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Axiom Systems for Classical First-Order Logic

1. Introduction

Having described the syntax and semantics for classical first-order logic, the next task is to offer a deductive account of argument validity in CFOL.

We approach this task in steps. First, we axiomatize classical quantifier logic (CQL), which is a major "fragment" of CFOL, then we axiomatize classical first-order logic. This is not entirely crazy; we have already adopted this strategy in concentrating on the SL-fragment of CFOL in previous chapters. The language for CQL is like the language of CFOL, except that it lacks identity.

We further fragmentize our chief meta-theorems for CQL. First, we prove the results for classical predicate logic (CPL), then we prove the results about CQL. The difference is not the axiom system but the underlying language – CPL is like the language of CQL, except that it lacks function signs [except zero-place, which are functionally identical to proper nouns].

We also adopt three convenient, and natural, simplifications of classical first-order logic (and its fragments).

Simplification #1:	no open formulas in arguments;
Simplification #2:	no open formulas in derivations;
Simplification #3:	no constants in arguments.

Simplification #1 is natural if we agree that, although an open formula $\mathbb{F}$ has a meaning, which contributes to the meaning of any super-formula of $\mathbb{F}$, it does not strictly speaking have a denotation (i.e., truth-value). We can therefore claim that, since validity amounts to truth-preservation, validity does not strictly apply to open argument forms. Simplification #2 is the natural extension of #1. Simplification #3 is natural if we agree that constants are purely intra-derivational devices [∃O, UD, etc.], and accordingly do not appear in any proper argument we wish to analyze. In this connection, recall that constants are <u>not</u> proper nouns, but rather unquantified variables (alternatively, *ad hoc* names).

2. An Axiom System for CQL – AS1+Q

The axiom system we discuss is obtained by taking axiom system AS1 for CSL, and adding rules for quantification, and is accordingly called AS1+Q. Notice that all the rules are restricted to closed formulas of the language $\mathbb{L}$.

AS1 Rules:

In the following, α, β, γ are <u>closed</u> formulas.

(R1) $\hookrightarrow \alpha\rightarrow(\beta\rightarrow\alpha)$
(R2) $\hookrightarrow [\alpha\rightarrow(\beta\rightarrow\gamma)]\rightarrow[(\alpha\rightarrow\beta)\rightarrow(\alpha\rightarrow\gamma)]$
(R3) $\hookrightarrow (\sim\alpha\rightarrow\sim\beta)\rightarrow(\beta\rightarrow\alpha)$
(R4) $\alpha, \alpha\rightarrow\beta \hookrightarrow \beta$

New Rules:

In the following, 𝔽 and 𝔾 are formulas, ν is a variable, c is a constant, and τ is a singular term.

(R5)	↪ ∀ν𝔽 → 𝔽[τ/ν]	τ is a <u>closed</u> singular term; at most ν is free in 𝔽
(R6)	↪ 𝔽 → ∀ν𝔽	𝔽 is closed
(R7)	↪ ∀ν(𝔽→𝔾)→(∀ν𝔽→∀ν𝔾)	at most ν is free in 𝔽 and 𝔾
(R8)	π(𝔽) ↪ ∀ν𝔽[ν/c]	π(𝔽) is a prior sub-sequence that <u>proves</u> 𝔽 [closed]

If one is inclined to give names to rules, the following might be considered.

(R1)	repetition	Rep
(R2)	conditional distribution	→Dist
(R3)	(non-intuitionistic) contraposition	Contra
(R4)	modus ponens	MP
(R5)	universal elimination (out)	∀O
(R6)	trivial quantification	TrivQ
(R7)	universal distribution	∀Dist
(R8)	(provable) generalization	Gen

3. Simple Examples of Derivations <u>in</u> AS1+Q

1. ⊢ " x(Fx ® Fx)

(1)	Fa→(Fa→Fa.→Fa)	R1	Rep
(2)	Fa→(Fa→Fa.→Fa) .→. (Fa→.Fa→Fa)→(Fa→Fa)	R2	→Dist
(3)	(Fa→.Fa→Fa)→(Fa→Fa)	1,2,R4	MP
(4)	(Fa→.Fa→Fa)	R1	Rep
(5)	Fa→Fa	3,4,R4	MP
(6)	∀x(Fx→Fx)	1-5,R8	Gen

In this derivation, first notice that lines 1-5 simply repeat an earlier proof in AS1, where ‘Fa’ is substituted for ‘P’. Notice, in particular, that the sequence 1-5 is a proof of ‘Fa→Fa’ – every line follows by a rule, and the last line is ‘Fa→Fa’. Accordingly, 1-5 can be used in combination with R8 (Gen) to produce line 6.

2. ⊢ " x " y(Rxy ® Rxy)

(1)	Rab→(Rab→Rab.→Rab)	R1	Rep
(2)	Rab→(Rab→Rab.→Rab) .→. (Rab→.Rab→Rab)→(Rab→Rab)	R2	→Dist
(3)	(Rab→.Rab→Rab)→(Rab→Rab)	1,2,R4	MP
(4)	(Rab→.Rab→Rab)	R1	Rep
(5)	Rab→Rab	3,4,R4	MP
(6)	∀y(Ray→Ray)	1-5,R8	Gen
(7)	∀x∀y(Rxy→Rxy)	1-6,R8	Gen

First notice that lines 1-5 are formally similar to Example 1; this time, these lines are a proof of ‘Rab→Rab’. Accordingly, we are entitled to apply R8 (Gen), to obtain ‘∀y(Ray→Ray)’. Next, we notice that lines 1-6 prove ‘∀y(Ray→Ray)’, so 1-6 can be used in combination with R8 (Gen) to produce line 7. [Notice that R8 also entitles us to infer '$\forall y \forall y(Ryy \rightarrow Ryy)$' at line 7; this is a harmless oddity of R8.]

3. {" x(Fx® Gx), Fa} ⊢ Ga

(1)	$\forall x(Fx \rightarrow Gx)$	Pr	
(2)	Fa	Pr	
(3)	$Fa \rightarrow Ga$	1,R5	$\forall$O
(4)	Ga	2,3,R4	MP

4. {" x(Fx® Gx), " xFx} ⊢ " xGx

(1)	$\forall x(Fx \rightarrow Gx)$	Pr	
(2)	$\forall xFx$	Pr	
(3)	$\forall x(Fx \rightarrow Gx) \rightarrow . \forall xFx \rightarrow \forall xGx$	R7	$\forall$Dist
(4)	$\forall xFx \rightarrow \forall xGx$	1,3,R4	MP
(5)	$\forall xGx$	2,4,R4	MP

4. The Deduction Theorem for AS1+Q

The first major theorem is the deduction theorem for AS1+Q. We have already proven DT for AS1. This does not automatically transfer to AS1+Q. The reason is that AS1+Q has a new multi-place rule, R8, which requires us to amend the proof of DT with a special case pertaining to R8 (GEN). In the following proof, which mostly reproduces the proof of DT for AS1 [i.e., lines 1-39], 'GEN[β]' means 'β follows from previous lines by (provable) generalization (R8). The only genuinely new part of the proof is lines 40-43, which employ a new supporting lemma (D4).

DT: **GÈ{a}⊢b ® G⊢a®b**

(1)	~~SHOW~~: $\forall\Gamma\forall\alpha\forall\beta\{\Gamma\cup\{\alpha\}\vdash\beta \longrightarrow \Gamma\vdash\alpha\rightarrow\beta\}$	Def $\vdash$
(2)	~~SHOW~~: $\forall\Gamma\forall\alpha\forall\beta\{\exists d[dD\beta/\Gamma\cup\{\alpha\}] \longrightarrow \Gamma\vdash\alpha\rightarrow\beta\}$	3,QL
(3)	~~SHOW~~: $\forall d\forall\Gamma\forall\alpha\forall\beta\{dD\beta/\Gamma\cup\{\alpha\} \longrightarrow \Gamma\vdash\alpha\rightarrow\beta\}$	4+G14
(4)	~~SHOW~~: $\forall n{:}\forall d\forall\Gamma\forall\alpha\forall\beta\{dD\beta/\Gamma\cup\{\alpha\}/n \longrightarrow \Gamma\vdash\alpha\rightarrow\beta\}$	SMI
	IH:	
(5)	| $\forall k<n{:}\forall d\forall\Gamma\forall\alpha\forall\beta\{dD\beta/\Gamma\cup\{\alpha\}/k \longrightarrow \Gamma\vdash\alpha\rightarrow\beta\}$	As
	IS:	
(6)	| ~~SHOW~~: $\forall d\forall\Gamma\forall\alpha\forall\beta\{dD\beta/\Gamma\cup\{\alpha\}/n \longrightarrow \Gamma\vdash\alpha\rightarrow\beta\}$	U4CD
(7)	| | $dD\beta/\Gamma\cup\{\alpha\}/n$	As
(8)	| | ~~SHOW~~: $\Gamma\vdash\alpha\rightarrow\beta$	SC
(9)	| | $\beta = d_n$	7,Def derives/n
(10)	| | $\forall\delta\in d$: Ax[δ] or $\delta\in\Gamma\cup\{\alpha\}$ or MP[δ] or GEN[β]	7,Def AS1+Q, derives [b]
(11)	| | $\beta\in d$	9,ST
(12)	| | Ax[β] or $\beta\in\Gamma\cup\{\alpha\}$ or MP[β] or GEN[β]	10,11,QL
(13)	| | c1: Ax[b]	As
(14)	| | | $\Gamma\vdash\alpha\rightarrow\beta$	13,D1

(15)	c2: $\beta\in\Gamma\cup\{\alpha\}$	As
(16)	$\beta\in\Gamma$ or $\beta=\alpha$	15,ST
(17)	c1: $\beta\in\Gamma$	As
(18)	$\Gamma\vdash\alpha\to\beta$	17,D2
(19)	c2: $\beta=\alpha$	As
(20)	$\Gamma\vdash\alpha\to\alpha$	D3
(21)	$\Gamma\vdash\alpha\to\beta$	19,20,IL
(22)	c3: MP[β]	As
(23)	$\exists j,k<n\exists\gamma$: $d_j=\gamma\to\beta$ & $d_k=\gamma$	9,22,Def MP[]
(24)	$j<n$ & $d_j=\gamma\to\beta$	23,∃&O
(25)	$k<n$ & $d_k=\gamma$	23,∃&O
(26)	~~SHOW~~: $\langle d_i: i\leqslant j\rangle$ D $\gamma\to\beta$ / $\Gamma\cup\{\alpha\}$ / j	Def D/n [&D]
(27)	a:~~SHOW~~: len$\langle d_i: i\leqslant j\rangle = j$	ST
(28)	b:~~SHOW~~: last$\langle d_i: i\leqslant j\rangle=\gamma\to\beta$	24b,29,IL
(29)	last$\langle d_i: i\leqslant j\rangle = d_j$	ST
(30)	c:~~SHOW~~: $\langle d_i: i\leqslant j\rangle$D$\Gamma\cup\{\alpha\}$	Def dDΓ
(31)	~~SHOW~~: $\forall\delta\in\langle d_i: i\leqslant j\rangle$: Ax[$\delta$] or $\delta\in\Gamma\cup\{\alpha\}$ or MP[δ] or GEN[δ]	UCD
(32)	$\delta\in\langle d_i: i\leqslant j\rangle$	As
(33)	~~SHOW~~: Ax[δ] or $\delta\in\Gamma\cup\{\alpha\}$ or MP[δ] or GEN[δ]	10,34,QL
(34)	$\delta\in d$	32,ST
(35)	~~SHOW~~: $\langle d_i: i\leqslant k\rangle$ D γ / $\Gamma\cup\{\alpha\}$ / j	Def D/n
(36)	similar to derivation lines 26-34	
(37)	$\Gamma\vdash\alpha\to(\gamma\to\beta)$	24a,26,IH
(38)	$\Gamma\vdash\alpha\to\gamma$	25a,35,IH
(39)	$\Gamma\vdash\alpha\to\beta$	37,38,D4
(40)	c4: GEN[β]	As
(41)	$\vdash\beta$	40, D5
(42)	$\vdash\alpha\to\beta$	41, earlier result about AS1 (prefix principle)
(43)	$\Gamma\vdash\alpha\to\beta$	42, GenTh($\vdash$)

5. Lemmas Supporting The Deduction Theorem

The proof of the Deduction Theorem appeals to five lemmas. (D1)–(D4) have already been proved in connection with SL. The remaining one – (D5) – is proven below.

(D1) β is an axiom $\longrightarrow$ $\Gamma\vdash\alpha\to\beta$

(D2) $\beta\in\Gamma$ $\longrightarrow$ $\Gamma\vdash\alpha\to\beta$

(D3) $\Gamma\vdash\alpha\to\alpha$

(D4) $\Gamma\vdash\alpha\to(\gamma\to\beta)$ & $\Gamma\vdash\alpha\to\gamma$.$\longrightarrow$ $\Gamma\vdash\alpha\to\beta$

(D5) GEN[α] $\longrightarrow$ $\vdash\alpha$

(1)	~~SHOW~~: GEN[α] $\longrightarrow$ $\vdash\alpha$	CD
(2)	| GEN[α]	As
(3)	| ~~SHOW~~: $\vdash\alpha$	4, Def($\vdash$)
(4)	| ~~SHOW~~: $\exists\pi$: π proves α	7,QL
(5)	| | $\exists d\exists\pi\exists\mathbb{F}\exists c\exists v\{\pi\subseteq\langle d_i: i<n\rangle$ & π proves $\mathbb{F}$ & $\alpha=\forall v\mathbb{F}[v/c]\}$	2, Def GEN[]
(6)	| | | $\pi_0\subseteq\langle d_i: i<n\rangle$ & π proves $\mathbb{F}$ & $\alpha=\forall v\mathbb{F}[c/v]$	41,$\exists$O
(7)	| | | $\pi_0+\langle\alpha\rangle$ proves α	♮

♮ α is clearly the last line of $\pi_0+\langle\alpha\rangle$, so the question is whether $\pi_0+\langle\alpha\rangle$ is a proof, which is the question whether every line follows by a rule. Let δ be a line in $\pi+\langle\alpha\rangle$. Then either $\delta\in\pi$ or $\delta=\alpha$. In the first case, by hypothesis π is a proof, so δ follows by a rule. In the second case, by hypothesis α – i.e., $\forall v\mathbb{F}[v/c]$ – follows from π by R8, so α follows by a rule.

6. The Universal Derivation Theorem

The next key task is to prove that universal derivation is an admissible rule – a result we call the Universal Derivation Theorem. [We could correspondingly call the Deduction Theorem the Conditional Derivation Theorem, since it demonstrates that conditional derivation is admissible.] Recall that the Universal Derivation show-rule (UD) tells us that showing $\mathbb{F}[c/v]$, where c is *new*, is *tantamount* to showing $\forall v\mathbb{F}$. The following theorem is the axiomatic counterpart of UD. It says that if constant c does not occur in any formula in set Γ, then if one can deduce $\mathbb{F}[c/v]$ from Γ, then one can deduce $\forall v\mathbb{F}$ from Γ. In other words,

if c does not occur in any formula in Γ, then
if $\Gamma\vdash\mathbb{F}[c/v]$, then
$\Gamma\vdash\forall v\mathbb{F}$

In order to simplify our notation in the proof, we employ the following shorthand, where it is understood that c is a constant, and Γ is a set of formulas.

$c\in\gamma$	$=_{df}$	c occurs in γ
$c\in{*}\Gamma$	$=_{df}$	$\exists\gamma\{\gamma\in\Gamma$ & $c\in\gamma\}$
$c\notin{*}\Gamma$	$=_{df}$	$\sim\exists\gamma\{\gamma\in\Gamma$ & $c\in\gamma\}$

Applying this notation, and restoring all the implicit universal quantifiers, the Universal Derivation Theorem can written thus.

UDT: **" G " 𝔽 " c " v{c Ï * G ® . G⊢ 𝔽[c/v] ® G⊢ " v𝔽}**

Note carefully that, just as with the Deduction Theorem (a.k.a. the Conditional Derivation Theorem), the Universal Derivation Theorem does not say that the derivation of $\mathbb{F}[c/v]$ <u>is</u> a derivation of $\forall v\mathbb{F}$; rather it only says that a derivation exists, without saying what the derivation looks like.

The following is a formal proof. As before,

d D ϕ / Γ	$=_{df}$	d is a derivation of ϕ from Γ
d D ϕ / Γ / n	$=_{df}$	d is an n-long derivation of ϕ from Γ

(1)	~~SHOW~~: ∀Γ∀𝔽∀c∀v{c∉*Γ ⟶. Γ⊢𝔽[c/v] ⟶ Γ⊢∀v𝔽}	Def ⊢
(2)	~~SHOW~~: ∀Γ∀𝔽∀c∀v{c∉*Γ ⟶. ∃d[dD𝔽[c/v]/Γ] ⟶ Γ⊢∀v𝔽}	QL
(3)	~~SHOW~~: ∀d∀Γ∀𝔽∀c∀v{c∉*Γ ⟶. dD𝔽[c/v]/Γ ⟶ Γ⊢∀v𝔽}	G14
(4)	~~SHOW~~: ∀n:∀d∀Γ∀𝔽∀c∀v{c∉*Γ ⟶. dD𝔽[c/v]/Γ/n ⟶ Γ⊢∀v𝔽}	SMI
(5)	∀k<n:∀d∀Γ∀𝔽∀c∀v{c∉*Γ ⟶. dD𝔽[c/v]/Γ/k ⟶ Γ⊢∀v𝔽}	As [IH]
(6)	~~SHOW~~: ∀d∀Γ∀𝔽∀c∀v{c∉*Γ ⟶. dD𝔽[c/v]/Γ/n ⟶ Γ⊢∀v𝔽}	U5CCD
(7)	c ∉* Γ	As
(8)	ó D 𝔽[c/v] / Γ / n	As
(9)	~~SHOW~~: Γ⊢∀v𝔽	11-52,SC
(10)	ó$_n$ = 𝔽[c/v]	8,Def Dα/Γ/n
(11)	Ax{𝔽[c/v]} or 𝔽[c/v] ∈ Γ or GEN{𝔽[c/v]} or MP{𝔽[c/v]}	8,10,Def Dα/Γ/n
(12)	c1: Ax{𝔽[c/v]}	As
(13)	⟨𝔽[c/v]⟩ proves 𝔽[c/v]	12, Def Ax, Def proves
(14)	⟨𝔽[c/v], ∀v𝔽⟩ D ∀v𝔽 / Γ	♮

♮ By hypothesis (12), 𝔽[c/v] is an axiom, so the sequence ⟨𝔽[c/v]⟩ proves 𝔽[c/v], so we can apply rule R8 to this sequence to obtain ∀v𝔽[c/v][v/c]. But ∀v𝔽[c/v][v/c] = ∀v𝔽. Thus, the sequence ⟨𝔽[c/v], ∀v𝔽⟩ proves ∀v𝔽. Since every proof is automatically a derivation from any set, we have ⟨𝔽[c/v], ∀v𝔽⟩ derives ∀v𝔽 from Γ.

(15)	∃d{d D ∀v𝔽 / Γ}	14,QL
(16)	Γ⊢∀v𝔽	15, Def ⊢
(17)	c2: 𝔽[c/v] ∈ Γ	As
(18)	~~SHOW~~: v is not free in 𝔽	ID
(19)	v is free in 𝔽	As
(20)	~~SHOW~~: ×	DD
(21)	c ∈ 𝔽[c/v]	19, Def [c/v]
(22)	×	7,17,21,QL
(23)	𝔽 = 𝔽[c/v]	18, Def [c/v]
(24)	𝔽→∀v𝔽 is an instance of R6	18, Def R6
(25)	𝔽[c/v]→∀v𝔽 is an instance of R6	23,24,IL
(26)	⟨𝔽[c/v], 𝔽[c/v]→∀v𝔽, ∀v𝔽⟩ D ∀v𝔽 / Γ	♮

♮ By 17, 𝔽[c/v] ∈ Γ. By 25, 𝔽[c/v]→∀v𝔽 is an instance of R6. By R4 (MP), ∀v𝔽 follows from 𝔽[c/v] and 𝔽[c/v]→∀v𝔽. Thus, the sequence ⟨𝔽[c/v], 𝔽[c/v]→∀v𝔽, ∀v𝔽⟩ derives ∀v𝔽 from Γ.

(27)	∃d{d D ∀v𝔽 / Γ}	26,QL
(28)	Γ⊢∀v𝔽	27, Def ⊢
(29)	c3: GEN{𝔽[c/v]}	As
(30)	⊢ 𝔽[c/v]	29, D5
(31)	𝔽[c/v] ⊢ ∀v𝔽	30, def(Gen)
(32)	⊢ ∀v𝔽	30,31, GenTh(⊢) ♯
(33)	Γ⊢∀v𝔽	32,GenTh(⊢)

(34)	c4: MP{𝔽[c/v]}	As
(35)	∃j,k<n,∃𝔸: d_j=𝔸→𝔽[c/v] & $ó_k$=𝔸	34,Def MP[]
(36)	j<n & $ó_j$=𝔸→𝔽[c/v]	35,∃&O
(37)	k<n & $ó_k$=𝔸	35,∃&O
(38)	~~SHOW~~: ⟨$ó_i$: i⩽j⟩ D 𝔸→𝔽[c/v] / Γ/ j	39,40,42, &I, Def D/n
(39)	a:~~SHOW~~: len⟨$ó_i$: i⩽j⟩ = j	ST
(40)	b:~~SHOW~~: last⟨$ó_i$: i⩽j⟩=𝔸→𝔽[c/v]	36b,40,IL
(41)	last⟨$ó_i$: i⩽j⟩ = $ó_j$	ST
(42)	c:~~SHOW~~: ⟨$ó_i$: i⩽j⟩DΓ	43,44,GenTh(⊢)
(43)	⟨$ó_i$: i⩽j⟩ ⊆ ó	ST
(44)	ó D Γ	8, Def D
(45)	~~SHOW~~: ⟨$ó_i$: i⩽k⟩ D 𝔸 / Γ / k	Def D/n
(46)	similar to derivation 39-45	
(47)	v is not free in 𝔸	37b, Simplification #2
(48)	𝔸[c/v] = 𝔸	47,Def [c/v]
(49)	(𝔸→𝔽)[c/v] = (𝔸[c/v]→𝔽[c/v])	GenSubTh ##
(50)	𝔸→𝔽[c/v] = (𝔸→𝔽)[c/v]	48,49,IL
(51)	⟨$ó_i$: i⩽j⟩ D (𝔸→𝔽)[c/v] / Γ/ j	38,50,IL
(52)	Γ⊢∀v(𝔸→𝔽)	36a,52,5(IH),QL
(53)	𝔸→∀v𝔸 is an instance of R6	47, Def R6
(54)	⟨$ó_i$: i⩽k⟩+⟨𝔸→∀v𝔸⟩+⟨∀v𝔸⟩ D ∀v𝔸 / Γ	51,53,inspection
(55)	∃d{d D ∀v𝔸 / Γ}	54,QL
(56)	Γ⊢∀v𝔸	55, Def ⊢
(57)	Γ⊢∀v𝔽	52,56,UD1###

GenTh(⊢) means 'by a general theorem about ⊢' [There are many such theorems.]
The General Substitution Theorem (GenSubTh) is presented in its own section.

Supporting Lemma:

UD1: G⊢ " v𝔽 & G⊢ " v(𝔽 ® 𝔾) . ® G⊢ " v𝔾

(1)	~~SHOW~~: Γ⊢∀v𝔽 & Γ⊢∀v(𝔽→𝔾) .→ Γ⊢∀v𝔾	&CD
(2)	Γ⊢∀v𝔽	As
(3)	Γ⊢∀v(𝔽→𝔾)	As
(4)	~~SHOW~~: Γ⊢∀v𝔾	DD
(5)	Ax[∀v(𝔽→𝔾)→(∀v𝔽→∀v𝔾)]	R7
(6)	⊢∀v(𝔽→𝔾)→(∀v𝔽→∀v𝔾)	5,G5
(7)	Γ⊢∀v(𝔽→𝔾)→(∀v𝔽→∀v𝔾)	6,G2
(8)	Γ⊢∀v𝔽→∀v𝔾	3,7,MPP
(9)	Γ⊢∀v𝔾	2,8,MPP

7. Exercises for Chapter 16

1. Derivations (and Proofs) in Axiom System AS1+Q

Given a valid argument form, give a derivation of the conclusion from the premises **in** Axiom System AS1+Q.

2. Deduction Theorem for AS1+Q

3. Universal Derivation Theorem

2. Appendix

1. A Very General Substitution Theorem About Semantics of CFOL

We next prove a very general theorem about CFOL. It provides two different corollaries that are important in later proofs.

> Th
>
> Let $\mathbb{F}$ be any formula. Let υ_1 and υ_2 be admissible valuations. Let $\langle x_1, x_2, \dots\rangle$ be a sequence of variables of $\mathbb{L}$, and let $\langle c_1, c_2, \dots\rangle$ be an equally-long sequence of constants of $\mathbb{L}$.
>
> Suppose $\forall i$: $\upsilon_1(x_i) = \upsilon_2(c_i)$.
>
> Suppose $\forall\varepsilon\{\text{Atomic}[\varepsilon] \longrightarrow.\ \forall i[\varepsilon \neq x_i] \longrightarrow \upsilon_1(\varepsilon) = \upsilon_2(\varepsilon)\}$
>
> In other words, υ_1 and υ_2 agree on all symbols except (perhaps) the variables x_1, x_2, … By analogy with our earlier predicate $\approx$, we abbreviate this as follows.
>
> $$\upsilon_1 \approx^* \upsilon_1$$
>
> Then:
>
> $$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$$
>
> where $\varepsilon^* =_{df}$ the result of substituting c_i for x_i in ε, for i=1, 2, …

Proof (by induction on formula formation):

Base Case:

(1)	$\mathbb{F}$ is an atomic formula.	As
(2)	~~SHOW~~: $\forall\upsilon_1, \upsilon_2, x, c\ \{\upsilon_1 \approx^* \upsilon_2 \ \longrightarrow\!\!.\ \forall i\{\upsilon_1(x_i)=\upsilon_2(c_i)\} \longrightarrow \upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)\}$	U4CCD
(3)	$\upsilon_1 \approx^* \upsilon_2$	As
(4)	i.e.: $\forall\varepsilon\{\text{Atomic}[\varepsilon] \longrightarrow\!\!.\ \forall i[\varepsilon \neq x_i] \longrightarrow \upsilon_1(\varepsilon)=\upsilon_2(\varepsilon)\}$	3, Def $\approx^*$
(5)	$\forall i\{\upsilon_1(x_i)=\upsilon_2(c_i)\}$	As
(6)	~~SHOW~~: $\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$	
(7)	$\mathbb{F} = \mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle$	1, Def Atomic formula, ∃O
(8)	$\upsilon_1(\mathbb{F}) = \upsilon_1(\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle) = \upsilon_1(\mathbb{P})\langle\upsilon_1(\tau_1), \ldots, \upsilon_1(\tau_k)\rangle$	7,IL / Def CFOL-val
(9)	$\mathbb{F}^* \quad = \quad [\mathbb{P}\langle\tau_1, \ldots, \tau_k\rangle]^*$	7,IL
(10)	$\qquad = \quad \mathbb{P}\langle\tau_1^*, \ldots, \tau_k^*\rangle$	GenSubTh
(11)	$\upsilon_2(\mathbb{F}^*) \quad = \quad \upsilon_2(\mathbb{P})\langle\upsilon_2(\tau_1^*),\ldots,\upsilon_2(\tau_k^*)\rangle$	9-10,IL,Def CFOL-val
(12)	$[\forall i \leqslant k]$: Atomic$[\tau_i]$	*

*This is ok if we are doing classical predicate logic; however, if we wish to consider classical function logic, then a more general proof is required at this point.

(13)	~~SHOW~~: $[\forall i \leqslant k]$: $\upsilon_1(\tau_k) = \upsilon_2(\tau_k^*)$	SC
(14)	$\exists i[\tau_k = x_i]$ or $\sim\exists i[\tau_k = x_i]$	SL
(15)	c1: $\exists i[\tau_k = x_i]$	As
(16)	$\upsilon_1(\tau_k) \ = \ \upsilon_1(x_i) \ = \ \upsilon_2(c_i)$	5,15,IL
(17)	$\tau_k^* = c_i$	15, Def(ε^*)
(18)	$\upsilon_2(\tau_k^*) \ = \ \upsilon_2(c_i)$	17,IL
(19)	$\upsilon_1(\tau_k) = v_2(\tau_k^*)$	16,18,IL
(20)	c2: $\sim\exists i[\tau_k = x_i]$	As
(21)	$\upsilon_1(\tau_k) = \upsilon_2(\tau_k)$	4,12,20,QL
(22)	$\tau_k^* = \tau_k$	12,20, Def(ε^*)
(23)	$\upsilon_2(\tau_k^*) \ = \ \upsilon_2(\tau_k)$	22,IL
(24)	$\upsilon_1(\tau_k) = v_2(\tau_k^*)$	21,23,IL
(25)	Atomic$[\mathbb{P}]$ & $\mathbb{P} \neq x$	*

* It is presumed that every predicate is atomic, and no predicate is a variable.

(26)	$\upsilon_1(\mathbb{P}) \quad = \quad \upsilon_2(\mathbb{P})$	4,25,QL
(27)	$\upsilon_1(\mathbb{F}) \quad = \quad \upsilon_2(\mathbb{P})\langle\upsilon_1(\tau_1),\ldots,\upsilon_1(\tau_k)\rangle$	8,26,IL
(28)	$\qquad = \quad \upsilon_2(\mathbb{P})\langle\upsilon_2(\tau_1^*),\ldots,\upsilon_2(\tau_k^*)\rangle$	4,12,IL
(29)	$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$	11,27-28,IL

Inductive Case 1 ($\sim$)

Given the form of the formula to be shown, it suffices to do the following conditional derivation.

(1)	$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$	As
(2)	~~SHOW~~: $\upsilon_1(\sim\mathbb{F}) = \upsilon_2([\sim\mathbb{F}][c/x])$	3-7,IL
(3)	$[\sim\mathbb{F}]^* = \sim\mathbb{F}^*$	GenSubTh
(4)	$\upsilon_2([\sim\mathbb{F}]^*) = \upsilon_2(\sim\mathbb{F}^*)$	3,IL
(5)	$\upsilon_1(\sim\mathbb{F}) = \sim\upsilon_1(\mathbb{F})$	Def CFOL-val
(6)	$\sim\upsilon_1(\mathbb{F}) = \sim\upsilon_2(\mathbb{F}^*)$	1,5,IL
(7)	$\upsilon_2(\sim\mathbb{F}^*) = \sim\upsilon_2(\mathbb{F}^*)$	Def CFOL-val

Inductive Case 2 ($\rightarrow$)

Given the form of the formula to be shown, it suffices to show the following.

(1)	$\upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)$	As
(2)	$\upsilon_1(\mathbb{G}) = \upsilon_2(\mathbb{G}^*)$	As
(3)	~~SHOW~~: $\upsilon_1(\mathbb{F}\rightarrow\mathbb{G}) = \upsilon_2([\mathbb{F}\rightarrow\mathbb{G}]^*)$	5-8,IL
(4)	$[\mathbb{F}\rightarrow\mathbb{G}]^* = \mathbb{F}^*\rightarrow\mathbb{G}^*$	GenSubTh
(5)	$\upsilon_2([\mathbb{F}\rightarrow\mathbb{G}]^*) = \upsilon_2(\mathbb{F}^*\rightarrow\mathbb{G}^*)$	4,IL
(6)	$\upsilon_1(\mathbb{F}\rightarrow\mathbb{G}) = \upsilon_1(\mathbb{F})\rightarrow\upsilon_1(\mathbb{G})$	Def CFOL-val
(7)	$\upsilon_1(\mathbb{F})\rightarrow\upsilon_1(\mathbb{G}) = \upsilon_2(\mathbb{F}^*)\rightarrow\upsilon_2(\mathbb{G}^*)$	1,2,6,IL
(8)	$\upsilon_2(\mathbb{F}^*\rightarrow\mathbb{G}^*) = \upsilon_2(\mathbb{F}^*)\rightarrow\upsilon_2(\mathbb{G}^*)$	Def CFOL-val

Inductive Case 3 ($\forall$)

(1)	$\forall\upsilon_1, \upsilon_2\{\upsilon_1 \approx^* \upsilon_2 \;\longrightarrow.\; \forall i\{\upsilon_1(x_i)=\upsilon_2(c_i)\} \;\longrightarrow\; \upsilon_1(\mathbb{F}) = \upsilon_2(\mathbb{F}^*)\}$	As
(2)	~~SHOW~~: $\forall\upsilon_1, \upsilon_2, y\ \{\upsilon_1 \approx^* \upsilon_2 \;\longrightarrow.\; \forall i\{\upsilon_1(x_i)=\upsilon_2(c_i)\} \;\longrightarrow\; \upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)\}$	U2CCD
(3)	$\upsilon_1 \approx^* \upsilon_2$	As
(4)	$\forall i\{\upsilon_1(x_i)=\upsilon_2(c_i)\}$	As
(5)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)$	6-17,SC
(6)	$\exists i[y = x_i]$ or $\sim\exists i[y = x_i]$	SL
(7)	c1: $\exists i[y = x_i]$	As
(8)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)$	7,9,IL
(9)	~~SHOW~~: $\upsilon_1(\forall x_i\mathbb{F}) = \upsilon_2([\forall x_i\mathbb{F}]^*)$	10,11,IL
(10)	$[\forall x_i\mathbb{F}]^* = \forall x_i\mathbb{F}$	y is not free in $\forall y\mathbb{F}$
(11)	$\upsilon_1(\forall x_i\mathbb{F}) = \upsilon_2(\forall x_i\mathbb{F})$	3,Lemma**
(12)	c2: $x \neq y$	As
(13)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2([\forall y\mathbb{F}]^*)$	14,15,IL
(14)	$[\forall y\mathbb{F}]^* = \forall y\mathbb{F}^*$	12, Lemma**
(15)	~~SHOW~~: $\upsilon_1(\forall y\mathbb{F}) = \upsilon_2(\forall y\mathbb{F}^*)$	16,17,18,GenTh(υ),IL
(16)	$\upsilon_1(\forall y\mathbb{F}) = T \leftrightarrow \forall\upsilon\{\upsilon \approx_y \upsilon_1 \rightarrow \upsilon(\mathbb{F}) = T\}$	Def CFOL-val (alt)
(17)	$\upsilon_2(\forall y\mathbb{F}^*) = T \leftrightarrow \forall\upsilon\{\upsilon \approx_y \upsilon_2 \rightarrow \upsilon(\mathbb{F}^*) = T\}$	Def CFOL-val (alt)
(18)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_y \upsilon_1 \rightarrow \upsilon(\mathbb{F}) = T\} \leftrightarrow \forall\upsilon\{\upsilon \approx_y \upsilon_2 \rightarrow \upsilon(\mathbb{F}^*) = T\}$	19,48,SL

(19)	~~SHOW~~: $\rightarrow$	CD
(20)	$\forall\upsilon\{\upsilon \approx_y \upsilon_1 \rightarrow \upsilon(\mathbb{F}) = T\}$	As
(21)	~~SHOW~~: $\forall\upsilon\{\upsilon \approx_y \upsilon_2 \rightarrow \upsilon(\mathbb{F}^*) = T\}$	UCD
(22)	$\upsilon_3 \approx_y \upsilon_2$	As
(23)	~~SHOW~~: $\upsilon_3(\mathbb{F}^*) = T$	
(24)	let $\upsilon_4(y) = \upsilon_3(y)$ & $\upsilon_4(\varepsilon) = \upsilon_1(\varepsilon)$ if $\varepsilon \neq y$ and Atomic[ε]	ST,$\exists$O
(25)	$\upsilon_4 \approx_y \upsilon_1$	24b, Def($\approx$)
(26)	$\upsilon_4(\mathbb{F}) = T$	20,25,QL
(27)	~~SHOW~~: $\upsilon_4 \approx_x \upsilon_3$	Def($\approx$)
(28)	~~SHOW~~: $\forall\varepsilon\{$Atomic[ε] $\rightarrow.$ $\varepsilon \neq x \rightarrow \upsilon_4(\varepsilon) = \upsilon_3(\varepsilon)\}$	UCCD
(29)	Atomic[ε]	As
(30)	$\varepsilon \neq x$	As
(31)	~~SHOW~~: $\upsilon_4(\varepsilon) = \upsilon_3(\varepsilon)$	SC,32-39
(32)	$\varepsilon = y$ or $\varepsilon \neq y$	SL
(33)	c1: $\varepsilon = y$	As
(34)	$\upsilon_4(\varepsilon) = \upsilon_4(y) = \upsilon_3(y) = \upsilon_3(\varepsilon)$	33,IL / 24a,IL / IL
(35)	c2: $\varepsilon \neq y$	As
(36)	$\upsilon_4(\varepsilon) = \upsilon_1(\varepsilon)$	25,29,35,Def($\approx$)
(37)	$\upsilon_1(\varepsilon) = \upsilon_2(\varepsilon)$	3,29,30,Def($\approx$)
(38)	$\upsilon_2(\varepsilon) = \upsilon_3(\varepsilon)$	22,29,35,Def($\approx$)
(39)	$\upsilon_4(\varepsilon) = \upsilon_3(\varepsilon)$	36-38,IL
(40)	~~SHOW~~: $\upsilon_4(x) = \upsilon_3(c)$	42,43,45,IL
(41)	$x \neq y$	12 (reminder)
(42)	$\upsilon_4(x) = \upsilon_1(x)$	24b,41
(43)	$\upsilon_1(x) = \upsilon_2(c)$	4 (reminder)
(44)	Atomic[c] & $c \neq y$	presumed
(45)	$\upsilon_2(c) = \upsilon_3(v)$	22,44,Def($\approx$)
(46)	$\upsilon_4(\mathbb{F}) = \upsilon_3(\mathbb{F}^*)$	1(IH),27,40,QL
(47)	$\upsilon_3(\mathbb{F}^*)$	26,46,IL

(48)	~~SHOW~~: $\leftarrow$	CD
	Proof is very similar to 19-47 [exercise]	