17 Soundness Theorem for System As1+Q

1. Introduction

Having described the axiom system AS1+Q, our next task is to show that it is sound with respect to the semantics of CFOL. This means we have to show the following, for every Γ and α

$$\Gamma \vdash \alpha \;\longrightarrow\; \Gamma \vDash \alpha$$

where $\vdash$ is defined relative to AS1+Q, and $\vDash$ is defined relative to CFOL.

2. Overall Construction of the Proof of Soundness

The proof that AS1+Q is sound w.r.t. the standard CFOL-semantics is the same in overall construction to the corresponding proof about AS1. After setting up the strong induction, one gets to the key juncture. One has assumed that α is derivable from Γ. So α is the last line of the derivation $\langle\delta_1, \ldots, \delta_m\rangle$ from Γ. Given the definition of derivation, either α is a premise, or follows by a rule. The first case is easy to settle, just like in AS1. The second case divides into as many cases as their are rules of deduction. In the case of AS1+Q, there are 8 rules, four of which are identical in form to AS1. Accordingly, the first four cases have in effect already been settled in the proof of the soundness of AS1.

That leaves four more cases. Each of these cases reduces to proving an appropriate lemma. We provide those lemmas in what follows.

[Exercise] The reader is invited to construct the overall derivation, being careful to set it up so that the following lemmas suffice; note in particular the form of the lemma concerning R8.

3. Rule R5 is Valid

(1)	~~SHOW~~: $\vDash \forall\nu\mathbb{F}\to\mathbb{F}[c/\nu]$	Def($\vDash$)
(2)	~~SHOW~~: $\forall\upsilon[\upsilon(\forall\nu\mathbb{F}\to\mathbb{F}[c/\nu]) = T]$	UD
(3)	~~SHOW~~: $\upsilon(\forall\nu\mathbb{F}\to\mathbb{F}[c/\nu]) = T$	♮

♮ Given some simple facts about admissible valuations and the truth-function for $\to$, in order to show (3), it suffices to show the following, where c is an arbitrary closed singular term.

(4)	~~SHOW~~: $\upsilon(\forall\nu\mathbb{F}) = T \;\longrightarrow\; \upsilon(\mathbb{F}[c/\nu]) = T$	CD
(5)	| $\upsilon(\forall\nu\mathbb{F}) = T$	As
(6)	| ~~SHOW~~: $\upsilon(\mathbb{F}[c/\nu]) = T$	10,11,IL
(7)	| | $\forall\upsilon'\{\upsilon' \approx_\nu \upsilon \;\longrightarrow\; \upsilon'(\mathbb{F})=T\}$	8, Def CFOL-val
(8)	| | let $\upsilon_0(\nu)=\upsilon(c)$ & $\forall\varepsilon\{\text{Atomic}[\varepsilon]$ & $\varepsilon\neq\nu \;.\!\!\longrightarrow \upsilon_0(\varepsilon)=\upsilon(\varepsilon)\}$	ST+∃O
(9)	| | $\upsilon_0 \approx_\nu \upsilon$	8b, Def $\approx$
(10)	| | $\upsilon_0(\mathbb{F})=T$	7,9,QL
(11)	| | $\upsilon_0(\mathbb{F}) = \upsilon(\mathbb{F}[c/\nu])$	8a,9,Subst Lemma

4. Rule R6 is Valid

(1)	ν is not free in $\mathbb{F}$	As
(2)	~~SHOW~~: $\vDash \mathbb{F}\to\forall\nu\mathbb{F}$	Def($\vDash$)
(3)	~~SHOW~~: $\forall\upsilon[\upsilon(\mathbb{F}\to\forall\nu\mathbb{F}) = \mathrm{T}]$	UD
(4)	~~SHOW~~: $\upsilon(\mathbb{F}\to\forall\nu\mathbb{F}) = \mathrm{T}$	♮

♮ Given some simple facts about admissible valuations and the truth-function for $\to$, in order to show (3), it suffices to show the following, where $\mathbb{F}$ is a closed formula.

(4)	~~SHOW~~: $\upsilon(\mathbb{F}) = \mathrm{T} \;\longrightarrow\; \upsilon(\forall\nu\mathbb{F}) = \mathrm{T}$	CD
(5)	$\upsilon(\mathbb{F}) = \mathrm{T}$	As
(6)	~~SHOW~~: $\upsilon(\forall\nu\mathbb{F}) = \mathrm{T}$	Def (CFOL-val)
(7)	$\forall\upsilon'\{\upsilon' \approx_\nu \upsilon \;\longrightarrow\; \upsilon'(\mathbb{F})=\mathrm{T}\}$	UCD
(8)	$\upsilon' \approx_\nu \upsilon$	As
(9)	~~SHOW~~: $\upsilon'(\mathbb{F})=\mathrm{T}$	1,8, Bound Variable Lemma (below)

5. Rule R7 is Valid

(1)	~~SHOW~~: $\vDash \forall\nu(\mathbb{F}\to\mathbb{G})\to(\forall\nu\mathbb{F}\to\forall\nu\mathbb{G})$	Def($\vDash$)
(2)	~~SHOW~~: $\forall\upsilon[\upsilon(\forall\nu(\mathbb{F}\to\mathbb{G})\to(\forall\nu\mathbb{F}\to\forall\nu\mathbb{G})) = \mathrm{T}]$	UD
(3)	~~SHOW~~: $\upsilon(\forall\nu(\mathbb{F}\to\mathbb{G})\to(\forall\nu\mathbb{F}\to\forall\nu\mathbb{G})) = \mathrm{T}$	♮

♮ Given some simple facts about admissible valuations and the truth-function for $\to$, in order to show (3), it suffices to show the following.

(4)	~~SHOW~~: $\upsilon(\forall\nu(\mathbb{F}\to\mathbb{G})) = \mathrm{T} \;\longrightarrow\!. \; \upsilon(\forall\nu\mathbb{F}) = \mathrm{T} \;\longrightarrow\; \upsilon(\forall\nu\mathbb{G})) = \mathrm{T}$	CCD
(5)	$\upsilon(\forall\nu(\mathbb{F}\to\mathbb{G})) = \mathrm{T}$	As
(6)	$\upsilon(\forall\nu\mathbb{F}) = \mathrm{T}$	As
(7)	~~SHOW~~: $(\forall\nu\mathbb{G})) = \mathrm{T}$	Def (CFOL-val)
(8)	~~SHOW~~: $\forall\upsilon'\{\upsilon' \approx_\nu \upsilon \;\longrightarrow\; \upsilon'(\mathbb{G})=\mathrm{T}\}$	UCD
(9)	$\upsilon_0 \approx_\nu \upsilon$	As
(10)	~~SHOW~~: $\upsilon_0(\mathbb{G})=\mathrm{T}$	DD
(11)	$\forall\upsilon'\{\upsilon' \approx_\nu \upsilon \;\longrightarrow\; \upsilon'(\mathbb{F}\to\mathbb{G})=\mathrm{T}\}$	5, Def (CFOL-val)
(12)	$\forall\upsilon'\{\upsilon' \approx_\nu \upsilon \;\longrightarrow\; \upsilon'(\mathbb{F})=\mathrm{T}\}$	6, Def (CFOL-val)
(13)	$\upsilon_0(\mathbb{F}\to\mathbb{G}) = \mathrm{T}$	9,11,QL
(14)	$\upsilon_0(\mathbb{F}) = \mathrm{T}$	9,12,QL
(15)	$\upsilon_0(\mathbb{G}) = \mathrm{T}$	13,14,earlier result about CSL

6. Rule R8 is Validity-Preserving

(1)	every occurrence of c in $\mathbb{F}$ is free for ν	As
(2)	$\mathbb{F}[\nu/c]$ is the result of replacing every occurrence of c in $\mathbb{F}$ by ν	As
(3)	~~SHOW~~: $\vDash\mathbb{F} \longrightarrow\ \vDash\forall\nu\mathbb{F}[\nu/c]$	CD
(4)	$\vDash\mathbb{F}$	As
(5)	i.e.: $\forall\upsilon[\upsilon(\mathbb{F}) = T]$	4,Def($\vDash$)
(6)	~~SHOW~~: $\vDash\forall\nu\mathbb{F}[\nu/c]$	Def($\vDash$)
(7)	~~SHOW~~: $\forall\upsilon[\upsilon(\forall\nu\mathbb{F}[\nu/c]) = T]$	UD
(8)	~~SHOW~~: $\upsilon_0(\forall\nu\mathbb{F}[\nu/c]) = T$	Def(CFOL-val)
(9)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_\nu \upsilon_0 \longrightarrow \upsilon(\mathbb{F}[\nu/c])=T\}$	UCD
(10)	$\upsilon_1 \approx_\nu \upsilon_0$	As
(11)	~~SHOW~~: $\upsilon_1(\mathbb{F}[\nu/c])=T$	5,17,IL
(12)	let υ_2 be such that: $\upsilon_2(c) = \upsilon_1(\nu)$; otherwise $\upsilon_2 = \upsilon_1$ for simples	ST,∃O
(13)	$\upsilon_2(c) \approx_\nu \upsilon_1(\nu)$	12b,Def($\approx$)
(14)	$\forall\phi\{\upsilon_1(\phi) = \upsilon_2(\phi[c/\nu])\}$	12a,13,Subst Lemma
(15)	$\upsilon_1(\mathbb{F}[\nu/c]) = \upsilon_2(\mathbb{F}[\nu/c][c/\nu])$	14,QL
(16)	claim: $\mathbb{F}[\nu/c][c/\nu] = \mathbb{F}$	♮
(17)	$\upsilon_1(\mathbb{F}[\nu/c]) = \upsilon_2(\mathbb{F})$	15,16,IL

♮ Intuitively, this is the argument: By requirement of Rule R8, $\mathbb{F}$ is closed, so $\mathbb{F}$ has no free occurrence of ν. Also by requirement of R8, c is free for ν in $\mathbb{F}$. Consider an arbitrary occurrence **o** of c in $\mathbb{F}$. When one does the first substitution, producing $\mathbb{F}[\nu/c]$, **o** is replaced by an occurrence **o**¢ of ν. Since c is free for ν, **o**¢ is free. Accordingly, when one does the second substitution, producing $\mathbb{F}[\nu/c][c/\nu]$, **o**¢ is replaced by an occurrence **o**¢¢¢ of c. Thus, every occurrence of c is replaced by an occurrence of ν, which in turn is replaced by an occurrence of c. It is evident that the resulting formula is identical to the original formula. Ultimately, a formal proof requires induction, and appeals to the official (inductive) definition of substitution [see Section 7].

7. Formal Definition of Substitution

Many of theorems we wish to prove involve substituting one expression for another. For this reason, it is useful to have a formal definition from which one can make logical deductions. As with many formal syntactic notions, substitution is officially defined inductively.

Definition of [t/v]

Df

if τ is an atomic singular term, then:

$$\tau[t/v] = t \quad \text{if} \quad \tau = v$$
$$= \tau \quad \text{if} \quad \tau \neq v$$

if τ is a molecular singular term, then:

$\tau = f\langle\tau_1, \ldots, \tau_k\rangle$ (for some f, $\tau_1, \ldots, \tau_k$), and
$\tau[t/v] = f\langle\tau_1[t/v], \ldots, \tau_k[t/v]\rangle$

if ϕ is an atomic formula, then:

$\phi = \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle$ (for some $\mathbb{P}$, $\tau_1, \ldots, \tau_k$), and
$\phi[t/v] = \mathbb{P}\langle\tau_1[t/v], \ldots, \tau_k[t/v]\rangle$

if ϕ is a molecular formula, then:
either:
$\phi = {\sim}\beta$ (for some β), in which case
$\phi[t/v] = {\sim}\beta[t/v]$
or:
$\phi = \alpha{\rightarrow}\beta$ (for some α, β), in which case
$\phi[t/v] = \alpha[t/v] \rightarrow \beta[t/v]$
or:
$\phi = \forall x\mathbb{F}$ (for some x, $\mathbb{F}$), in which case
either:
x=w, in which case
$\phi[t/v] = \phi$
or:
x≠w, in which case
$\phi[t/v] = \forall x\mathbb{F}[t/v]$

8. The Substitution Lemma

The Substitution Lemma – which is very important both in the proof of soundness and in the (later) proof of completeness – concerns the semantics of CFOL.

> Th
>
> Let $\mathbb{F}$ be a formula. Let υ_1 and υ_2 be admissible valuations, let x be a variable, and let c be any closed singular term; for any expression ε, let ε* = ε[c/x], the latter being defined as usual. Then:
>
> $$\upsilon_1 \approx_x \upsilon_2 \longrightarrow.\ \upsilon_1(x)=\upsilon_2(c) \longrightarrow \upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$$

Recall that ≈ is defined as follows.

$$\upsilon_1 \approx_x \upsilon_2 \quad =_{df} \quad \forall\varepsilon\{\text{Simple}[\varepsilon] \longrightarrow.\ \{\varepsilon \neq x \longrightarrow \upsilon_1(\varepsilon) = \upsilon_2(\varepsilon)\}\}$$

Here, 'Simple[ε]' means that ε is a syntactically atomic *expression* of $\mathbb{L}$, which is to say a symbol (other than punctuation). In FOL's the simple expressions are variables, constants, proper nouns, predicate letters, and function signs.

For clarity, we divide the proof into two segments – one for singular terms, the other for formulas – . the first of which feeds into the second.

SINGULAR TERMS:

(1)	~~SHOW~~: $\forall\upsilon_1, \upsilon_2, x, c\ \{\upsilon_1 \approx_x \upsilon_2 \longrightarrow.\ \forall\tau\{\upsilon_1(x)=\upsilon_2(c) \longrightarrow \upsilon_1(\tau) = \upsilon_2(\tau^*)\}\}$	UCUCD
(2)	$\upsilon_1 \approx_x \upsilon_2$	
	i.e., $\forall\varepsilon\{\text{Simple}[\varepsilon] \longrightarrow.\ \{\varepsilon \neq x \longrightarrow \upsilon_1(\varepsilon) = \upsilon_2(\varepsilon)\}\}$	As
(3)	$\upsilon_1(x)=\upsilon_2(c)$	As
(4)	~~SHOW~~: $\upsilon_1(\tau) = \upsilon_2(\tau^*)$	Induction on term formation
	Base Case:	
(5)	τ is an atomic singular term, and hence simple	As
(6)	~~SHOW~~: $\upsilon_1(\tau) = \upsilon_2(\tau^*)$	separation of cases, 7-17
(7)	$\tau = x$ or $\tau \neq x$	SL
(8)	c1: $\tau = x$	As
(9)	$\upsilon_1(\tau) = \upsilon_1(x) = \upsilon_2(c)$	3,8,IL
(10)	$\tau^* = c$	8, Def[c/x]
(11)	$\upsilon_2(\tau^*) = \upsilon_2(c)$	10,IL
(12)	$\upsilon_1(\tau) = v_2(\tau^*)$	10,11,IL
(13)	c2: $\tau \neq x$	As
(14)	$\upsilon_1(\tau) = \upsilon_2(\tau)$	2,5,QL
(15)	$\tau^* = \tau$	13, Def[c/x]
(16)	$\upsilon_2(\tau^*) = \upsilon_2(\tau)$	15,IL
(17)	$\upsilon_1(\tau) = v_2(\tau^*)$	15,16,IL

Inductive Case:

(18)	$\upsilon_1(\tau_1) = \upsilon_2(\tau_1{*}), \ldots, \upsilon_1(\tau_k) = \upsilon_2(\tau_k{*})$	As(IH)
(19)	~~SHOW~~: $\upsilon_1(f\langle\tau_1, \ldots, \tau_k\rangle) = \upsilon_2(f\langle\tau_1{*}, \ldots, \tau_k{*}\rangle$	21,22,IL
(20)	$\upsilon_1(f\langle\tau_1, \ldots, \tau_k\rangle) \; = \; \upsilon_1(f)\langle\upsilon_1(\tau_1), \ldots, \upsilon_1(\tau_k)\rangle)$	Def CFOL-val
(21)	$\upsilon_1(f) = \upsilon_2(f)$	2,♮

♮ every function sign is simple, and no function sign is a variable

(22)	$\upsilon_1(f)\langle\upsilon_1(\tau_1), \ldots, \upsilon_1(\tau_k)\rangle \; = \; \upsilon_2(f)\langle\upsilon_2(\tau_1{*}), \ldots, \upsilon_2(\tau_k){*}\rangle$	18,21,IL

FORMULAS (Proof by induction on formula formation):

(1)	~~SHOW~~: $\forall\mathbb{F}{:}\forall\upsilon_1, \upsilon_2, \mathrm{x}, \mathrm{c}\, \{\upsilon_1 \approx_{\mathrm{x}} \upsilon_2 \;\longrightarrow.\; \upsilon_1(\mathrm{x}){=}\upsilon_2(\mathrm{c}) \;\longrightarrow\; \upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}{*})\}$	Induction on formula formation

Base Case:

(2)	$\mathbb{F}$ is an atomic formula.	As
(3)	~~SHOW~~: $\forall\upsilon_1, \upsilon_2, \mathrm{x}, \mathrm{c}\, \{\upsilon_1 \approx_{\mathrm{x}} \upsilon_2 \;\longrightarrow.\; \upsilon_1(\mathrm{x}){=}\upsilon_2(\mathrm{c}) \;\longrightarrow\; \upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}{*})\}$	U4CCD
(4)	$\upsilon_1 \approx_{\mathrm{x}} \upsilon_2$	As
(5)	i.e.: $\forall\varepsilon\{\mathrm{Simple}[\varepsilon] \longrightarrow.\; \varepsilon{\neq}\mathrm{x} \longrightarrow \upsilon_1(\varepsilon){=}\upsilon_2(\varepsilon)\}$	3, Def $\approx$
(6)	$\upsilon_1(\mathrm{x}){=}\upsilon_2(\mathrm{c})$	As
(7)	~~SHOW~~: $\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}{*})$	
(8)	$\mathbb{F} = \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle$	1, Def Atomic formula, ∃O ♮

♮ Note that given the categorial approach to grammar, this includes the case in which $\mathbb{P}$ is the special (logical) predicate '='. In that case $\mathbb{P}\langle\tau_1, \tau_2\rangle \; = \; [=]\langle\tau_1, \tau_2\rangle = \ulcorner\tau_1{=}\tau_2\urcorner$.

(9)	$\upsilon_1(\mathbb{F}) = \upsilon_1(\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle) = \upsilon_1(\mathbb{P})\langle\upsilon_1(\tau_1), \ldots, \upsilon_1(\tau_k)\rangle$	7,IL / Def CFOL-val
(10)	$\mathbb{F}{*} \quad = \quad [\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle]{*}$	7,IL
(11)	$\quad = \quad \mathbb{P}\langle\tau_1{*}, \ldots, \tau_k{*}\rangle$	GenSubTh
(12)	$\upsilon_2(\mathbb{F}{*}) \quad = \quad \upsilon_2(\mathbb{P})\langle\upsilon_2(\tau_1{*}),\ldots,\upsilon_2(\tau_k{*})\rangle$	9-10,IL,Def CFOL-val
(13)	$[\forall i{\leqslant}k]{:}\; \upsilon_1(\tau_k) = \mathrm{v}_2(\tau_k{*})$	shown above
(14)	$\mathrm{Simple}[\mathbb{P}]\ \&\ \mathbb{P}{\neq}\mathrm{x}$	♮

♮ every predicate is simple, and no predicate is a variable.

(15)	$\upsilon_1(\mathbb{P}) \quad = \quad \upsilon_2(\mathbb{P})$	4,13,QL
(16)	$\upsilon_1(\mathbb{F}) \quad = \quad \upsilon_2(\mathbb{P})\langle\upsilon_1(\tau_1),\ldots,\upsilon_1(\tau_k)\rangle$	8,14,IL
(17)	$\quad = \quad \upsilon_2(\mathbb{P})\langle\upsilon_2(\tau_1{*}),\ldots,\upsilon_2(\tau_k{*})\rangle$	12,IL
(18)	$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}{*})$	11,14-16,IL

Inductive Case 1 ($\sim$)

Given the form of the formula to be shown, it suffices to do the following conditional derivation.

(1)	$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}{*})$	As
(2)	~~SHOW~~: $\upsilon_1({\sim}\mathbb{F}) = \upsilon_2([{\sim}\mathbb{F}]{*})$	3-7,IL
(3)	$[{\sim}\mathbb{F}]{*} = {\sim}\mathbb{F}{*}$	Def [c/x]
(4)	$\upsilon_2([{\sim}\mathbb{F}]{*}) = \upsilon_2({\sim}\mathbb{F}{*})$	3,IL
(5)	$\upsilon_1({\sim}\mathbb{F}) = {\sim}\upsilon_1(\mathbb{F})$	Def CFOL-val
(6)	${\sim}\upsilon_1(\mathbb{F}) = {\sim}\upsilon_2(\mathbb{F}{*})$	1,5,IL
(7)	$\upsilon_2({\sim}\mathbb{F}{*}) = {\sim}\upsilon_2(\mathbb{F}{*})$	Def CFOL-val

Inductive Case 2 ($\rightarrow$)
Given the form of the formula to be shown, it suffices to show the following.

(1)	$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$	As
(2)	$\upsilon_1(\mathbb{G}) = \upsilon_2(\mathbb{G}^*)$	As
(3)	~~SHOW~~: $\upsilon_1(\mathbb{F}\rightarrow\mathbb{G}) = \upsilon_2([\mathbb{F}\rightarrow\mathbb{G}]^*)$	5-8,IL
(4)	$[\mathbb{F}\rightarrow\mathbb{G}]^* = \mathbb{F}^*\rightarrow\mathbb{G}^*$	Def [c/x]
(5)	$\upsilon_2([\mathbb{F}\rightarrow\mathbb{G}]^*) = \upsilon_2(\mathbb{F}^*\rightarrow\mathbb{G}^*)$	4,IL
(6)	$\upsilon_1(\mathbb{F}\rightarrow\mathbb{G}) = \upsilon_1(\mathbb{F})\rightarrow\upsilon_1(\mathbb{G})$	Def CFOL-val
(7)	$\upsilon_1(\mathbb{F})\rightarrow\upsilon_1(\mathbb{G}) = \upsilon_2(\mathbb{F}^*)\rightarrow\upsilon_2(\mathbb{G}^*)$	1,2,6,IL
(8)	$\upsilon_2(\mathbb{F}^*\rightarrow\mathbb{G}^*) = \upsilon_2(\mathbb{F}^*)\rightarrow\upsilon_2(\mathbb{G}^*)$	Def CFOL-val

Inductive Case 3 ($\forall$)

(1)	$\forall\upsilon, \upsilon', x, c\ \{\upsilon \approx_x \upsilon' \longrightarrow\!.\ \ \upsilon(x)=\upsilon'(c) \longrightarrow \upsilon(\mathbb{F}) = \upsilon'(\mathbb{F}^*)\}$	As
(2)	~~SHOW~~: $\forall\upsilon_1, \upsilon_2, x, c, y\ \{\upsilon_1 \approx_x \upsilon_2 \longrightarrow\!.\ \ \upsilon_1(x)=\upsilon_2(c) \longrightarrow \upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)\}$	
(3)	$\upsilon_1 \approx_x \upsilon_2$	As
(4)	$\upsilon_1(x)=\upsilon_2(c)$	As
(5)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)$	SC,6-
(6)	$x = y$ or $x \neq y$	SL
(7)	c1: $x = y$	As
(8)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)$	7,9,IL
(9)	~~SHOW~~: $\upsilon_1(\forall x\mathbb{F}) = \upsilon_2([\forall x\mathbb{F}]^*)$	10,11,IL
(10)	$[\forall x\mathbb{F}]^* = \forall x\mathbb{F}$	Def [c/x]
(11)	$\upsilon_1(\forall x\mathbb{F}) = \upsilon_2(\forall x\mathbb{F})$	3 + Bound Variable Lemma (below)
(12)	c2: $x \neq y$	As
(13)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)$	14,15,IL
(14)	$[\forall y\mathbb{F}]^* = \forall y\mathbb{F}^*$	12, Def [c/x]
(15)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2(\forall y\mathbb{F}^*)$	16,17,18,GenTh(υ),IL
(16)	$\upsilon_1(\forall y\mathbb{F}) = T \leftrightarrow \forall\upsilon\{\upsilon \approx_y \upsilon_1 \rightarrow \upsilon(\mathbb{F}) = T\}$	Def CFOL-val (alt)
(17)	$\upsilon_2(\forall y\mathbb{F}^*) = T \leftrightarrow \forall\upsilon\{\upsilon \approx_y \upsilon_2 \rightarrow \upsilon(\mathbb{F}^*) = T\}$	Def CFOL-val (alt)
(18)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_y \upsilon_1 \rightarrow \upsilon(\mathbb{F}) = T\} \leftrightarrow \forall\upsilon\{\upsilon \approx_y \upsilon_2 \rightarrow \upsilon(\mathbb{F}^*) = T\}$	19,48,SL

(19)	~~SHOW~~: $\rightarrow$	CD
(20)	$\forall\upsilon\{\upsilon \approx_y \upsilon_1 \rightarrow \upsilon(\mathbb{F}) = T\}$	As
(21)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_y \upsilon_2 \rightarrow \upsilon(\mathbb{F}^*) = T\}$	UCD
(22)	$\upsilon_3 \approx_y \upsilon_2$	As
(23)	~~SHOW~~: $\upsilon_3(\mathbb{F}^*) = T$	
(24)	let $\upsilon_4(y) = \upsilon_3(y)$ & $\upsilon_4(\varepsilon) = \upsilon_1(\varepsilon)$ if $\varepsilon \neq y$ and Simple[ε]	ST,$\exists$O
(25)	$\upsilon_4 \approx_y \upsilon_1$	24b, Def($\approx$)
(26)	$\upsilon_4(\mathbb{F}) = T$	20,25,QL
(27)	~~SHOW~~: $\upsilon_4 \approx_x \upsilon_3$	Def($\approx$)
(28)	~~SHOW~~: $\forall\varepsilon\{$Simple[ε] $\rightarrow.\ \varepsilon \neq x \rightarrow \upsilon_4(\varepsilon) = \upsilon_3(\varepsilon)\}$	UCCD
(29)	Simple[ε]	As
(30)	$\varepsilon \neq x$	As
(31)	~~SHOW~~: $\upsilon_4(\varepsilon) = \upsilon_3(\varepsilon)$	SC,32-39
(32)	$\varepsilon = y$ or $\varepsilon \neq y$	SL
(33)	c1: $\varepsilon = y$	As
(34)	$\upsilon_4(\varepsilon) = \upsilon_4(y) = \upsilon_3(y) = \upsilon_3(\varepsilon)$	33,IL / 24a,IL / IL
(35)	c2: $\varepsilon \neq y$	As
(36)	$\upsilon_4(\varepsilon) = \upsilon_1(\varepsilon)$	25,29,35,Def($\approx$)
(37)	$\upsilon_1(\varepsilon) = \upsilon_2(\varepsilon)$	3,29,30,Def($\approx$)
(38)	$\upsilon_2(\varepsilon) = \upsilon_3(\varepsilon)$	22,29,35,Def($\approx$)
(39)	$\upsilon_4(\varepsilon) = \upsilon_3(\varepsilon)$	36-38,IL
(40)	~~SHOW~~: $\upsilon_4(x) = \upsilon_3(c)$	42,43,45,IL
(41)	$x \neq y$	12 (reminder)
(42)	$\upsilon_4(x) = \upsilon_1(x)$	24b,41
(43)	$\upsilon_1(x) = \upsilon_2(c)$	4 (reminder)
(44)	Atomic[c] & $c \neq y$	presumed
(45)	$\upsilon_2(c) = \upsilon_3(v)$	22,44,Def($\approx$)
(46)	$\upsilon_4(\mathbb{F}) = \upsilon_3(\mathbb{F}^*)$	1(IH),27,40,QL
(47)	$\upsilon_3(\mathbb{F}^*)$	26,46,IL

(48) ~~SHOW~~: $\leftarrow$ CD

Proof is very similar to 19-47 [**exercise**]

9. The Substitution/Quantification Lemma

The next lemma, which follows fairly directly from The Substitution Lemma, shows that, under certain circumstances, a universally quantified formula is naturally related to its closed substitution instances. First we define a subordinate notion.

> Def
>
> Let υ be a valuation from $\mathbb{L}$ into U, and let u be an element of U. Then:
>
> u *has a name according to* υ [a υ-name] $\;=_{\text{df}}\;$ $\exists\tau\{\text{closed}[\tau] \;\&\; \upsilon(\tau)=\text{u}\}$

The Substitution/Quantification Lemma

> Th (SubQ)
>
> Suppose every object u in U has a name according to υ. Then:
>
> $\upsilon(\forall v\mathbb{F}) = T \;\longleftrightarrow\; \forall\tau\{\text{closed}[\tau] \longrightarrow \upsilon(\mathbb{F}[\tau/v]) = T\}$

In other words, if every object has a υ-name, then a universal formula υ verifies $\forall v\mathbb{F}$if and only if υ verifies every (closed) substitution instance of $\mathbb{F}$.

We divide the proof into two natural halves. Notice that the first half does not employ the hypothesis [that every object in U has a υ-name]; indeed, the first half simply amounts to the soundness of Rule R5. The converse, however, does employ this hypothesis.

(1)	~~SHOW~~: $\upsilon_0(\forall x\mathbb{F}) = T \longrightarrow \forall\tau\{\text{closed}[\tau] \longrightarrow \upsilon_0(\mathbb{F}[\tau/x]) = T\}$	CUCD
(2)	$\vert$ $\upsilon_0(\forall x\mathbb{F}) = T$	As
(3)	$\vert$ closed[τ]	As
(4)	$\vert$ ~~SHOW~~: $\upsilon_0(\mathbb{F}[\tau/x]) = T$	ID
(5)	$\vert\;\vert$ $\forall\upsilon\{\upsilon \approx_x \upsilon_0 \longrightarrow \upsilon(\mathbb{F})=T\}$	2, Def CFOL-val
(6)	$\vert\;\vert$ let $\upsilon_1(x)=\upsilon_0(\tau)$ & $\forall\varepsilon\{\text{Simple}[\varepsilon] \;\&\; \varepsilon\neq x \;.\!\!\longrightarrow \upsilon_1(\varepsilon)=\upsilon_0(\varepsilon)\}$	ST+∃O
(7)	$\vert\;\vert$ $\upsilon_1 \approx_x \upsilon_0$	6b, Def ≈
(8)	$\vert\;\vert$ $\upsilon_1(\mathbb{F})=T$	5,7,QL
(9)	$\vert\;\vert$ $\upsilon_1(\mathbb{F}) = \upsilon_0(\mathbb{F}[\tau/x])$	3,6a,7, Subst Lemma
(10)	$\vert\;\vert$ $\upsilon_0(\mathbb{F}[\tau/x]) = T$	8,9,IL

(1)	$\forall u\{u\in U \longrightarrow \exists\tau\{\text{closed}[\tau]\ \&\ \upsilon_0(\tau)=u\}\}$	hyp
(2)	~~SHOW~~: $\forall\tau\{\text{closed}[\tau] \longrightarrow \upsilon(\mathbb{F}[\tau/x]) = T\} \longrightarrow \upsilon_0(\forall x\mathbb{F}) = T$	CD
(3)	$\forall\tau\{\text{closed}[\tau] \longrightarrow \upsilon_0(\mathbb{F}[\tau/x]) = T\}$	As
(4)	~~SHOW~~: $\upsilon_0(\forall x\mathbb{F}) = T$	Def CFOL-val
(5)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_x \upsilon_0 \longrightarrow \upsilon(\mathbb{F})=T\}$	UCD
(6)	$\upsilon_1 \approx_x \upsilon_0$	As
(7)	~~SHOW~~: $\upsilon_1(\mathbb{F})=T$	15,16,IL
(8)	$\exists u\{u\in U\ \&\ u=\upsilon_1(x)\}$	Def CFOL-val
(9)	$u\in U$	8,∃&O
(10)	$u = \upsilon_1(x)$	8,∃&O
(11)	$\exists\tau\{\text{closed}[\tau]\ \&\ \upsilon_0(\tau)=u\}$	1,9,QL
(12)	$\text{closed}[\tau]$	11,∃&O
(13)	$\upsilon_0(\tau)=u$	11,∃&O
(14)	$\upsilon_0(\tau) = \upsilon_1(x)$	10,13,IL
(15)	$\upsilon_1(\mathbb{F}) = \upsilon_0(\mathbb{F}[\tau/x])$	6,12,14,Subst Lemma
(16)	$\upsilon_0(\mathbb{F}[\tau/x]) = T$	3,12,QL

10. The Bound Variable Lemma

Suppose variable v is not free in formula $\mathbb{F}$ [for example, ‘x’ is not free in ‘∀xFx’]. Then, inutitively, the semantic value of $\mathbb{F}$ [e.g., ‘∀xFx’] should not depend upon the semantic value of v [e.g., ‘x’]. The following theorem corroborates this intution.

> Let ϕ be any formula. Let υ_1 and υ_2 be admissible valuations, let x be any variable that does not occur free in ϕ; then:
>
> $$\upsilon_1 \approx_x \upsilon_2 \longrightarrow \upsilon_1(\phi) = \upsilon_2(\phi)$$

Although the theorem is “obvious”, in some sense, the proof is quite challenging, and is left as an exercise for the reader.

2. Appendix

(1)	~~SHOW~~: $\upsilon_1 \approx_x \upsilon_2 \;\longrightarrow\; \upsilon_1(\phi) = \upsilon_2(\phi)$	CD
(2)	$\upsilon_1 \approx_x \upsilon_2$	As
(3)	~~SHOW~~: $\upsilon_1(\phi) = \upsilon_2(\phi)$	SC, 4-
(4)	ϕ is atomic, or ϕ is molecular (i.e., not atomic)	SL
(5)	c1: ϕ is atomic	As
(6)	~~SHOW~~: $\upsilon_1(\phi) = \upsilon_2(\phi)$	7-11,IL
(7)	$\phi = \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle$ (some $\mathbb{P}, \tau_1, \ldots, \tau_k$)	Def(atomic),∃O
(8)	$\upsilon_1(\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle) = \upsilon_1(\mathbb{P})\langle\upsilon_1(\tau_1), \ldots, \upsilon_1(\tau_k)\rangle$	Def(CFOL-val)
(9)	$\upsilon_2(\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle) = \upsilon_2(\mathbb{P})\langle\upsilon_2(\tau_1), \ldots, \upsilon_2(\tau_k)\rangle$	Def(CFOL-val)
(10)	$\upsilon_1(\mathbb{P}) = \upsilon_2(\mathbb{P})$	2, Def($\approx$)$\natural$
	$\natural$ every predicate is atomic, and no predicate is a variable	
(11)	$\upsilon_1(\tau_1) = \upsilon_2(\tau_1), \ldots, \upsilon_1(\tau_k) = \upsilon_2(\tau_k)$	IH$\natural$
	$\natural$	
(12)	c2: ϕ is molecular	As
(13)	~~SHOW~~: $\upsilon_1(\phi) = \upsilon_2(\phi)$	SC,14-
(14)	ϕ is a negation, or a conditional, or a universal	12,Def(molecular for $\mathbb{L}$)
(15)	c2.1: ϕ is a negation	As
(16)	~~SHOW~~: $\upsilon_1(\phi) = \upsilon_2(\phi)$	
(17)	$\phi = \sim\beta$	15, Def(negation), ∃O
(18)	$\upsilon_1(\phi) = \sim\langle\upsilon_1(\beta)\rangle$	Def(CFOL-val)
(19)	$\upsilon_2(\phi) = \sim\langle\upsilon_2(\beta)\rangle$	Def(CFOL-val)
(20)	$\upsilon_1(\beta) = \upsilon_2(\beta)$	IH
(21)	c2.2: ϕ is a conditional	As
(22)	similar to case 2.1	
(23)	c2.3: ϕ is a universal	
(24)	~~SHOW~~: $\upsilon_1(\phi) = \upsilon_2(\phi)$	
(25)	$\phi = \forall y\mathbb{F}$ (some y, $\mathbb{F}$)	23,Def(universal), ∃O
(26)	y=x, or y≠x	SL
(27)	c1: y=x	As
(28)	~~SHOW~~: $\upsilon_1(\forall x\mathbb{F}) = \upsilon_2(\forall x\mathbb{F})$	29-31,ST
(29)	$\upsilon_1(\forall x\mathbb{F}) = \min\{\upsilon(\mathbb{F}) : \upsilon \approx_x \upsilon_1\}$	def CFOL-val
(30)	$\upsilon_2(\forall x\mathbb{F}) = \min\{\upsilon(\mathbb{F}) : \upsilon \approx_x \upsilon_2\}$	def CFOL-val
(31)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_x \upsilon_1 \leftrightarrow \upsilon \approx_x \upsilon_2\}$	2, routine [**exercise**]
(32)	c2: y≠x	As
(33)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2(\forall y\mathbb{F})$	
(34)		