
15

Validity in Classical First-Order Logic

1.	Introduction.....	2
2.	Argument Validity.....	3
3.	An Intermediate Notion between Truth and Validity.....	3
4.	Closures of Formulas.....	5
5.	Examples of Valid/Invalid Formulas in CFOL	5
6.	Examples of Valid/Invalid Arguments in CFOL	6
7.	Examples of Proofs about Validity	6
8.	Definitions and Lemmas	8
2.	Study Questions for Chapters 13-15	9
1.	Syntax of Simple First-Order Languages; Definitions.....	9
2.	Syntax of Simple First-Order Languages; Examples.....	9
3.	Semantics of Simple First-Order Languages; Definitions	9
4.	Semantics of Simple First-Order Languages; Examples	10
5.	Validity in Classical First-Order Logic.....	10
6.	Answers.....	11

1. Introduction

Having defined the class of admissible valuations for CFOL, we can now define semantic validity and semantic entailment in the context of CFOL. Recall the following definitions.

1. **a is semantically valid**

$$\models \alpha \quad =_{\text{df}} \quad \forall v \{v \in V \rightarrow v(\alpha) = T\}$$

2. **a is semantically contra-valid**

$$\alpha \models \quad =_{\text{df}} \quad \sim \exists v \{v \in V \ \& \ v(\alpha) = T\}$$

3. **G is unfalsifiable**

$$\models \Gamma \quad =_{\text{df}} \quad \sim \exists v \{v \in V \ \& \ \forall x (x \in \Gamma \rightarrow v(x) = F)\}$$

4. **G is unverifiable**

$$\Gamma \models \quad =_{\text{df}} \quad \sim \exists v \{v \in V \ \& \ \forall x (x \in \Gamma \rightarrow v(x) = T)\}$$

5. **G semantically entails a**

$$\Gamma \models \alpha \quad =_{\text{df}} \quad \forall v \{v \in V \rightarrow . \forall x (x \in \Gamma \rightarrow v(x) = T) \rightarrow v(\alpha) = T\}$$

6. **G semantically entails D**

$$\begin{aligned} \Gamma \models \Delta &\quad =_{\text{df}} \quad \forall v \{v \in V \rightarrow . \forall x (x \in \Gamma \rightarrow v(x) = T) \rightarrow \exists x (x \in \Delta \ \& \ v(x) = T)\} \\ &\quad \leftrightarrow \quad \sim \exists v \{v \in V \ \& \ \forall x (x \in \Gamma \rightarrow v(x) = T) \ \& \ \forall x (x \in \Delta \rightarrow v(x) = F)\} \end{aligned}$$

Recall that the double turnstile ‘ \models ’ is used ambiguously, although all notions can be reduced to the 5th notion, as follows.

$$\models \alpha \quad \leftrightarrow \quad \emptyset \models \{\alpha\}$$

$$\alpha \models \quad \leftrightarrow \quad \{\alpha\} \models \emptyset$$

$$\models \Gamma \quad \leftrightarrow \quad \emptyset \models \Gamma$$

$$\Gamma \models \quad \leftrightarrow \quad \Gamma \models \emptyset$$

$$\Gamma \models \alpha \quad \leftrightarrow \quad \Gamma \models \{\alpha\}$$

2. Argument Validity

At this point, we define the obvious correlated notion of a valid argument, as follows.

(D) An argument is an ordered pair (Γ/α) , where Γ is a set of formulas, and α is a formula.

(D) (Γ/α) is valid $\equiv_{df} \Gamma \models \alpha$.

(t) terminology:

Γ are individually called premises of (Γ/α) ;
 α is called the conclusion of (Γ/α) .

(n) notation:

$(\alpha_1; \alpha_2; \dots; \alpha_n/\beta) \equiv_{df} (\{\alpha_1, \alpha_2, \dots, \alpha_n\}/\beta)$

3. An Intermediate Notion between Truth and Validity

Every CFOL valuation v is uniquely determined by a pair α/ω , where α is an interpretation function, and ω is an assignment function. We describe this functional relation between α/ω and v simply as follows.

$$v = \text{val}(\alpha/\omega)$$

Given this functional relation, we can define a multi-place truth-predicate as follows.

$$\alpha \text{ is true in } \alpha/\omega \equiv_{df} v(\alpha)=T \quad [\text{where } v = \text{val}(\alpha/\omega)]$$

Since the assignment function ω is to a large extent arbitrary, we can “supervaluate” over ω to obtain the notion of truth-in- α , also called validity-in- α . We shall reserve the concept of truth for valuations, so we use the latter term in this connection. It is officially defined as follows.

$$\alpha \text{ is valid in } \alpha \equiv_{df} \alpha \text{ is true in } \alpha/\omega \text{ for any assignment function } \omega.$$

In other words, α is valid in α if and only if α is true in α irrespective of the respective semantic values of its free variables. For example,

$$\text{'Rxy' is valid in } \alpha$$

if and only if ‘Rxy’ is true no matter how we assign objects in U to ‘x’ and ‘y’.

The following are examples.

Let L be a quantified language with two non-logical signs as follows:

R , a 2-place predicate;
 p , a proper noun;

Let $U = \{0,1\}$;
Let $\alpha(R) = \{\langle 0,1 \rangle, \langle 1,1 \rangle\}$;
Let $\alpha(p) = 1$

- e1: $\alpha = 'Rxp'$; α is valid in α ;
- e2: $\alpha = 'Rpx'$; α is not valid in α ;
- e3: $\alpha = '\exists y Rxy'$; α is valid in α ;
- e4: $\alpha = '\exists y Ryx'$; α is not valid in α ;

For the sake of illustration, we do e1, leaving the remaining three as exercises; to avoid clutter we occasionally drop quotes.

(1)	$U = \{0,1\}$	Pr.
(2)	$\alpha(R) = \{\langle 0,1 \rangle, \langle 1,1 \rangle\}$	Pr.
(3)	$\alpha(p) = 1$	Pr.
(4)	SHOW: ' Rxp ' is valid in α	Def valid in α
(5)	SHOW: $\forall \omega['Rxp' is true in \alpha/\omega]$	UD
(6)	SHOW: ' Rxp ' is true in α/ω	Def of true in α/ω
(7)	$v = val(\alpha/\omega)$	As
(8)	SHOW: $v(Rxp) = T$	15-25,SC
(9)	$v(Rxp) = v(R)\langle v(x), v(p) \rangle$	category requirement
(10)	$v(R) = I(R) = \{\langle 0,1 \rangle, \langle 1,1 \rangle\}$	Def CFOL-val, 2, 7
(11)	$v(x) = \omega(x)$	Def CFOL-val, 7
(12)	$v(p) = \alpha(p) = 1$	Def CFOL-val, 3,7
(13)	$\omega(x) \in U$	Def assignment
(14)	$\omega(x) \in \{0,1\}$	1,13,IL
(15)	$\omega(x)=0$ or $\omega(x)=1$	14, ST
(16)	c1: $\omega(x)=0$	As.
(17)	$\langle 0,1 \rangle \in \{\langle 0,1 \rangle, \langle 1,1 \rangle\}$	ST
(18)	$\langle v(x), v(p) \rangle \in v(R)$	10-12,16,17,IL
(19)	$v(R)\langle v(x), v(p) \rangle = T$	18, S/F convention*
(20)	$v(Rxp) = T$	9,19,IL
(21)	c2: $\omega(x)=1$	As.
(22)	$\langle 1,1 \rangle \in \{\langle 0,1 \rangle, \langle 1,1 \rangle\}$	ST
(23)	$\langle v(x), v(p) \rangle \in v(R)$	10-12,21,22,IL
(24)	$v(R)\langle v(x), v(p) \rangle = T$	23, S/F convention*
(25)	$v(Rxp) = T$	9,24,IL

*The S/F [set/function] convention is the convention that allows us to treat a subset S of U and its associated characteristic function as interchangeable; thus, $S(u)=T$ if $u \in S$; $S(u)=F$ if $u \notin S$.

4. Closures of Formulas

When we say ‘ Fx ’ is valid in α , we are saying that no matter how we interpret ‘ x ’, the formula ‘ Fx ’ is true in α . Of course, that is also how we understand the truth of formula ‘ $\forall x Fx$ ’ in α . The latter formula is referred to as the (universal) closure of ‘ Fx ’. The following is our general definition.

- (D) Let L be a FOL, and let \mathbb{F} be a formula of L . Then by *a closure* of \mathbb{F} is meant any formula \mathbb{F}' such that $\mathbb{F}' = \forall v_1, \dots, \forall v_k \mathbb{F}$ (where v_1, \dots, v_k are variables), and such that no variable occurs free in \mathbb{F}' .

This defines the notion of *a* closure. We can also define the notion of *the* closure, if we wish.

- (D) Let L be a FOL, and let \mathbb{F} be a formula of L . Then by *the closure* of \mathbb{F} is meant the formula $\forall v_1, \dots, \forall v_k \mathbb{F}$ such that v_1, \dots, v_k are ordered lexicographically and are all the free variables in \mathbb{F} .

Note: we mean to include the degenerate case in which $k=0$. This means that we count any closed formula as *the* closure of itself, and hence *a* closure of itself.

Examples:

- $\forall x Fx$ is a closure of Fx and the closure of Fx
- $\forall y \forall x Fx$ is a closure of Fx but not the closure of Fx

In relation to the notion(s) of closure, we have the following theorem.

- (T) \mathbb{F} is valid in $\alpha \iff$ every closure of \mathbb{F} is true in α
 \mathbb{F} is valid in $\alpha \iff$ the closure of \mathbb{F} is true in α

5. Examples of Valid/Invalid Formulas in CFOL

In the present section, we consider examples of semantically valid and invalid formulas in the context of CFOL. First, we note the following theorem about the relation between semantic validity and validity in an interpretation.

- (T) α is valid $\iff \alpha$ is valid in α for any interpretation function α .

Proof: left as an exercise for the student.

Examples:

Valid Formulas:

- (1) $Fx \rightarrow Fx; Rxy \rightarrow Rxy; \text{etc.}$
- (2) $Fx \rightarrow \sim \sim Fx; Rxy \rightarrow \sim \sim Rxy; \text{etc.}$
- (3) $\forall x(Fx \rightarrow Fx); \forall y \forall x(Fx \rightarrow Fx); \text{etc.}$
- (4) $\forall x(Fx \rightarrow \sim \sim Fx); \forall y \forall x(Fx \rightarrow \sim \sim Fx); \text{etc.}$
- (5) $\forall x(Fx \rightarrow \exists y Fy)$
- (6) $\exists x(Fx \rightarrow \forall y Fy)$
- (7) $\exists x \forall y Rxy \rightarrow \forall y \exists x Rxy$

Invalid Formulas:

- (1) $Fx \rightarrow \sim Fy$
- (2) $Fx \rightarrow Fy$
- (3) $Fx \rightarrow \sim \sim Fy$
- (4) $Fx \rightarrow \forall x Fx$
- (5) $\exists x Fx \rightarrow Fx$
- (6) $\forall x \exists y Rxy \rightarrow \exists y \forall x Rxy$

6. Examples of Valid/Invalid Arguments in CFOL

In the present section, we examine examples of valid arguments of CFOL.

Valid Arguments:

- (1) $Fx \rightarrow Gx ; Fx / Gx$
- (2) $Fx \rightarrow Gx ; \sim Gx / \sim Fx$
- (3) $\forall x Fx / Fy$
- (4) $Fx / \exists y Fy$
- (5) $\forall x(Fx \rightarrow Gx) ; Fy / Gy$

Invalid Arguments:

- (1) $Fx \rightarrow Gx ; Gx / Fx$
- (2) $Fx \rightarrow Gx ; \sim Fx / \sim Gx$
- (3) $Fx / \forall y Fy$
- (4) $\exists x Fx / Fy$
- (5) $\forall x Fx \rightarrow \forall x Gx / \forall x(Fx \rightarrow Gx)$

7. Examples of Proofs about Validity

Note: we generally drop single quotes in metalinguistic literal names.

#1:

- | | | |
|-----|--|-------------------|
| (1) | $\text{SHOW: } \models Fx \rightarrow Fx$ | Def valid |
| (2) | $\text{SHOW: } \forall v[v(Fx \rightarrow Fx) = T]$ | UD |
| (3) | $\text{SHOW: } v(Fx \rightarrow Fx) = T$ | 4,5,IL |
| (4) | $\quad \quad v(Fx \rightarrow Fx) = v(Fx) \rightarrow v(Fx)$ | Def CFOL-val |
| (5) | $\quad \quad \forall p[p \rightarrow p = T]$ | Def \rightarrow |
| | [‘ p ’ ranges over truth-values] | |

#2:

(1)	SHOW: $\#Fx \rightarrow \forall x Fx$	Def $\vDash (\neg)$
(2)	SHOW: $\sim \forall v [v(Fx \rightarrow \forall x Fx) = T]$	3, L01,IL
(3)	SHOW: $\exists v [v(Fx \rightarrow \forall x Fx) = F]$	22,QL
(4)	let: $U = \{0,1\}$ & $0 \neq 1$	ST+ $\exists O$
(5)	let: $v_0(F) = \{1\}$	ST+ $\exists O$
(6)	let: $\forall v [v(v) = 1]$	ST+ $\exists O$
(7)	$v_0(Fx \rightarrow \forall x Fx) = v_0(Fx) \rightarrow v_0(\forall x Fx)$	Def CFOL-val
(8)	$v_0(x) \in v_0(F)$	5,6,ST,IL
(9)	$v_0(F)(v_0(x)) = T$	8, S/F convention
(10)	$v_0(Fx) = v_0(F)(v_0(x))$	Def CFOL-val
(11)	$v_0(Fx) = T$	9,10,IL
(12)	let: $v_1(x) = 0$ & $\forall \varepsilon \{\text{Atomic}[\varepsilon] \& \varepsilon \neq 'x' . \rightarrow v_1(\varepsilon) = v_0(\varepsilon)\}$	ST+ $\exists O$
(13)	$v_1 \approx_x v_0$	12b, Def \approx
(14)	Atomic['F'] & 'F' \neq 'x'	literal inspection, Def Atomic
(15)	$v_1(F) = v_0(F)$	12,14,QL
(16)	$v_1(Fx) = v_1(F)(v_1(x))$	Def CFOL-val
(17)	$v_1(Fx) = \{1\}\langle 0 \rangle$	5,12a,16,IL
(18)	$\{1\}\langle 0 \rangle = F$	ST + S/F convention
(19)	$v_1(Fx) = F$	17,18,IL
(20)	$\sim \forall v' \{v' \approx_x v_0 \rightarrow v'(Fx) = T\}$	13,19,L01,QL
(21)	$v_0(\forall x Fx) = F$	20, Def CFOL-val
(22)	$v_0(Fx \rightarrow \forall x Fx) = T \rightarrow F = F$	7,11,21,Def \rightarrow ,IL

Recall that a ‘let’ hypothesis is a combination of a (presumed to be true) existential formula, and an application of existential-elimination. In the formal context, the ‘let’ is superfluous, given the annotation. Similarly, in the informal context, the annotation is superfluous, given the presence of ‘let’. We include both for maximal clarity.

#3:

(1)	SHOW: $(\forall x Fx/Fy)$ is valid	Def (Γ/α) is valid
(2)	SHOW: $\{\forall x Fx\} \vDash Fy$	L02
(3)	SHOW: $\forall x Fx \vDash Fy$	Def $\alpha \vDash \beta$
(4)	SHOW: $\forall v \{v(\forall x Fx) = T\} \rightarrow v(Fy) = T\}$	UCD
(5)	$v(\forall x Fx) = T$	As
(6)	SHOW: $v(Fy) = T$	DD
(7)	$\forall v' \{v' \approx_x v \rightarrow v'(Fx) = T\}$	5, Def CFOL-val
(8)	let: $\forall \varepsilon \{\text{Atomic}[\varepsilon] \& \varepsilon \neq 'x' . \rightarrow v_1(\varepsilon) = v(\varepsilon)\} \& v_1(x) = v(y)$	ST+ $\exists O$
(9)	$v_1 \approx_x v$	8a, Def \approx
(10)	$v_1(Fx) = T$	7,9,QL
(11)	$v_1(Fx) = v_1(F)(v_1(x))$	Def CFOL-val
(12)	$v(Fy) = v(F)(v(y))$	Def CFOL-val
(13)	Atomic['F'] & 'F' \neq 'x'	literal inspection, Def Atomic
(14)	$v_1(F) = v(F)$	8a,13,QL
(15)	$v(F)(v(y)) = v_1(F)(v_1(x))$	8b,14,IL
(16)	$v(F)(v(y)) = T$	10,11,15,IL
(17)	$v(Fy) = v(F)(v(y))$	Def CFOL-val
(18)	$v(Fy) = T$	16,17,IL

8. Definitions and Lemmas

Def CFOL-val:

Where \mathbb{L} is an FOL, a CFOL admissible valuation (semantic evaluation) is a function v that assigns semantic items of U to the syntactic items of \mathbb{L} , subject to the following conditions.

$$\begin{aligned} N^* &= U \\ S^* &= V \\ \text{if } K = (K_1, \dots, K_m \rightarrow K_0), \text{ then } K^* &= (K_1^*, \dots, K_m^* \rightarrow K_0^*) \\ \forall \varepsilon \{ \text{cat}(\varepsilon) = K \rightarrow \text{cat}(v(\varepsilon)) = K^* \} \\ \forall \phi \forall \varepsilon_1 \dots \forall \varepsilon_k \{ v[\phi(\varepsilon_1, \dots, \varepsilon_k)] &= v(\phi)(v(\varepsilon_1), \dots, v(\varepsilon_k)) \} \end{aligned}$$

except for \forall and \exists , which are treated as follows.

$$\begin{aligned} v(\forall v \mathbb{F}) &= \min\{v'(\mathbb{F}): v' \approx_v v\} \\ v(\exists v \mathbb{F}) &= \max\{v'(\mathbb{F}): v' \approx_v v\} \end{aligned}$$

alternatively:

$$\begin{array}{ll} v(\forall v \mathbb{F}) = T & \text{iff } \forall v' \{ v' \approx_v v \rightarrow v'(\mathbb{F}) = T \} \\ v(\forall v \mathbb{F}) = F \text{ iff } & \sim \forall v' \{ v' \approx_v v \rightarrow v'(\mathbb{F}) = T \} \\ v(\exists v \mathbb{F}) = T \text{ iff } & \exists v' \{ v' \approx_v v \ \& \ v'(\mathbb{F}) = T \} \\ v(\exists v \mathbb{F}) = F \text{ iff } & \sim \exists v' \{ v' \approx_v v \ \& \ v'(\mathbb{F}) = T \} \end{array}$$

Def \approx : $v_1 \approx_v v_2 \quad =_{df} \quad \forall \varepsilon \{ \text{Atomic}[\varepsilon] \ \& \ \varepsilon \neq v \ . \rightarrow v_1(\varepsilon) = v_2(\varepsilon) \}$

Various:	$\text{Atomic}[\varepsilon] \quad =_{df} \quad \varepsilon$ is atomic; i.e., ε has no (grammatically relevant) proper parts
	$\text{Prop}[\varepsilon] \quad =_{df} \quad \varepsilon$ is a proper symbol of \mathbb{L} , which is to say an atomic non-logical symbol [in FOL, a proper noun, function sign, or predicate other than ' $=$ '].
	$\text{Con}[\varepsilon] \quad =_{df} \quad \varepsilon$ is a constant
	$\text{Var}[\varepsilon] \quad =_{df} \quad \varepsilon$ is a variable

Def \circledast : $T \rightarrow T = T; T \rightarrow F = F; F \rightarrow T = T; F \rightarrow F = T.$

S/F convention:

The S/F [set/function] convention is the convention that allows us to treat a subset S of U and its associated characteristic function as notationally interchangeable; thus:

$$\begin{aligned} S(u) = T &\text{ iff } u \in S; \\ S(u) = F &\text{ iff } u \notin S. \end{aligned}$$

Literal Inspection:

By a *literal name* is meant a (literal) object language expression surrounded by single quotes. If a noun phrase \square is a *literal name*, then the lexical material between the quotes is a literal representation of the referent, $\text{ref}(\square)$. Accordingly, \square can be directly inspected in order to determine facts about $\text{ref}(\square)$.

L01: " u " $a \{ u(a) = T \text{ xor } u(a) = F \}$

L02: { " $x F x$ } $\vdash Fy \quad \Leftarrow \quad " x F x \vdash Fy$

2. Study Questions for Chapters 13-15

1. Syntax of Simple First-Order Languages; Definitions

Describe the rules of formation for a simple first-order language, including reference to the following:

- (1) singular terms
 - (a) atomic singular terms
 - (b) molecular singular terms

- (2) formulas
 - (a) atomic formulas
 - (b) molecular formulas

You need not explicitly specify the vocabulary.

2. Syntax of Simple First-Order Languages; Examples

For each of the following strings, say whether it is a singular term, a formula, or is ill-formed, relative to the *official* (orthographically explicit) language of CFOL.

- (1) $\forall x\#P\flat\flat x\#$
- (2) $(\forall xP \rightarrow P\#)$
- (3) $f\flat\#\#f\flat\flat\#x\#a\#$
- (4) $\forall xf\flat\flat f\flat x a$

In this connection, recall the following:

$x, x\#, x\#\#, \dots$, etc., are variables;
 $a, a\#, a\#\#, \dots$, etc., are constants;
 $f, f\#\#, \dots$, etc., $f\flat, f\flat\#\#, f\flat\#\#\#$, etc. are function signs;
 $P, P\#\#, \dots$, etc., $P\flat, P\flat\#\#, P\flat\#\#\#$, etc. are predicates.

3. Semantics of Simple First-Order Languages; Definitions

Define the following:

semantic evaluation
 truth-valuation
 interpretation
 assignment function
 designation function

4. Semantics of Simple First-Order Languages; Examples

Consider a simple FOL with the following non-logical vocabulary.

- R: a 2-place predicate
- F: a 1-place predicate
- \square : a proper noun

Consider the following universe, interpretation, and assignment.

- (a) $U = \{0,1\}$
- (b) $\alpha(R) = \{\langle 0,0 \rangle, \langle 0,1 \rangle\}$
 $\alpha(F) = \{1\}$
 $\alpha(\square) = 0$
- (c) $\omega(v) = 1$, for every variable v

For each of the following formulas, say whether it is true or false relative to α/ω .

- (1) $R\square y$
- (2) $\forall x(\sim Fx \rightarrow \forall y Rxy)$

5. Validity in Classical First-Order Logic

1. Demonstrate that the following argument is invalid by constructing a counterexample; you need merely specify the universe and interpretation.

$$\forall x Fx \rightarrow \forall x Gx ; Fa / Ga$$

$$\begin{aligned} U &=? \\ a &=? \end{aligned}$$

2. Prove that the following argument forms are valid.

$$\begin{aligned} \forall x Fx / Fa \\ \forall x(Fx \rightarrow Gx) / \forall x Fx \rightarrow \forall x Gx \\ Fa / \forall x Fa \end{aligned}$$

3. Prove the following [where ν maps \mathbb{L} into U]

$$\begin{aligned} \nu(\forall x Fx) = T &\leftrightarrow \forall u \{u \in U \rightarrow u \in \nu(F)\} \\ \nu(\exists x Fx) = T &\leftrightarrow \exists u \{u \in U \& u \in \nu(F)\} \end{aligned}$$

4. Prove that the following argument forms are invalid.

$$\begin{aligned} \forall x Fx \rightarrow \forall x Gx / \forall x(Fx \rightarrow Gx) \\ \exists x Fx ; \exists x Gx / \exists x(Fx \& Gx) \\ \exists x Fx / \forall x Fx \end{aligned}$$

6. Answers

1. Syntax of CQL; Definitions

(1) singular terms:

(a) atomic singular terms:

1. every variable is an atomic singular term;
2. every constant is an atomic singular term;
3. every proper noun is an atomic singular term;
4. nothing else is an atomic singular term.

(b) singular terms:

1. every atomic singular term is a singular term;
2. if ω is an n-place function sign, and τ_1, \dots, τ_n are singular terms, then $\omega(\tau_1, \dots, \tau_n)$ is a singular term;
3. nothing else is a singular term.

(2) formulas:

(a) atomic formulas:

1. if P is an n-place predicate, other than '=', and τ_1, \dots, τ_n are singular terms, then $P(\tau_1, \dots, \tau_n)$ is an atomic formula;
2. if τ_1 and τ_2 are singular terms, then $[\tau_1 = \tau_2]$ is an atomic formula;
3. nothing else is an atomic formula.

(b) formulas

1. every atomic formula is a formula;
2. if F and G are formulas, then so are $(F \rightarrow G)$, $(F \vee G)$, $(F \& G)$, $(F \leftrightarrow G)$, $\sim F$;
3. if F is a formula, and v is a variable, then $\forall v F$ and $\exists v F$ are formulas;
4. nothing else is a formula.

2. Syntax of CQL; Examples

(1)	$\forall x\#P\#\#x\#$	ill-formed
(2)	$(\forall x P \rightarrow P\#)$	formula
(3)	$f\#\#f\#\#x\#a\#$	term
(4)	$\forall x f\#\#f\#\#x\#a\#$	ill-formed

3. Semantics of CFOL; Definitions

semantic evaluation on \mathbb{L}	assigns a semantic value to every	grammatical expression of \mathbb{L} ;
truth-valuation on \mathbb{L}	assigns a truth-value to every	formula of \mathbb{L} ;
interpretation on \mathbb{L}	assigns a semantic value to every	proper symbol of \mathbb{L} ;
assignment function on \mathbb{L}	assigns an element of U to every	constant and variable;
designation function on \mathbb{L}	assigns an element of U to every	singular term of \mathbb{L} .

4. Semantics of CFOL; Examples

- (1) True
- (2) True

5. Validity in CFOL

1.

$$\begin{aligned} U &= \{0,1\} \\ \alpha(F) &= \{0\}; \alpha(G) = \{1\} \\ \omega(x) &= 0; \omega(a) = 0 \end{aligned}$$

In the following, we assume ‘P’ is a one-place predicate. Usually, literal strings are written without their quotes, in order to avoid clutter.

3.1a:

(1)	$v(\forall xPx) = T$	As
(2)	SHOW: $\forall u \{u \in U \rightarrow u \in v(P)\}$	UCD
(3)	$u \in U$	As
(4)	SHOW: $u \in v(P)$	ID
(5)	$u \notin v(P)$	As
(6)	SHOW: \times	10,14,15,IL
(7)	$\forall v' \{v' \approx_x v \rightarrow v'(Px) = T\}$	1,Def CFOL-val
(8)	let: $v_0(x) = u \& \forall \varepsilon \{\text{Atomic}[\varepsilon] \& \varepsilon \neq x \rightarrow v_0(\varepsilon) = v(\varepsilon)\}$	ST+ $\exists O$
(9)	$v_0 \approx_x v$	8b, Def \approx
(10)	$v_0(Px) = T$	7,9,QL
(11)	Atomic[‘P’] & ‘P’ ≠ ‘x’	inspection, Def At[]
(12)	$v_0(P) = v(P)$	8b,11,QL
(13)	$v_0(Px) = v_0(P) \langle v_0(x) \rangle$	Def CFOL-val
(14)	$v_0(Px) = v(P) \langle u \rangle$	8a,12,13,IL
(15)	$v(P) \langle u \rangle = F$	5, S/F convention

3.1b:

(1)	$\forall u \{ u \in U \rightarrow u \in v(P) \}$	As
(2)	SHOW: $v(\forall x Px) = T$	Def CFOL-val
(3)	SHOW: $\forall v' \{ v' \approx_x v \rightarrow v'(Px) = T \}$	UCD
(4)	$v_0 \approx_x v$	As
(5)	SHOW: $v_0(Px) = T$	ID
(6)	$v_0(Px) \neq T$	As
(7)	SHOW: \times	14,16,SL
(8)	$v_0(Px) = F$	6,LO
(9)	$v_0(Px) = v_0(P) \langle v_0(x) \rangle$	Def CFOL-val
(10)	$v_0(P) \langle v_0(x) \rangle = F$	8,9,IL
(11)	$v_0(x) \notin v_0(P)$	10, S/F convention
(12)	Atomic['P'] & 'P' ≠ 'x'	inspection, Def At[]
(13)	$v_0(P) = v(P)$	4,12,Def ≈
(14)	$v_0(x) \notin v(P)$	11,14,IL
(15)	$v_0(x) \in U$	Def CFOL-val
(16)	$v_0(x) \in v(P)$	1,15,QL

3.1b:

(1)	$\forall u \{ u \in U \rightarrow u \in v(P) \}$	As
(2)	SHOW: $v(\forall x Px) = T$	Def CFOL-val
(3)	SHOW: $\forall v' \{ v' \approx_x v \rightarrow v'(Px) = T \}$	UCD
(4)	$v_0 \approx_x v$	As
(5)	SHOW: $v_0(Px) = T$	6,11,SL
(6)	$v_0(Px) = T \leftrightarrow v_0(x) \in v_0(P)$	Def CFOL-val
(7)	$v_0(x) \in U$	Def CFOL-val
(8)	$v_0(x) \in v(P)$	1,7,QL
(9)	Atomic['P'] & 'P' ≠ 'x'	inspection, Def At[]
(10)	$v_0(P) = v(P)$	4,9,Def ≈
(11)	$v_0(x) \in v_0(P)$	8,10,IL

3.2a:

(1)	$v(\exists x Px) = T$	As
(2)	SHOW: $\exists u \{ u \in U \ \& \ u \in v(P) \}$	12,13,QL
(3)	$\exists v' \{ v' \approx_x v \ \& \ v'(Px) = T \}$	1, Def CFOL-val
(4)	$v_0 \approx_x v$	3,∃&O
(5)	$v_0(Px) = T$	3,∃&O
(6)	$v_0(Px) = v_0(P) \langle v_0(x) \rangle$	Def CFOL-val
(7)	$v_0(P) \langle v_0(x) \rangle = T$	5,6,IL
(8)	$\forall \varepsilon \{ \text{Atomic}[\varepsilon] \ \& \ \varepsilon \neq x \ .\rightarrow \ v_0(\varepsilon) = v(\varepsilon) \}$	4, Def ≈
(9)	Atomic['P'] & 'P' ≠ 'x'	inspection, Def At[]
(10)	$v_0(P) = v(P)$	8,9,QL
(11)	$v(P) \langle v_0(x) \rangle = T$	7,10,IL
(12)	$v_0(x) \in v(P)$	11, C/F convention
(13)	$v_0(x) \in U$	Def CFOL-val

3.2b:

(1)	$\exists u \{u \in U \ \& \ u \in v(P)\}$	As
(2)	SHOW: $v(\exists x P x) = T$	Def CFOL-val
(3)	SHOW: $\exists v' \{v' \approx_x v \ \& \ v'(P x) = T\}$	7,13,QL
(4)	$u \in U$	1, $\exists \& O$
(5)	$u \in v(P)$	1, $\exists \& O$
(6)	let $v_0(x) = u \ \& \ \forall \varepsilon \{\text{Atomic}[\varepsilon] \ \& \ \varepsilon \neq x \ . \rightarrow \ v_0(\varepsilon) = v(\varepsilon)\}$	ST+ $\exists O$
(7)	$v_0 \approx_x v$	6b, Def \approx
(8)	Atomic['P'] & 'P' ≠ 'x'	inspection, Def At[]
(9)	$v_0(P) = v(P)$	6b,8,QL
(10)	$v_0(x) \in v_0(P)$	5,6a,9,IL
(11)	$v_0(P) \langle v_0(x) \rangle = T$	10,S/F convention
(12)	$v_0(P x) = v_0(P) \langle v_0(x) \rangle$	Def CFOL-val
(13)	$v_0(P x) = T$	11,12,IL