ = \frac{g^4 e^4}{4 (p_1^2 - p_3^2)} \text{Tr} [\gamma^\mu (p_1^2 + m) \gamma^\nu (p_3^2 + m)] \text{Tr} [\gamma_\mu (p_2^2 + M) \gamma_\nu (p_4^2 + M)] \]

\[ = \gamma^\mu p_\mu , \quad m = m_e , \quad M = m_{\mu} \]

Let us evaluate the first \( \text{Tr} : \text{Tr} [\gamma^\mu (p_1^2 + m) \gamma^\nu (p_3^2 + m)] \)

\[ = \text{Tr} (\gamma^\mu \gamma^\nu p_3) + m [\text{Tr} (\gamma^\mu p_3^2)] + m^2 \text{Tr} (\gamma^\mu \gamma^\nu ) \]

Use following Trace identities:

\[ \text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4 \gamma^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} + g^{\rho\sigma} g^{\mu\nu} \]

\[ \text{Tr} (\gamma^\mu \gamma^\nu) = 4 \gamma^\mu \gamma^\nu \]

Trace of odd number of \( \gamma \) matrices is 0
\[\text{LECTURE X11}\]

\[\text{Tr} (\sigma^\mu \sigma^\nu \gamma_5 \gamma_3 \gamma_3) = \frac{1}{4} \left( g^{\mu\nu} - g^{\mu\nu} + g^{\mu\nu} \right) \]

\[= \frac{1}{4} \left[ (\gamma_3^\mu \gamma_3^\nu - g^{\mu\nu} (\gamma_3^\mu \gamma_3^\nu + \gamma_3^\mu \gamma_3^\nu) \right] \]

\[\Rightarrow \text{Tr} [\gamma_3^\mu (\gamma_3 + m) \gamma_3^\nu (\gamma_3 + m) = \frac{1}{4} \left[ \gamma_3^\mu \gamma_3^\nu + \gamma_3^\mu \gamma_3^\nu + g^{\mu\nu} (m^2 - \gamma_3^3 \gamma_3^3) \right] \]

\[\text{Similarly,} \]

\[\text{Tr} [\gamma_3^\mu (\gamma_3 + M) \gamma_3^\nu (\gamma_3 + M) = \frac{1}{4} \left[ \gamma_3^\mu \gamma_3^\nu + \gamma_3^\mu \gamma_3^\nu + g^{\mu\nu} (M^2 - \gamma_3^3 \gamma_3^3) \right] \]

\[\langle |M|^2 \rangle = \frac{8g_e^4}{(p_4 \cdot p_3)^4} \left[ (p_1 \cdot p_2) (p_3 \cdot p_4) + (p_1 \cdot p_4) (p_3 \cdot p_2) - M^2 (p_1 \cdot p_3) - M^2 (p_2 \cdot p_4) + 2m^2 M^2 \right] \]

Choose to work in LAB frame.

\[p_1 = (E_1, p_1) \quad p_2 = (M, 0) : \text{ Assume negligible recoil} \]

\[\Rightarrow \quad |p_1| = |p_3| = |p_1|, \quad p_3 = (E_1, p_3), \quad p_4 = (M, 0) \]

\[\left( p_1 - p_3 \right)^2 = - (p_1 - p_3) \cdot (p_1 - p_3) = -2|p_1|^2 - 2|p_1|^2 \cos \theta = -4|p_1|^2 \sin^2 \theta/2 \]

\[p_4 \cdot p_3 = \frac{E}{M} - p_1 \cdot p_3 = 2|p_1|^2 \sin^2 \theta/2 + m^2 \]

\[p_2 \cdot p_3 \cdot p_4 = \frac{M^2}{2} \]

\[d\sigma \over d\Omega = \frac{1}{(8\pi M)^2} \langle |M|^2 \rangle \]

\[8 \frac{g_e^4}{(4|p_1|^2 \sin^2 \theta/2)^2} \left[ M^2 E^2 + M^2 (2|p_1|^2 \sin^2 \theta/2 + m^2) - m^2 M^2 + 2m^2 M^2 \right] \]

\[= \langle |M|^2 \rangle \]
LECTURE XIII

\[ \langle |M|^2 \rangle = \frac{g e^4 M^2}{|p|^4 \sin^4 \theta/2} \left[ m^2 + |p|^2 \cos^2 \theta/2 \right] \]

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2 (\frac{\hbar^2 c^2}{\alpha^2})}{4 |p|^4 \sin^4 \theta/2} \left[ m^2 + |p|^2 \cos^2 \theta/2 \right] \]

If \( |p| \ll m \Rightarrow \text{Rutherford scattering!} \)

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2 m^2 (\frac{\hbar^2 c^2}{\alpha^2})}{4 |p|^4 \sin^4 \theta/2} \]

If \( |p| \gg m \), then:

\[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2 (\frac{\hbar^2 c^2}{\alpha^2}) \cos^2 \theta/2}{4 |p|^2 \sin^4 \theta/2} \]

Use: \( q^2 = 4E^2 \sin^2 \theta/2 \).

Then \[ \frac{d\sigma}{d\Omega} = \frac{\alpha^2 (\frac{\hbar^2 c^2}{\alpha^2}) E^2 \cos^2 \theta/2}{q^4} \]

Mott Scattering.
Continue discussion of $e^{-\mu} \rightarrow e^{-\mu}$

Now assume $m = 0$ but $M$ can recoil

results can carry over to $e^{-\mu} \rightarrow e^{-\mu}$

$$\langle |M|^2 \rangle = \frac{8q^4e^4}{(p_1 \cdot p_2)(p_3 \cdot p_4)} \left[ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_3)M^2 \right]$$

$p_1 = (E, \vec{P}) \quad p_2 = (M, 0) \quad p_3 = (E', \vec{P'}) \quad p_4 = p_1 + p_2 - p_3$

$p_2^2 = 0 \quad p_3^2 = 0 \quad E = |\vec{P}| \quad E' = |\vec{P'}|

(p_1 \cdot p_3)(p_1 \cdot p_3) = q^2 = -2(p_1 \cdot p_3) = -2EE'(1 + \cos \theta) = -4EE'\sin^2\theta/2$

(p_3 \cdot p_4) = (p_3 \cdot p_1) + (p_3 \cdot p_2) \quad (p_2 \cdot p_4) = (p_1 \cdot p_2) - (p_1 \cdot p_3)

$$\Rightarrow \frac{8q^4e^4}{q^4} \left[ -\frac{q^2}{2} \{ (p_1 \cdot p_2) - (p_2 \cdot p_3) \} - M^2 \right] + 2(p_1 \cdot p_2)(p_2 \cdot p_3)$$

$p_1 \cdot p_2 = ME \quad p_2 \cdot p_3 = ME'$

$$\Rightarrow \frac{8q^4e^4}{q^4} \left[ -\frac{q^2}{2} \{ ME - ME' - M^2 \} + 2M^2EE' \right]$$

$$= \frac{8q^4e^4}{q^4} \left[ 2M^2EE'\cos^2\theta/2 - \frac{q^2}{2M} \{ M^2(E-E') \} \right]$$

$q = p_1 - p_3 = p_4 - p_2 \Rightarrow p_4^2 = (q + p_2)^2 = q^2 + 2p_2 \cdot q + p_2^2$

$$\Rightarrow q^2 = -2p_2 \cdot q = -2M(E-E')$$

$$\Rightarrow E - E' = -q^2/2M = \frac{4EE'/2M}{\sin^2\theta/2}$$

$$\Rightarrow \langle |M|^2 \rangle = \frac{8q^4e^4}{q^4} 2M^2EE' \left[ \cos^2\theta/2 - \frac{q^2}{2M^2} \sin^2\theta/2 \right]$$
Lecture XI

For fixed target scattering with recoil:

\[
\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi M}\right)^2 \frac{E'^2}{E^2} \langle 1M^2 \rangle \\
= \frac{\alpha^2}{4E^2\sin^4\theta/2} \frac{E'}{E} \left[ \cos^2\theta - \frac{g^2}{2} \frac{\sin^2\theta}{2M^2} \right]
\]

The first term is the Mott term.
The second term is a new term due to the target magnetic moment. Let us see why?

Suppose the electron had charge but no spin.
Then \((\partial \mu \partial + m^2)\phi = -V\phi\) where \(V\) contains \(A\mu\)

\[
V \propto A^\mu \cdot j_{\mu}^{fi} = ie (\phi^* \partial \mu \phi_i - \partial \mu \phi^* \phi_i) A^\mu
\]

\[
j_{\mu}^{fi} \propto (P_i + P_f) e^{i (P_f - P_i) \cdot x}
\]

For spin \(1/2\), \(j_{\mu}^{fi} \propto \bar{u}_f \gamma^\mu U_i \frac{1}{2m} \bar{u}_f [(P_i + P_f)^\mu + i \sigma^\mu_\nu (P_i - P_f)_\nu]u_i\)

where \(\sigma^\mu_\nu = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)\). The nonrelativistic limit of \(\frac{\sigma^\mu_\nu (P_i - P_f)_\nu}{2} \to \frac{\sigma \cdot B}{2}\).

Thus: magnetic moment = \(-e/2m\)

\(\Rightarrow\) predict for magnetic moment of a fundamental fermion.
\[ \Gamma_{\mu} = F_1(q^2) \delta^{\mu}_\nu + \frac{K}{2M} F_2(q^2) i \sigma^{\mu\nu} q_{\nu} \]

\( F_1 \) and \( F_2 \) are functions of \( q^2 \), the only independent Lorentz scalar available at the proton vertex.

As \( q^2 \to 0 \), proton becomes a Dirac particle

\[ \Rightarrow F_1(0) = F_2(0) = 1 \]

Proton magnetic moment \( = 1 + K/2M \Rightarrow K = 1.99 \)

For the neutron, \( F_1(0) = 0 \) \& \( F_2(0) = -1.91 \)

\[ \frac{d\sigma}{d\Omega} \propto \cos^2\theta/2 \left[ \frac{F_1^2 - K^2 q^2}{4M^2} \right] + \sin^2\theta/2 \left[ -\frac{K^2}{2} \left( F_1 + K F_2 \right)^2 \right] \]

\( F_1 \cdot F_2 \) interference terms are hard to measure.

Instead define \( G_E = F_1 + K F_2 \) \& \( G_M = F_1 + K F_2 \) \( \tau = q^2/4M^2 \)

\[ \frac{d\sigma}{d\Omega} \propto \cos^2\theta/2 \left[ \frac{G_E^2 + CG_M^2}{1 + \tau} \right] + \sin^2\theta/2 \left[ 2 CG_M^2 \right] \]

One can measure \( G_E \) \& \( G_M \) by varying \( q^2 \) \& \( \theta \).
Lecture XIV

\[ \mu^- \]

\[ \mu^+ \]

\[ e^+ e^- \rightarrow \mu^+ \mu^- \]

\[ M = (2\pi)^4 \int \left[ \bar{u}^s(p_3)(ig_\gamma^\mu)u^s(p_4) \right] \frac{i \gamma^\nu}{q^2} \left[ \bar{u}^s(p_2)(ig_\gamma^\nu)u^s(p_1) \right] \]

\[ \delta(q + p_1 + p_2 - q) \delta(q - p_3 - p_4) \]

\[ = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{i \alpha^2}{(p_1 + p_2)^2} \left[ \bar{u}(3) \gamma^\mu \gamma(4) \right] \left[ \bar{u}(2) \gamma^\nu u(1) \right] \]

Must evaluate \( \text{Tr}[\gamma^\mu (p_4 - M) \gamma^\nu (p_3 + M)] \text{Tr}[\gamma^\nu (p_1 + m) \gamma^\mu (p_2 - m)] \)

For \( E \gg M : \langle 1M^2 \rangle = 9e^4 \{ 1 + \cos^2 \theta \} \)

\[ \frac{d\sigma}{dS_2} \left( \text{com} \right) = \frac{\alpha^2}{16E^2} (1 + \cos^2 \theta) \text{ with} \]

\[ p_1 = (E, \overrightarrow{p}) , \quad p_2 = (E, -\overrightarrow{p}) , \quad p_3 = (E, \overrightarrow{p'}) , \quad p_4 = (E, -\overrightarrow{p'}) \]

\[ \mu^+ \overrightarrow{q} \quad \forall e^+ e^- \rightarrow \bar{q} \overline{q} \]

\[ e^+ \quad e^- \overrightarrow{p} \quad \Rightarrow \quad ig_\gamma^\mu = i \gamma^\mu g_\gamma \]

Must sum over all species of quarks possible

\[ R = \frac{\sigma(e^+ e^- \rightarrow \bar{q} \overline{q})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)} \]
If total beam energy is less than 3 GeV, then there are 3 kinds of quarks, \(u, d, s\).
\[
\epsilon \delta i^2 = \left(\frac{2}{3}\right)^2 + 2 \left(\frac{1}{3}\right)^2 = \frac{2}{3}.
\]
But observed \(R\) is 2.11.

Direct evidence that there are 3 colors!

Above \(E_{beam} = 3.05\) GeV, \(R = 10/3 \Rightarrow \) new charmed quark of charge \(2/3\).

Above \(E_{beam} = 5\) GeV, \(R = 11/3 \Rightarrow \) b quark.
\(R = 11/3\) until \(\sim 70\) GeV \(\Rightarrow\) no new light quarks.

\[
\frac{d\sigma}{d\Omega} = \frac{x^2\left(1 + \cos^2\theta\right)}{16E^2}.
\]

Simple explanation for angular distribution.

Consider the operators:
\[
P_L = \frac{1}{2}\left(1 - \gamma^5\right), \quad P_R = \frac{1}{2}\left(1 + \gamma^5\right)
\]

\[
P_L + P_R = I, \quad P_L P_R = 0, \quad P_L P_L = P_L, \quad P_R P_R = P_R.
\]

\[
\frac{1}{4}(1 - \gamma^5)^2 = \frac{1}{4}(1 + 2\gamma^5 + \gamma^5\gamma^5) = \frac{1}{2}(1 - \gamma^5)
\]

Recall: if \(|n\rangle = \langle n|\), define \(|i\rangle < i | = P_i^n\).
Then \(\sum P_i = I\), \(P_i P_j = \delta_{ij} P_j\).

\(\Rightarrow P_L \& P_R\) form a complete set of projection operators.
e^+e^- \rightarrow \mu^+\mu^- \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2} (1 + \cos^2\theta) \\
(for \ E \gg m, M) \\

-Simple explanation for angular distribution \\

Consider the operators 

\[ P_L = \frac{1}{2}(1 - \gamma^5) \quad P_R = \frac{1}{2}(1 + \gamma^5) \]

\[ P_L + P_R = I \quad P_L P_R = 0 \quad P_L P_L = P_L \quad P_R P_R = P_R \]

\[ P_L P_L = \frac{1}{4}(1 - \gamma^5)^2 = \frac{1}{4}(1 - 2\gamma^5 + \gamma^5^2) = \frac{1}{2}(1 - \gamma^5) = P_L \]

\[ P_R P_R = (1 - \gamma^5)(1 + \gamma^5) = 1 - (\gamma^5)^2 = 0 \]

\[ P_L P_L = \frac{1}{4}(1 - \gamma^5)^2 = \frac{1}{4}(1 - 2\gamma^5 + \gamma^5^2) = \frac{1}{2}(1 - \gamma^5) = P_L \]

Recall: if \[ |V\rangle = \sum_i m_i \langle n|V\rangle \], define \[ |i\rangle\langle i| = P_i \]
with \[ \sum_i P_i = I \quad \text{j, } P_i P_j = \delta_{ij} P_j \]

\[ \Rightarrow P_L \text{ & } P_R \text{ form a complete set of projection operators} \]

What do they project? 

\[ \gamma^5 U(p) = \begin{bmatrix} \frac{p \cdot \sigma}{E+m} & 0 \\ 0 & \frac{p \cdot \sigma}{E-m} \end{bmatrix} U \]

Consider the \( m = 0 \) case:

\[ \gamma^5 U = (p \cdot \vec{E}) U = \pm U \Rightarrow U \text{ is an eigenfunction} \]

With eigenvalues \[ \pm 1 = \lambda \]

\[ \text{If } (p \cdot \vec{E}) U = +U \Rightarrow \frac{1}{2} (1 - \gamma^5) U = 0 \]

\[ \text{If } (p \cdot \vec{E}) U = -U \Rightarrow \frac{1}{2} (1 - \gamma^5) U = U \]

\[ \Rightarrow P_L = \frac{1}{2}(1 - \gamma^5) \text{ project } \lambda = -1 \text{ state & } m = 0 \text{ fermion, } \]

\[ \lambda \text{ is the helicity or handedness} \]
LECTURE XV

$P_L$ & $P_R$ are projection operators or chirality

Chirality $\leftrightarrow$ helicity iff $m=0$

$\mathcal{H}$ commutes with $H$ i.e. conserved $\mathcal{H}$ but

NOT LORENTZ INVARIANT

- $P_L$ & $P_R$ are Lorentz invariant but NOT CONSERVED
- $U_L$ & $U_R$ do not satisfy the Dirac Eqs.

A freely propagating $U_R$ will develop an $U_L$ component

(Unless $m=0$)

$U = U_L + U_R$
$\bar{U} = \bar{U}_L + \bar{U}_R$  ($U_L = P_L U, \text{etc.}$)

$\bar{U}_L = U^+_L \gamma^0 = U^+_L \left( 1 - \gamma^5 \right) \gamma^0 = \bar{U}_L \left( 1 + \gamma^5 \right)$

($\gamma^5 = i \gamma^0 \gamma^5, \quad \gamma^5 \gamma^0 = -\gamma^0 \gamma^5$)

$\bar{U}_L \gamma^\mu U_R = \frac{1}{4} \bar{U}_L \left( 1 + \gamma^5 \right) \gamma^\mu \left( 1 + \gamma^5 \right) u = \frac{1}{4} \bar{U}_L \gamma^\mu \left( 1 + \gamma^5 \right) \left( 1 + \gamma^5 \right) u$

$\Rightarrow \bar{U}_L \gamma^\mu u = \left( \bar{U}_L + \bar{U}_R \right) \gamma^\mu \left( U_L + U_R \right) = \bar{U}_L \gamma^\mu U_L + \bar{U}_R \gamma^\mu U_R$

$\Rightarrow$ Helicity is conserved at each vertex.

\[ e^- e^+ \rightarrow \mu^+ \mu^- \]
\[ e^- e^+ \rightarrow \mu^- \mu^- \]
Consider \( e^+e^- \rightarrow \mu^+\mu^- \)

If \( \gamma^* \) has spin 1 along a certain axis, what is the probability of finding \( \pm 1 \) along an axis at angle \( \theta \)?

Matrix elements for finite rotations.

The generator of rotations by angle \( \theta \) about \( \gamma \)-axis is \( \exp \left( i \Theta J_y / \hbar \right) \)

Define \( d^j_{mm'} = \langle j m' | \exp \left( i \Theta J_y / \hbar \right) | j m \rangle \)

For \( j = \frac{1}{2} \), \( d^j_{mm'} \)

\[
\begin{pmatrix}
\cos \Theta/2 & \sin \Theta/2 \\
\sin \Theta/2 & \cos \Theta/2
\end{pmatrix} \begin{pmatrix} +1/2 \\ -1/2 \end{pmatrix}
\]

For \( j = 1 \),

4 amplitudes:

\[ (+1 \rightarrow +1, \; +1 \rightarrow -1, \; -1 \rightarrow +1, \; -1 \rightarrow -1) \]

\[
\sum d^2(\theta) = 2 \times \frac{1}{4} (1 + \cos \theta)^2 + 2 \times \frac{1}{4} (1 - \cos \theta)^2
\]

\[
= 1 + \cos^2 \theta
\]
LECTURE XV

**ELECTROWEAK INTERACTIONS**: We will take a novel approach: We will first introduce the electroweak Lagrangian via local gauge invariance and then study quantitative aspects of weak interactions.

Weak interactions \( \Rightarrow \) "long" lifetime

\[ \pi^- \rightarrow \mu^- \bar{\nu}_\mu : \tau_\pi \approx 2 \times 10^{-8} \text{s} \]
\[ \mu^- \rightarrow e^- \nu_e \bar{\nu}_\mu : \tau_\mu \approx 2 \times 10^{-6} \text{s} \]

Fermi proposed a "current-current" Lagrangian to describe weak interactions.

\[ A_\rho = e (\bar{u}_p \sigma^\mu u_p \frac{-i}{q^2}) e (\bar{u}_e \gamma_\mu u_e) \]
\[ = -\frac{e^2}{q^2} (\gamma^\mu)_\rho (j_\mu) e \]

Consider \( n \rightarrow p + e^- + \nu_e \): 4-Fermi interaction.

\[ A_W = G (\bar{u}_n^\mu u_p) (\bar{u}_e \gamma_\mu u_e) \]
\[ J^\mu_+ \quad J^- \mu \]

This theory successfully dealt with nuclear \& muon decay.

\( J^\mu_+ , J^- \mu \) are 4-vectors \& \( A_W \) is a Lorentz scalar.
LECTURE XV

1956: Weak interactions violate parity
\[ A_W \text{ has pseudo-scalar terms.} \]

This can be accomplished by introducing an axial-vector current \( \Rightarrow \)
\[ A_W \sim C_V \overline{u}_e \gamma^\mu u_e + C_A \overline{u}_e \gamma^\mu 5 u_e \]
Change nomenclature \( \Rightarrow \overline{u}_e \equiv \overline{\nu}_e \)
\[ J^\mu = C_V \overline{\nu}_e \gamma^\mu e + C_A \overline{u}_e \gamma^\mu 5 e \]

Experiments showed that \( C_V/C_A = -1 \Rightarrow \) Maximal
\( \Rightarrow \) V-A universal interaction.

In 1950s, theorists began to see patterns that could lead to electroweak unification:

For example:
\[ e \overline{\nu}_e \rightarrow J^\mu = \overline{\nu}_e \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) e \]
\[ \frac{\gamma^\mu - \gamma^\mu \gamma^5}{2} = \frac{\gamma^\mu + \gamma^5 \gamma^\mu}{2} \Rightarrow \left( \frac{1 + \gamma^5}{2} \right) \gamma^\mu = \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) \]

Thus,
\[ \left( \frac{1-\gamma^5}{2} \right)^2 = \frac{1 - 2\gamma^5 + (\gamma^5)^2}{4} = \frac{1 - \gamma^5}{2} : (\gamma^5)^2 = 1 \]

Therefore:
\[ \left( \frac{1 + \gamma^5}{2} \right) \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) = \gamma^\mu \left( \frac{1 - \gamma^5}{2} \right)^2 = \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) \]
\( \langle j^\mu \rangle_e = \overline{\nu}_e \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) e = \overline{\nu}_e \left( \frac{1+\gamma^5}{2} \right) \gamma^\mu \left( \frac{1-\gamma^5}{2} \right) e \)

\[ u_L = \left( \frac{1-\gamma^5}{2} \right) u \quad u_R = \left( \frac{1+\gamma^5}{2} \right) u \]

\[ \overline{u}_L = u_L^+ \gamma^0 = u \left( \frac{1+\gamma^5}{2} \right) \]

\[ \Rightarrow \langle j^\mu \rangle_e = \overline{\nu}_L \gamma^\mu e_L \implies \text{CHARGED CURRENT CONSISTS ONLY OF LEFT-CHIRAL STATES} \]

\[ j^\mu = e \overline{\nu}_e \gamma^\mu e = \nu (e_L + e_R) \gamma^\mu (e_L + e_R) \]

But \( \overline{e}_L \gamma^\mu e_R = \overline{e}_L \gamma^\mu e_L = 0 \)

\[ \implies j^\mu = e \overline{\nu}_L \gamma^\mu e_L + e \overline{\nu}_R \gamma^\mu e_R \]

i.e. the electromagnetic current consists of left- and right-handed states of equal strength.

But the weak interaction current only couples to left-chiral states and does not couple at all to the right-chiral states \( \Rightarrow \text{Maximal parity-violation} \)

**NOTE:**
- \( e_L \neq e_R \) do not represent different particles
- \( u_L \neq u_R \) do not satisfy the Dirac Eqn.