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Completeness Theorem for Axiom System AS1+Q for CPL

1. Introduction

As mentioned earlier, in proving the major theorem, we proceed in small steps. First, we prove that AS1+Q is complete for Classical Predicate Logic (CPL), then we prove that it is complete for Classical Quantifier Logic (CQL). After that, we consider an axiom system for Classical First-Order Logic, and show that it is complete for CFOL.

In proving completeness of AS1+Q for CPL, we proceed in a manner fairly similar to, but not exactly like, our proof that AS1 is complete for CSL.

(1) We prove that every (deductively) consistent set can be extended to a maximal consistent set. The CPL-construction is slightly different from the SL-construction.

(2) We prove that the maximal consistent set constructed in the manner specified in Part 1 is verifiable (semantically consistent).

(3) We employ the two negation theorems – NT1 and NT2 – to prove that every CPL-valid argument is AS1+Q-valid [meaning that its conclusion can be derived from its premises using the rules of AS1].

2. The Construction of the Sequence áG_1, G_2, …ñ

Let $\mathbb{L}$ be the official language of CPL. Let $\langle \varepsilon_1, \varepsilon_2, \ldots \rangle$ be an enumeration of the <u>closed</u> formulas of $\mathbb{L}$. Let $\langle c_1, c_2, \ldots \rangle$ be an enumeration of the constants of $\mathbb{L}$. Suppose additionally that both enumerations are one-to-one, meaning that $\varepsilon_i = \varepsilon_j$ only if i=j, and $c_i = c_j$ only if i=j.

Suppose that Γ is a consistent set of closed formulas of $\mathbb{L}$; further suppose that Γ does not contain any constants (recall Simplification #3).

The construction of $\langle \Gamma_1, \Gamma_2, \ldots \rangle$ goes as follows.

(1) $\Gamma_1 \quad = \quad \Gamma$

(2) if $\Gamma_n \cup \{\varepsilon_n\} \nvdash$
then:
$\Gamma_{n+1} \quad = \quad \Gamma_n \cup \{\varepsilon_n\}$

if $\Gamma_n \cup \{\varepsilon_n\} \vdash$
then:
if ε_n is <u>not</u> a universal formula [i.e., $\varepsilon \neq \forall v\phi$, for any v, ϕ]
then:
$\Gamma_{n+1} \quad = \quad \Gamma_n \cup \{\sim\varepsilon_n\}$

if ε_n is a universal formula,
then:
$\varepsilon = \forall v\phi$ (for some v, ϕ), and
$\Gamma_{n+1} \quad = \quad \Gamma_n \cup \{\sim\forall v\phi\} \cup \{\sim\phi[c/v]\}$
where c is the* first constant (relative to $\langle c_1, c_2, \ldots \rangle$) not occurring in Γ_n.

*By hypothesis, Γ contains no constants. Given the definition of $\langle\Gamma_1, \Gamma_2, \ldots\rangle$, Γ_n has finitely-many formulas <u>not</u> in Γ, and each formula has finitely-many symbols, so at most finitely-many constants appear in Γ_n. It follows, by ST, that there is a first constant not in Γ_n. Thus, the description 'the first constant not in Γ_n' is referentially proper.

3. Informal Account and Justification of the Construction

As the reader can see, the construction of $\langle\Gamma_1, \Gamma_2, \ldots\rangle$ in Section 2 is very similar to the SL-construction. The only difference is the "extra" formula — $\sim\phi[c/v]$ — added in the third case.

The intuition is fairly straightforward. Everytime one adds a $\sim\forall$ formula to the expanding set, one adds a "corroborating" formula (also called a "witness"). This maneuver corresponds to the natural deduction rule $\sim\forall$O. Recall how this rule is formulated.

$$\frac{\sim\forall v\phi}{\sim\phi[c/v]}$$ where c does not appear earlier in the derivation [i.e., c is "new"]

So what we are doing in adding the extra formula is applying the rule $\sim\forall$O. Notice that the constant c substituted for v in ϕ is specified to be the first constant not appearing in Γ_n – in other words, c is "new".

One might naturally ask why this extra step is required. Why don't we just construct $\langle\Gamma_1, \Gamma_2, \ldots\rangle$ and Ω [$= \bigcup\{\Gamma_1, \Gamma_2, \ldots\}$] the same way we do in SL? Then the proof would exactly parallel the proof for SL. We show Γ is contained in Ω, and Ω is verifiable, so Γ is verifiable, then apply NT1 and NT2 to show completeness.

The answer is that the particular construction we choose makes showing that Ω is verifiable easier to accomplish. As it turns out, some maximal consistent sets are easier to show verifiable than others.

4. Every G_i in áG_1, G_2, …ñ is Consistent

Having constructed $\langle\Gamma_1, \Gamma_2, \ldots\rangle$, we next show every Γ_n is consistent, which is accomplished by (simple, weak) math induction.

(1)	~~SHOW~~: $\forall n[\Gamma_n \nvdash]$	MI [n⩾1]
	Base Case:	
(2)	~~SHOW~~: $\Gamma_1 \nvdash$	3,4,IL
(3)	$\Gamma_1 = \Gamma$	Def $\langle\Gamma_1, \Gamma_2, \ldots\rangle$
(4)	$\Gamma \nvdash$	by hypothesis
	Inductive Case:	
(5)	~~SHOW~~: $\forall n\{\Gamma_n \nvdash\ \rightarrow\ \Gamma_{n+1} \nvdash\}$	UCD
(6)	$\Gamma_n \nvdash$	As
(7)	~~SHOW~~: $\Gamma_{n+1} \nvdash$	SC

Given the definition of Γ_{n+1} in terms of Γ_n there are two cases to consider, the second of which divides into two cases.

(8)	c1: $\Gamma_n \cup \{\varepsilon_n\} \nvdash$	As
(9)	$\Gamma_{n+1} = \Gamma_n \cup \{\varepsilon_n\}$	8, Def $\langle\Gamma_1, \Gamma_2, \ldots\rangle$
(10)	$\Gamma_{n+1} \nvdash$	6,9,IL

(11)	| c2: $\Gamma_n \cup \{\varepsilon_n\} \vdash$	As
(12)	| c2.1: ε_n is not a universal	As
(13)	| | $\Gamma_{n+1} = \Gamma_n \cup \{\sim\varepsilon_n\}$	11, Def $\langle\Gamma_1, \Gamma_2, \ldots\rangle$
(14)	| | $\Gamma_n \cup \{\sim\varepsilon_n\} \nvdash$	11,earlier result about AS1♮

♮ AS1+Q contains the full deductive apparatus of AS1, so results about AS1 can simply be transferred to AS1+Q.

(15)	| | $\Gamma_{n+1} \nvdash$	13,14,IL
(16)	| c2.2: ε_n is a universal	As
(17)	| | $\varepsilon_n = \forall v\phi$	16,def(is a universal),$\exists$O
(18)	| | let c = the first constant not in Γ_n	♮see Section 2 for justification
(19)	| | $\Gamma_{n+1} = \Gamma_n \cup \{\sim\forall v\phi\} \cup \{\sim\phi[c/v]\}$	16,17,18,Def $\langle\Gamma_1, \Gamma_2, \ldots\rangle$

given 19, in order to show $\Gamma_{n+1} \nvdash$, the following suffices:

(20)	| | ~~SHOW~~: $\Gamma_n \cup \{\sim\forall v\phi\} \cup \{\sim\phi[c/v]\} \nvdash$	ID
(21)	| | | $\Gamma_n \cup \{\sim\forall v\phi\} \cup \{\sim\phi[c/v]\} \vdash$	As
(22)	| | | ~~SHOW~~: $\times$	6,29,SL
(23)	| | | | $\Gamma_n \vdash \sim\varepsilon_n$	11,NT1
(24)	| | | | $\Gamma_n \vdash \sim\forall v\phi$	17,23,IL
(25)	| | | | $\Gamma_n \cup \{\sim\forall v\phi\} \vdash \phi[c/v]$	20,NT1
(26)	| | | | $\Gamma_n \vdash \phi[c/v]$	24,25,GenTh($\vdash$)
(27)	| | | | $c \notin^* \Gamma_n$	18, description logic
(28)	| | | | $\Gamma_n \vdash \forall v\phi$	26,27,UDT
(29)	| | | | $\Gamma_n \vdash$	24,28,earlier result about AS1

5. The Construction of W, and Proof that it is Consistent

Next, given $\langle\Gamma_1, \Gamma_2, \ldots\rangle$, as defined above, we define Ω as follows.

$$\Omega \quad = \quad \bigcup\{\Gamma_1, \Gamma_2, \ldots\}$$

The proof that Ω is consistent duplicates the result for AS1, which we summarize here.

(1)	~~SHOW~~: $\Omega \nvdash$	DD
(2)	| $\Omega = \bigcup\{\Gamma_1, \Gamma_2, \ldots\}$	Def of Ω
(3)	| $\forall n\{\Gamma_i \subseteq \Gamma_{n+1}\}$	L03
(4)	| $\bigcup\{\Gamma_1, \Gamma_2, \ldots\} \nvdash$	3,L01,L02,QL
(5)	| $\Omega \nvdash$	2,4,IL

Subordinate Lemmas:

L01: $\forall n[\Gamma_n \nvdash]$

proven above

L02: $\forall n\{\Gamma_n \subseteq \Gamma_{n+1}\}$ & $\forall n[\Gamma_n \nvdash]$ $. \!\!\longrightarrow \bigcup\{\Gamma_1, \Gamma_2, \ldots\} \nvdash$

This result, proven earlier about AS1, transfers to AS1+Q.

L03: $\forall n\{\Gamma_n \subseteq \Gamma_{n+1}\}$

This result, proven earlier about AS1, transfers to AS1+Q.

6. Proof that W is Maximal Consistent w.r.t. Closed Formulas

Next, we prove that Ω is maximal consistent w.r.t. closed formulas, which may be defined as follows.

$$\mathrm{MC}[\Omega] \quad =_{\mathrm{df}} \quad \Omega \nvdash \;\&\; \forall\alpha\{\mathrm{closed}[\alpha] \longrightarrow \{\alpha \notin \Omega \longrightarrow. \; \Omega \cup \{\alpha\} \vdash\}\}$$

We have already shown the first conjunct – that Ω is consistent. The second conjunct is proven like the analogous result about AS1, which is reproduced here.

(1)	~~SHOW~~: $\forall\alpha\{\mathrm{closed}[\alpha] \longrightarrow \{\alpha \notin \Omega \longrightarrow. \; \Omega \cup \{\alpha\} \vdash\}\}$	UCCD
(2)	$\vert$ closed[α]	As
(3)	$\vert$ $\alpha \notin \Omega$	As
(4)	$\vert$ ~~SHOW~~: $\Omega \cup \{\alpha\} \vdash$	DD
(5)	$\vert$ $\sim\alpha \in \Omega$	3,L04,QL
(6)	$\vert$ $\Omega \vdash \sim\alpha$	5,GenTh($\vdash$)
(7)	$\vert$ $\Omega \cup \{\alpha\} \vdash \sim\alpha$	6,GenTh($\vdash$)
(8)	$\vert$ $\Omega \cup \{\alpha\} \vdash \alpha$	GenTh($\vdash$)
(9)	$\vert$ $\Omega \cup \{\alpha\} \vdash$	7,8, result about AS1

L04: " a{a Î W xor ~a Î W}

(0)	$\forall\alpha\{\text{closed}[\alpha] \longrightarrow \{\alpha\in\Omega \text{ xor } {\sim}\alpha\in\Omega\}\}$	UCD
(1)	closed[α]	As
(2)	~~SHOW~~: $\alpha\in\Omega$ xor ${\sim}\alpha\in\Omega$	3,16,Def(xor)
(3)	~~SHOW~~: $\alpha\in\Omega$ or ${\sim}\alpha\in\Omega$	5,10,SL
(4)	$\exists n[\alpha = \varepsilon_n]$	1, Def $\langle\varepsilon_1, \varepsilon_2, \ldots\rangle$
(5)	$\alpha = \varepsilon_n$	5,∃O
(6)	~~SHOW~~: $\Gamma_n\cup\{\varepsilon_n\}\nvdash \;\longrightarrow\; \alpha \in \Omega$	CD
(7)	$\Gamma_n\cup\{\varepsilon_n\}\vdash$	As
(8)	~~SHOW~~: $\alpha \in \Omega$	5,10,IL
(9)	$\Gamma_{n+1} = \Gamma_n\cup\{\varepsilon_n\}$	7, Def $\langle\Gamma_1, \Gamma_2, \ldots\rangle$
(10)	$\varepsilon_n \in \Omega$	9,Def Ω,ST
(11)	~~SHOW~~: $\Gamma_n\cup\{\varepsilon_n\}\vdash \;\longrightarrow\; {\sim}\alpha \in \Omega$	CD
(12)	$\Gamma_n\cup\{\varepsilon_n\}\vdash$	As
(13)	~~SHOW~~: ${\sim}\alpha \in \Omega$	5,15,IL
(14)	$\Gamma_{n+1} = \Gamma_n\cup\{{\sim}\varepsilon_n\}$	10, Def $\langle\Gamma_1, \Gamma_2, \ldots\rangle$
(15)	${\sim}\varepsilon_n \in \Omega$	14,Def Ω,ST
(16)	~~SHOW~~: ${\sim}\{\alpha\in\Omega \;\&\; {\sim}\alpha\in\Omega\}$	ID
(17)	$\alpha\in\Omega \;\&\; {\sim}\alpha\in\Omega$	As
(18)	~~SHOW~~: $\times$	21,22
(19)	$\Omega\vdash\alpha$	17a, since Ω is MC, it is deductively closed
(20)	$\Omega\vdash{\sim}\alpha$	17b, since Ω is MC, it is deductively closed
(21)	$\Omega\vdash$	19,20,result about AS1
(22)	$\Omega\nvdash$	shown earlier

7. The Substitution/Quantification Lemma About W

In the Chapter on Soundness, we proved the Substitution/Quantification Lemma, which is about admissible valuations in CFOL. In the present section, we prove an analogous lemma about the constructed set Ω.

L05: " v𝔽 Î W « " c{𝔽[c/v] Î W}

[$\longrightarrow$]

Suppose $\forall v\mathbb{F} \in \Omega$, and suppose c is a closed singular term. First, $\forall v\mathbb{F}\rightarrow\mathbb{F}[c/v]$ is an axiom of AS1+Q, so we have $\forall v\mathbb{F}\vdash\mathbb{F}[c/v]$, so we have $\Omega\vdash\mathbb{F}[c/v]$. Next, by an earlier theorem, Ω is maximal consistent, which by yet another earlier theorem entails that it is closed under deductive consequence. It follows that $\mathbb{F}[c/v] \in \Omega$.

(1)	~~SHOW~~: $\leftarrow$	CD
(2)	$\forall c\{\mathbb{F}[c/v] \in \Omega\}$	As
(3)	~~SHOW~~: $\forall v\mathbb{F} \in \Omega$	ID
(4)	$\forall v\mathbb{F} \notin \Omega$	As
(5)	~~SHOW~~: $\times$	DD
(6)	$\exists n[\forall v\mathbb{F} = \varepsilon_n]$	Def $\langle \varepsilon_1, \varepsilon_2, \ldots\rangle$
(7)	$\forall v\mathbb{F} = \varepsilon_n$	6,∃O
(8)	~~SHOW~~: $\Gamma_n \cup \{\forall v\mathbb{F}\} \vdash$	ID
(9)	$\Gamma_n \cup \{\forall v\mathbb{F}\} \nvdash$	As
(10)	~~SHOW~~: $\times$	4,12,SL
(11)	$\Gamma_n \cup \{\varepsilon_n\} \nvdash$	7,9,IL
(12)	$\Gamma_{n+1} = \Gamma_n \cup \{\varepsilon_n\}$	11,Def $\langle \Gamma_1, \Gamma_2, \ldots\rangle$
(13)	$\Gamma_{n+1} = \Gamma_n \cup \{\forall v\mathbb{F}\}$	7,12,IL
(14)	$\forall v\mathbb{F} \in \Omega$	13, Def Ω, ST
(15)	$\Gamma_{n+1} = \Gamma_n \cup \{\sim\forall v\mathbb{F}\} \cup \{\sim\mathbb{F}[c_0/v]\}$ (some c_0)	7,8,Def $\langle \Gamma_1, \Gamma_2, \ldots\rangle$
(16)	$\sim\mathbb{F}[c_0/v] \in \Omega$	15, Def Ω
(17)	$\mathbb{F}[c_0/v] \in \Omega$	2,QL
(18)	$\times$	16,17,L04

8. Showing that W is Verifiable

Our next step is to show that our constructed set Ω is verifiable, which is to say there is a CFOL-admissible valuation that verifies every formula in Ω, and hence every formula in our original set Γ. We proceed as follows.

First, we define the underlying domain of discourse (universe) U as follows.

$$U = C \cup N$$

where C = the set of constants of $\mathbb{L}$
and N = the set of proper nouns of $\mathbb{L}$

In other words,

U = the set of all the closed singular terms of $\mathbb{L}$

Next, define an interpretation function I, and partial assignment function a, as follows.

$a(c)$	=	c	for every constant c
I(n)	=	n	for every proper noun n
$I(\mathbb{P})$	=	$\{\langle u_1,\ldots,u_k\rangle : \mathbb{P}\langle u_1\ldots u_k\rangle \in \Omega\}$	for every k-place predicate $\mathbb{P}$, for every k

Now, one can routinely show that there is at least one admissible valuation that extends I and a [exercise]. Let ω be one such valuation. In other words:

ω is CFOL-admissible

ω(c)	=	$a(c)$	for every constant c
ω(n)	=	I(n)	for every proper noun n
$\omega(\mathbb{P})$	=	$I(\mathbb{P})$	for every k-place predicate $\mathbb{P}$, for every k

What we wish to show is that ω verifies every formula in Ω. We in fact prove that ω verifies a formula if and only if it is in Ω.

9. Aside on Induction on Complexity

In doing the proof that ω verifies every formula in Ω, we will employing a deductive technique called *induction on formula-complexity*.

First, the general notion of syntactic complexity may be defined as follows, where it is assumed that ε is a grammatically well-formed expression of $\mathbb{L}$.

$$\text{complexity}(\varepsilon) =_{df} \text{ the number of functor-occurrences in } \varepsilon$$

Note, in this connection, that we count quantifiers as two-place functors: $\ulcorner\forall v\phi\urcorner = \forall\langle v,\phi\rangle$.

Subordinate to this is the notion of *term-complexity*, which is defined just for singular terms, and *formula-complexity*, which is defined just for formulas, as follows.

$$\text{term-complexity}(\tau) =_{df} \text{ the number of function sign occurrences in } \tau$$

$$\text{formula-complexity}(\phi) =_{df} \text{ the number of logical operator occurrences in } \phi$$

Note, in this connection, that we count quantifier expressions (e.g., '$\forall$x', '$\forall$y') as one-place logical operators.

The following are examples.

Expression	**Syntactic Complexity**	**Term-Complexity**	**Formula-Complexity**
a	0	0	–
f(a)	1	1	–
f(f(a))	2	2	–
s(f(a), f(b))	3	3	–
P[a]	1	–	0
P[f(a)]	2	–	0
P[f(f(a))]	3	–	0
$Fx \rightarrow Gx$	3	–	1
$\forall x(Fx \rightarrow Gx)$	4	–	2
$\sim\forall x(Fx \rightarrow Gx)$	5	–	3

Now, every grammatical expression of a formal language $\mathbb{L}$ has a complexity, so in order to show:

$$\forall\varepsilon\mathbb{P}[\varepsilon]$$

[where ε is understood to be an expression of $\mathbb{L}$] we need merely show:

$$\forall n\forall\varepsilon\{\text{complexity}(\varepsilon) = n \longrightarrow \mathbb{P}[\varepsilon]\}$$

The latter can be shown either by weak induction or strong induction. Let us concentrate on strong induction, which is often more useful. Then the inductive case is officially written as follows.

Assume: $\forall k\{k<n \longrightarrow \forall\varepsilon\{\text{complexity}(\varepsilon) = k \longrightarrow \mathbb{P}[\varepsilon]\}\}$

Show: $\forall\varepsilon\{\text{complexity}(\varepsilon) = n \;\longrightarrow\; \mathbb{P}[\varepsilon]\}$

The inductive case can be simplified if we introduce the notion *simpler than*, defined as follows.

$$\varepsilon_1 < \varepsilon_2 \quad =_{df} \quad \text{complexity}(\varepsilon_1) < \text{complexity}(\varepsilon_2)$$

Specifically, appealing to some fairly obvious principles of arithmetic, we can instead write the inductive case as follows.

Assume: $\forall\varepsilon'\{\varepsilon' < \varepsilon \;\longrightarrow\; \mathbb{P}[\varepsilon']\}$

Show: $\mathbb{P}[\varepsilon]$

The general case may not be useful. It is often easier to do proofs about singular terms, or about formulas. That is where the notions of term-complexity and formula complexity arise. For example, in dealing with CPL, we can do many proofs using induction on formula-complexity. In that case, we can write the inductive case as follows, where ‘ϕ’ ranges over formulas of $\mathbb{L}$.

Assume: $\forall\phi\{\phi < \phi_0 \;\longrightarrow\; \mathbb{P}[\phi]\}$

Show: $\mathbb{P}[\phi_0]$

10. Proof that **w** Verifies the Formulas in **W**

(a1)	ω is CFOL-admissible	As
(a2)	$\omega(c) = a(c)$ for every constant c	As
(a3)	$\omega(n) = I(n)$ for every proper noun n	As
(a4)	$\omega(\mathbb{P}) = I(\mathbb{P})$ for every k-place predicate $\mathbb{P}$, for every k	As
(a5)	$a(c) = c$ for every constant c	As
(a6)	$I(n) = n$ for every proper noun n	As
(a7)	$I(\mathbb{P}) = \{\langle u_1,\ldots,u_k\rangle : \mathbb{P}\langle u_1\ldots u_k\rangle \in \Omega\}$ for every k-place predicate $\mathbb{P}$, for every k	As
	$I(\mathbb{P})\langle\tau\rangle \;\longleftrightarrow\; \mathbb{P}\langle\tau\rangle \in \Omega$	alternate description

(1)	~~SHOW~~: $\forall\phi\{\omega(\phi) = T \;\longleftrightarrow\; \phi \in \Omega\}$	induction on complexity of ϕ
(2)	$\forall\phi'\{\phi' < \phi \;\longrightarrow\; \{\omega(\phi') = T \;\longleftrightarrow\; \phi' \in \Omega\}\}$	IH
(3)	~~SHOW~~: $\omega(\phi) = T \;\longleftrightarrow\; \phi \in \Omega$	separation of cases
(4)	ϕ is atomic or molecular (i.e., not atomic)	SL
(5)	c1: ϕ is atomic	As
(6)	see below	
(7)	c2: ϕ is molecular	As
(8)	ϕ is a universal, or a negation, or a conditional	6, def CPL
(9)	c2.1: ϕ is a universal	As
(10)	see below	
(11)	c2.2: ϕ is a negation	As
(12)	see below	
(13)	c2.3: ϕ is a conditional	As
(14)	see below	

Subordinate Cases:

Case 1:

(1)	ϕ is atomic	As
(2)	~~SHOW~~: $\omega(\phi) = T \leftrightarrow \phi \in \Omega$	3,4,IL
(3)	$\phi = \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle$ (some $\mathbb{P}, \tau_1, \ldots, \tau_k$)	Def(atomic), ∃O
(4)	~~SHOW~~: $\omega(\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle) = T \leftrightarrow \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle \in \Omega$	5,8,9,SL
(5)	$\omega(\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle) = T \leftrightarrow \omega(\mathbb{P})\langle\omega(\tau_1), \ldots, \omega(\tau_k)\rangle = T$	a1, def(val)
(6)	$\omega(\mathbb{P}) = I(\mathbb{P})$	a4
(7)	$\forall i \leqslant k$: $\omega(\tau_i) = I(\tau_i) = \tau_i$	a2,a3
(8)	$\omega(\mathbb{P})\langle\omega(\tau_1), \ldots, \omega(\tau_k)\rangle = T \leftrightarrow I(\mathbb{P})\langle\tau_1, \ldots, \tau_k\rangle = T$	6,7,IL
(9)	$I(\mathbb{P})\langle\tau_1, \ldots, \tau_k\rangle = T \leftrightarrow \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle \in \Omega$	a7

Case 2.1:

(1)	ϕ is a universal	As
(2)	~~SHOW~~: $\omega(\phi) = T \leftrightarrow \phi \in \Omega$	3,4,IL
(3)	$\phi = \forall x\mathbb{F}$ (some x, $\mathbb{F}$)	def(universal), ∃O
(4)	~~SHOW~~: $\omega(\forall x\mathbb{F}) = T \leftrightarrow \forall x\mathbb{F} \in \Omega$	6,8,9,SL
(5)	every element of U has a name according to ω	
(6)	$\omega(\forall x\mathbb{F}) = T \leftrightarrow \forall c\{\omega(\mathbb{F}[c/v]) = T\}$	5,SubQ
(7)	$\forall c\{\mathbb{F}[c/v] < \forall x\mathbb{F}\}$	♮
(8)	$\forall c\{\omega(\mathbb{F}[c/v]) = T \leftrightarrow \mathbb{F}[c/v] \in \Omega\}$	7,IH,QL
(9)	$\forall x\mathbb{F} \in \Omega \leftrightarrow \forall c\{\mathbb{F}[c/v] \in \Omega\}$	SubQ for Ω

Case 2.2:

(1)	ϕ is a negation	As
(2)	~~SHOW~~: $\omega(\phi) = T \leftrightarrow \phi \in \Omega$	3,4,IL
(3)	$\phi = {\sim}\beta$ (some β)	def(negation), ∃O
(4)	~~SHOW~~: $\omega({\sim}\beta) = T \leftrightarrow {\sim}\beta \in \Omega$	5,8,9,SL
(5)	$\omega({\sim}\beta) = T \leftrightarrow \omega(\beta) = F$	def(val)
(6)	β is simpler than ${\sim}\beta$	obvious
(7)	$\omega(\beta) = T \leftrightarrow \beta \in \Omega$	6,IH
(8)	$\omega(\beta) = F \leftrightarrow \beta \notin \Omega$	7,def(val),SL
(9)	$\beta \notin \Omega \leftrightarrow {\sim}\beta \in \Omega$	L04

Case 2.3:

(1)	ϕ is a conditional	As
(2)	~~SHOW~~: $\omega(\phi) = T \leftrightarrow \phi \in \Omega$	3,4,IL
(3)	$\phi = \alpha\rightarrow\beta$ (some α, β)	def(conditional), ∃O
(4)	~~SHOW~~: $\omega(\alpha\rightarrow\beta) = T \leftrightarrow \alpha\rightarrow\beta \in \Omega$	5,7,9,10,SL
(5)	$\omega(\alpha\rightarrow\beta) = T \leftrightarrow.\ \omega(\alpha) = F$ or $\omega(\beta) = T$	def(val)
(6)	α is simpler than $\alpha\rightarrow\beta$	obvious
(7)	$\omega(\alpha) = F \leftrightarrow \alpha \notin \Omega$	6,IH
(8)	β is simpler than $\alpha\rightarrow\beta$	obvious
(9)	$\omega(\beta) = T \leftrightarrow \beta \in \Omega$	8,IH
(10)	$\alpha\rightarrow\beta \in \Omega \leftrightarrow.\ \alpha \notin \Omega$ or $\beta \in \Omega$	earlier result about AS1

2. Appendix – Completeness for CQL

1. Derivation System for CQL for Closed Formulas

Rules of CSL

(R1) / $\alpha\rightarrow(\beta\rightarrow\alpha)$
(R2) / $[\alpha\rightarrow(\beta\rightarrow\gamma)]\rightarrow[(\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\gamma)]$
(R3) / $(\sim\alpha\rightarrow\sim\beta)\rightarrow(\beta\rightarrow\alpha)$
(R4) α, $\alpha\rightarrow\beta$ / β

New Rules

(R5) / $\forall v\mathbb{F} \rightarrow \mathbb{F}[t/v]$ where t is closed
(R6) / $\mathbb{F} \rightarrow \forall v\mathbb{F}$ where v is not free in $\mathbb{F}$
(R7) / $\forall v(\mathbb{F}\rightarrow\mathbb{G})\rightarrow(\forall v\mathbb{F}\rightarrow\forall v\mathbb{G})$
(R8) $\pi(\mathbb{F}[c/v])$ / $\forall v\mathbb{F}$ where $\pi(\mathbb{F}[c/v])$ are prior lines that prove $\mathbb{F}[c/v]$

2. Completeness for Closed Argument Forms

Assume $\mathbb{L}$ is a CQL without constants; assume $\mathbb{L}+$ is $\mathbb{L}$ augmented by a denumerable set C of constants, one of which is c_0. Assume Γ is a consistent set of closed formulas of $\mathbb{L}$; assume $\langle\sigma_1, \sigma_2, \ldots\rangle$ is an enumeration of the closed formulas $\mathbb{L}+$.

[[Constants play the same role here that they do in derivations – they are *ad hoc* names of objects in the domain; they are like variables, except that they are not quantified.]]

Construct $\langle\Gamma_1, \Gamma_2, \ldots\rangle$, $\langle U_1, U_2, \ldots\rangle$, and $\langle D_1, D_2, \ldots\rangle$ inductively as follows.

Base Cases:

a) $U_1 = P \cup \{c_0\}$

b) $\Gamma_1 = \Gamma$

c) $D_1 = \varnothing$

Inductive Cases:

c1: if at least one constant in σ_n is already discarded $[\exists c\{c \in \sigma_n \ \&\ c \in D_n\}]$, then:

a) $D_{n+1} = D_n$

b) $U_{n+1} = U_n$

c) $\Gamma_{n+1} = \Gamma_n$

c2: otherwise:

c2.1: if every constant in σ_n is already used $[\forall c\{c \in \sigma_n \rightarrow c \in U_n\}]$, then:

a) $D_{n+1} = D_n$

b) $U_{n+1} = U_n$

c) c2.1.1: if $\Gamma_n \cup \{\sigma_n\} \nvdash$, then: $\Gamma_{n+1} = \Gamma_n \cup \{\sigma_n\}$

c2.1.2: otherwise: $\Gamma_{n+1} = \Gamma_n \cup \{{\sim}\sigma_n\}$

c2.2: if any constant c in σ_n is not already used [i.e., $c \in \sigma_n \ \&\ c \notin U_n$], then:

c2.2.1: if $\exists \alpha \in \Gamma_n\{\alpha = {\sim}\forall v\phi \ \ \&\ \ \sigma_n = {\sim}\phi[c/v]\}$, then:

a) $D_{n+1} = D_n$

b) $U_{n+1} = U_n \cup \{c\}$

c) $\Gamma_{n+1} = \Gamma_n \cup \{\sigma_n\}$

c2.2.2: otherwise:

a) $D_{n+1} = D_n \cup \{c\}$

b) $U_{n+1} = U_n$

c) $\Gamma_{n+1} = \Gamma_n$

$$\Omega = \bigcup\{\Gamma_1, \Gamma_2, \ldots\}$$

$$U = \bigcup\{U_1, U_2, \ldots\}$$

$\upsilon(\alpha)$	=	T	if	$\alpha^* \in \Omega$
$\upsilon(\alpha)$	=	F	if	$\alpha^* \notin \Omega$
$\upsilon(\tau)$	=	τ^*	if	τ is a term
$\upsilon(\acute{\omega})$	=	$\{\langle\langle u_1,\ldots,u_k\rangle, \acute{\omega}u_1\ldots u_k\rangle$: $u_1\ldots u_k \in \Omega\}$	if	$\acute{\omega}$ is a k-place function sign
$\upsilon(\mathbb{P})$	=	$\{\langle u_1,\ldots,u_k\rangle$: $\mathbb{P}u_1\ldots u_k \in \Omega\}$	if	$\mathbb{P}$ is a k-place predicate

ε^* $=_{df}$ the result of replacing every free variable in ε by c_0.

Show: $\forall n[\Gamma_n \nvdash]$. Then by the compactness of derivations, $\Omega \nvdash$.

Show: $\sim\exists\gamma\{\gamma\in\Gamma \;\&\; c\in\gamma\} \longrightarrow\!.\; \Gamma\cup\{\mathbb{F}[c/v]\}\vdash\alpha \longrightarrow \Gamma\cup\{\exists v\mathbb{F}\}\vdash\alpha$

Show: $\upsilon(\mathbb{P}\tau_1\ldots\tau_k) = [\upsilon(\mathbb{P})]\langle\upsilon(\tau_1),\ldots,\upsilon(\tau_k)\rangle$

Show: $\upsilon(\acute{\omega}\tau_1\ldots\tau_k) = [\upsilon(\acute{\omega})]\langle\upsilon(\tau_1),\ldots,\upsilon(\tau_k)\rangle$

Show: $\upsilon(\alpha\rightarrow\beta) = \upsilon(\alpha)\rightarrow\upsilon(\beta)$

Show: $\upsilon(\sim\alpha) = \sim\upsilon(\alpha)$

Show: $\upsilon(\forall x_k\mathbb{F}) = \min\{\upsilon'(\mathbb{F}): \upsilon' \approx_k \upsilon\}$

$\upsilon' \approx_k \upsilon$ $=_{df}$ $\forall i\{i\neq k \longrightarrow \upsilon'(x_i) = \upsilon(x_i)\}$

Practice Proof:

(1)	~~SHOW~~: $`\forall xFx' \in \Omega \leftrightarrow \forall u\{u\in U \rightarrow `Fu' \in \Omega\}$	$\leftrightarrow$D
(2)	~~SHOW~~: $\rightarrow$	CD
(3)	$`\forall xFx' \in \Omega$	As
(4)	~~SHOW~~: $\forall u\{u\in U \rightarrow `Fu' \in \Omega\}$	UCD
(5)	$c\in U$	As
(6)	~~SHOW~~: $`Fc' \in \Omega$	ID
(7)	$`Fc' \notin \Omega$	As
(8)	~~SHOW~~: ×	DD
(9)	$`{\sim}Fc' \in \Omega$	7,Lemma
(10)	×	3,9,Earlier Theorem
(11)	~~SHOW~~: $\leftarrow$	CD
(12)	$\forall c\{c\in U \rightarrow `Fc' \in \Omega\}$	As
(13)	~~SHOW~~: $`\forall xFx' \in \Omega$	ID
(14)	$`\forall xFx' \notin \Omega$	As
(15)	~~SHOW~~: ×	DD
(16)	$`{\sim}\forall xFx' \in \Omega$	13,Lemma
(17)	$\exists n\{`{\sim}\forall xFx' \in \Gamma_n\}$	16,Def Ω,ST
(18)	$`{\sim}\forall xFx' \in \Gamma_n$	17,$\exists$O
(19)	finite$[U_n]$	lemma
(20)	finite$[D_n]$	lemma
(21)	$\exists k\exists c\{k>n \;\&\; c\notin U_n\cup D_n \;\&\; `{\sim}Fc' = \sigma_k\}$	19,20,lemma
(22)	$k>n$	21,$\exists$&O
(23)	$c \notin U_n\cup D_n$	21,$\exists$&O
(24)	$`{\sim}Fc' = \sigma_k$	21,$\exists$&O
(25)	$`{\sim}Fc' \in \Gamma_{k+1}$	18,22,23,24,Def $\langle\Gamma_i\rangle$
(26)	$c \in U_{k+1}$	18,22,23,24,Def $\langle U_i\rangle$
(27)	$c \in U$	26, Def U
(28)	$`{\sim}Fc' \in \Omega$	25, Def Ω
(29)	$`Fc' \in \Omega$	12,28,QL
(30)	×	28,29,Lemma

Real Proof:

(1)	~~SHOW~~: $\forall v\mathbb{F} \in \Omega \leftrightarrow \forall u\{u\in U \rightarrow \mathbb{F}[u/v] \in \Omega\}$	$\leftrightarrow$D
(2)	~~SHOW~~: $\rightarrow$	CD
(3)	$\forall v\mathbb{F}$	As
(4)	~~SHOW~~: $\forall u\{u\in U \rightarrow \mathbb{F}[u/v] \in \Omega\}$	UCD
(5)	$c\in U$	As
(6)	~~SHOW~~: $\mathbb{F}[c/v] \in \Omega$	ID
(7)	$\mathbb{F}[c/v] \notin \Omega$	As
(8)	~~SHOW~~: $\times$	DD
(9)	$\sim\mathbb{F}[c/v] \in \Omega$	7,Lemma
(10)	$\times$	3,9,Earlier Theorem
(11)	~~SHOW~~: $\leftarrow$	CD
(12)	$\forall u\{u\in U \rightarrow \mathbb{F}[u/v] \in \Omega\}$	As
(13)	~~SHOW~~: $\forall v\mathbb{F} \in \Omega$	ID
(14)	$\forall v\mathbb{F} \notin \Omega$	As
(15)	~~SHOW~~: $\times$	DD
(16)	$\sim\forall v\mathbb{F} \in \Omega$	13,Lemma
(17)	$\exists n\{\sim\forall v\mathbb{F} \in \Gamma_n\}$	16,Def Ω,ST
(18)	$\sim\forall v\mathbb{F} \in \Gamma_n$	17,$\exists$O
(19)	finite$[U_n]$	lemma
(20)	finite$[D_n]$	lemma
(21)	$\exists k\exists c\{k>n \;\&\; c\notin U_n\cup D_n \;\&\; \sim\mathbb{F}[c/v] = \sigma_k\}$	19,20,lemma
(22)	$k>n$	21,$\exists$&O
(23)	$c \notin U_n\cup D_n$	21,$\exists$&O
(24)	$\sim\mathbb{F}[c/v] = \sigma_k$	21,$\exists$&O
(25)	$\sim\mathbb{F}[c/v] \in \Gamma_{k+1}$	18,22,23,24,Def $\langle\Gamma_i\rangle$
(26)	$c \in U_{k+1}$	18,22,23,24,Def $\langle U_i\rangle$
(27)	$c \in U$	26, Def U
(28)	$\sim\mathbb{F}[c/v] \in \Omega$	25, Def Ω
(29)	$\mathbb{F}[c/v] \in \Omega$	12,28,QL
(30)	$\times$	28,29,Lemma

$\sigma^1 \approx_k \sigma^2 \quad =_{df} \quad \forall i\{i \neq k \rightarrow \sigma^1{}_i = \sigma^2{}_i\}$

$\sigma[u/k] \quad =_{df}$ the result of substituting u for σ_k

For example, $\langle a,b,c,\ldots\rangle[d/2] = \langle a,d,c,\ldots\rangle$, and $\langle c,d,e,\ldots\rangle[a/3] = \langle c,d,a,\ldots\rangle$.

$\forall \upsilon \in V{:}\exists I \exists \sigma{:}\ \upsilon = \mathrm{val}(I,\sigma)$; in particular,

$\upsilon(\mathbb{P}) = I(\mathbb{P})$
$\upsilon(\acute{\omega}) = I(\acute{\omega})$
$\upsilon(\rho) = I(\rho)$
$\upsilon(x_k) = \sigma_k$

$I_n \quad = \quad$ the instantiations at n –
i.e., every formula obtained by instantiating every open formula in Γ_n to constant c_0, plus every formula obtained by multiple applications of $\forall O$ to formulas in Γ_n using singular terms in U_n.

$\forall \mathbb{F} \forall v_1 \ldots v_k\{\mathbb{F} \in \Gamma_n\ \&\ v_1,\ldots,v_k \text{ are free in } \mathbb{F}\ .\!\longrightarrow \mathbb{F}[c/v_1,\ldots,c/v_k] \in I_n\}$
$\forall \mathbb{F} \forall v_1 \ldots v_k \forall \tau_1 \ldots \tau_k\{\text{¸}\ \forall v_1 \ldots \forall v_k \mathbb{F}^1 \in \Gamma_n\ \&\ \tau_1,\ldots,\tau_k \in U_n\ .\!\longrightarrow \mathbb{F}[\tau_1/v_1,\ldots,\tau_k/v_k] \in I_n\}$

*We describe this by saying that σ_n <u>corroborates</u> $\sim\forall v\phi$.

3. Universal Derivation is Admissible

UDP: ~$g{g Î G & c Î g} ®. G⊢𝔽[c/v] ® G⊢ " v𝔽

(1)	~∃γ{γ∈Γ & c∈γ}	As
(2)	SHOW: Γ⊢𝔽[c/v] ⟶ Γ⊢∀v𝔽	3, lemma
(3)	SHOW: ∀n:∀d{dD𝔽[c/v]/Γ/n ⟶ Γ⊢∀v𝔽}	SMI
(4)	∀k<n:∀d{dD𝔽[c/v]/Γ/k ⟶ Γ⊢∀v𝔽}	As
(5)	SHOW: ∀d{dD𝔽[c/v]/Γ/n ⟶ Γ⊢∀v𝔽}	UCD
(6)	δD𝔽[c/v]/Γ/n	As
(7)	SHOW: Γ⊢∀v𝔽	Def ⊢
(8)	δ_n = 𝔽[c/v]	6,Def Dα/Γ/n
(9)	Ax{𝔽[c/v]} or 𝔽[c/v] ∈ Γ or GEN{𝔽[c/v]} or MP{𝔽[c/v]}	6,8,Def Dα/Γ/n
(10)	c1: Ax{𝔽[c/v]}	As
(11)	⟨𝔽[c/v]⟩ proves 𝔽[c/v]	7,9,Def proves
(12)	⟨𝔽[c/v], ∀v𝔽⟩DER∀v𝔽[c/v]/Γ	inspection
(13)	∃d{dDER∀v𝔽/Γ	11,QL
(14)	Γ⊢∀v𝔽	12, Def ⊢
(15)	c2: 𝔽[c/v] ∈ Γ	As
(16)	SHOW: v not free in 𝔽	ID
(17)	v is free in 𝔽	As
(18)	SHOW: ✕	
(19)	c ∈ 𝔽[c/v]	16, Def [c/v]
(20)	✕	1,14,18,QL
(21)	𝔽 = 𝔽[c/v]	14, Def [c/v]
(22)	⟨/ 𝔽→∀v𝔽⟩ ∈ R6	15, Def R6
(23)	⟨/ 𝔽[c/v]→∀v𝔽⟩ ∈ R6	20,21,IL
(24)	⟨𝔽[c/v], 𝔽[c/v]→∀v𝔽, ∀v𝔽⟩DER∀v𝔽[c/v]/Γ	22+R4+inspection
(25)	∃d{dDER∀v𝔽/Γ	23,QL
(26)	Γ⊢∀v𝔽	24, Def ⊢
(27)	c3: GEN{𝔽[c/v]}	As
(28)	∃δ′⊆δ: δ′ proves 𝔽′[c/v][c′/v′] & 𝔽[c/v]=∀v′𝔽′[c/v]	27,Def GEN [R8]
(29)	δ′+⟨∀v𝔽⟩ proves ∀v𝔽	
(30)	∃d{d proves ∀v𝔽}	29,QL
(31)	⊢∀v𝔽	30, Def ⊢
(32)	Γ⊢∀v𝔽	21,G?
(33)	c4: MP{𝔽[c/v]}	As
(34)	∃j,k<n,∃γ: d_j=γ→𝔽[c/v] & d_k=γ	33,Def MP[]
(35)	j<n & d_j=γ→𝔽[c/v]	34,∃&O
(36)	k<n & d_k=γ	34,∃&O
(37)	SHOW: ⟨d_i: i⩽j⟩ D γ→𝔽[c/v] / Γ/ j	Def D/n [&D]
(38)	a:SHOW: len⟨d_i: i⩽j⟩ = j	ST
(39)	b:SHOW: last⟨d_i: i⩽j⟩=γ→𝔽[c/v]	
(40)	last⟨d_i: i⩽j⟩ = d_j	ST
(41)	c:SHOW: ⟨d_i: i⩽j⟩DΓ	Def dDΓ
(42)	SHOW: ∀δ∈⟨d_i: i⩽j⟩: Ax[δ] or δ∈Γ or GEN[δ] or MP[δ]	UCD
(43)	δ∈⟨d_i: i⩽j⟩	As
(44)	SHOW: Ax[δ] or δ∈Γ∪{α} or GEN[δ] or MP[δ]	
(45)	δ∈d	32,ST
(46)	SHOW: ⟨d_i: i⩽k⟩ D γ / Γ∪{α} / k	Def D/n
(47)	similar to derivation of line 37	
(48)	Γ⊢∀v{γ→𝔽}	IH
(49)	Γ⊢∀v{γ}	IH
(50)	Γ⊢∀v𝔽	

4. Negation-Universal Elimination is Admissible; Existential Elimination is Admissible

L1: **GÈ{~ a}⊢ b ® GÈ{~b}⊢ a**

(1)	~~SHOW~~: $\Gamma\cup\{\sim\alpha\}\vdash\beta \longrightarrow \Gamma\cup\{\sim\beta\}\vdash\alpha$	CD
(2)	$\Gamma\cup\{\sim\alpha\}\vdash\beta$	As
(3)	~~SHOW~~: $\Gamma\cup\{\sim\beta\}\vdash\alpha$	CDT
(4)	~~SHOW~~: $\Gamma\vdash\sim\beta\rightarrow\alpha$	5,7,MPP
(5)	$\Gamma\vdash\sim\alpha\rightarrow\beta$	2,DT
(6)	$\vdash(\sim\alpha\rightarrow\beta)\rightarrow(\sim\beta\rightarrow\alpha)$	lemma
(7)	$\Gamma\vdash(\sim\alpha\rightarrow\beta)\rightarrow(\sim\beta\rightarrow\alpha)$	lemma

NUEP: **~$g{g Î G & c Î g} & c Ï a ®. GÈ{~𝔽[c/v]}⊢ a ® GÈ{~ " v𝔽}⊢ a**

(1)	$\sim\exists\gamma\{\gamma\in\Gamma \;\&\; c\in\gamma\}$	As
(2)	$c\notin\alpha$	As
(3)	~~SHOW~~: $\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash\alpha \longrightarrow \Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash\alpha$	CD
(4)	$\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash\alpha$	As
(5)	~~SHOW~~: $\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash\alpha$	DD
(6)	$\Gamma\cup\{\sim\alpha\}\vdash\mathbb{F}[c/v]$	3,L1
(7)	$\sim\exists\gamma\{\gamma\in\Gamma\cup\{\sim\alpha\} \;\&\; c\in\gamma\}$	1,2,ST
(8)	$\Gamma\cup\{\sim\alpha\}\vdash\forall v\mathbb{F}$	6,7,UDP
(9)	$\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash\alpha$	8,L1

Corollary: **~$g{g Î G & c Î g} ®. GÈ{~𝔽[c/v]}⊢ ® GÈ{~ " v𝔽}⊢**

(1)	$\sim\exists\gamma\{\gamma\in\Gamma \;\&\; c\in\gamma\}$	As
(2)	~~SHOW~~: $\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash \longrightarrow \Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash$	CD
(3)	$\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash$	As
(4)	~~SHOW~~: $\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash$	DD
(5)	$\forall\alpha[\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash\alpha]$	3,Def $\vdash$
(6)	$\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash P$	5,QL
(7)	$\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash\sim P$	5,QL
(8)	$c\notin P$	inspection
(9)	$c\notin\sim P$	inspection
(10)	$\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash P$	6,8,NUEP
(11)	$\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash\sim P$	7,9,NUEP
(12)	$\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash$	10,11,Lemma ?

Corollary: **¸ ~ " v𝔽 ¹ Î G & ~$g{g Î G & c Î g} & G⊬ .® GÈ{~𝔽[c/v]}⊬**

(1)	$\ulcorner\sim\forall v\mathbb{F}\urcorner\in\Gamma$	As
(2)	$\sim\exists\gamma\{\gamma\in\Gamma \;\&\; c\in\gamma\}$	As
(3)	$\Gamma\nvdash$	As
(4)	~~SHOW~~: $\Gamma\cup\{\sim\mathbb{F}[c/v]\}\nvdash$	CD
(5)	$\Gamma\cup\{\sim\mathbb{F}[c/v]\}\vdash$	As
(6)	~~SHOW~~: $\times$	3,9,SL
(7)	$\Gamma\cup\{\sim\forall v\mathbb{F}\}\vdash$	2,5,cor1
(8)	$\Gamma=\Gamma\cup\{\sim\forall v\mathbb{F}\}$	1,ST
(9)	$\Gamma\vdash$	7,8,IL

L2: **GÈ{a}⊢ b ® GÈ{~~a}⊢ b**

(1)	~~SHOW~~: $\Gamma\cup\{\alpha\}\vdash\beta \longrightarrow \Gamma\cup\{\sim\sim\alpha\}\vdash\beta$	CD
(2)	$\Gamma\cup\{\alpha\}\vdash\beta$	As
(3)	~~SHOW~~: $\Gamma\cup\{\sim\sim\alpha\}\vdash\beta$	DD
(4)	$\vdash\sim\sim\alpha\rightarrow\alpha$	lemma
(5)	$\Gamma\vdash\sim\sim\alpha\rightarrow\alpha$	4,G?
(6)	$\Gamma\cup\{\sim\sim\alpha\}\vdash\sim\sim\alpha$	G?
(7)	$\Gamma\cup\{\sim\sim\alpha\}\vdash\sim\sim\alpha\rightarrow\alpha$	G?
(8)	$\Gamma\cup\{\sim\sim\alpha\}\vdash\alpha$	6,7,MPP
(9)	$\Gamma\cup\{\sim\sim\alpha\}\cup\{\alpha\}\vdash\beta$	2,G?
(10)	$\Gamma\cup\{\sim\sim\alpha\}\vdash\beta$	8,9,G?

EEP: **~$g{g Î G & c Î g} & c Ï a ®. GÈ{𝔽[c/v]}⊢ a ® GÈ{$v𝔽}⊢ a**

(1)	$\sim\exists\gamma\{\gamma\in\Gamma$ & $c\in\gamma\}$	As
(2)	$c\notin\alpha$	As
(3)	~~SHOW~~: $\Gamma\cup\{\mathbb{F}[c/v]\}\vdash\alpha \longrightarrow \Gamma\cup\{\exists v\mathbb{F}\}\vdash\alpha$	CD
(4)	$\Gamma\cup\{\mathbb{F}[c/v]\}\vdash\alpha$	As
(5)	~~SHOW~~: $\Gamma\cup\{\exists v\mathbb{F}\}\vdash\alpha$	DD
(6)	$\Gamma\cup\{\sim\sim\mathbb{F}[c/v]\}\vdash\alpha$	4,L2
(7)	$\Gamma\cup\{\sim\alpha\}\vdash\sim\mathbb{F}[c/v]$	6,L1
(8)	$\sim\exists\gamma\{\gamma\in\Gamma\cup\{\sim\alpha\}$ & $c\in\gamma\}$	1,2,ST
(9)	$\Gamma\cup\{\sim\alpha\}\vdash\forall v\sim\mathbb{F}$	7,8,UDP
(10)	$\Gamma\cup\{\sim\forall v\sim\mathbb{F}\}\vdash\alpha$	9,L1
(11)	$\Gamma\cup\{\exists v\mathbb{F}\}\vdash\alpha$	10, Def $\exists$v