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1. General Definitions

1. Valuations

(D) Let \mathbb{L} be a formal language with sentences/formulas \mathbb{S} . Then a valuation on \mathbb{L} is, by definition, any function from \mathbb{S} into {T,F}. It is (of course!) assumed that T≠F.

2. Theorems about Valuations

Axiom 0: $T \neq F$

vt1: $\exists x[x=v(\alpha)] \rightarrow \alpha \in \mathbb{S}$

$\text{domain}(v) = \mathbb{S}$

vt2: $\alpha \in \mathbb{S} \rightarrow v(\alpha) \in \{T, F\}$

$\text{co-domain}(v) = \{T, F\}$

vt3: $\alpha \in \mathbb{S} \rightarrow . v(\alpha)=T \text{ xor } v(\alpha)=F$

vt2+Axiom 0

3. Semantic Characterization of Logic

In the following, we presuppose a given class \mathbb{V} of valuations, with respect to which \vDash is defined.

$v < \alpha$	$\stackrel{\text{df}}{=} v(\alpha)=T$	[v verifies α]
$v < \Gamma$	$\stackrel{\text{df}}{=} \forall x \{x \in \Gamma \rightarrow v < x\}$	[v verifies Γ]
$\models \alpha$	$\stackrel{\text{df}}{=} \forall v[v < \alpha]$	[α is semantically valid]
$\Gamma \models \alpha$	$\stackrel{\text{df}}{=} \forall v \{v < \Gamma \rightarrow v < \alpha\}$	[Γ semantically entails α]
$\Gamma \models$	$\stackrel{\text{df}}{=} \sim \exists v[v < \Gamma]$	[Γ is semantically inconsistent; Γ is unverifiable]
$\Gamma \not\models$	$\stackrel{\text{df}}{=} \exists v[v < \Gamma]$	[Γ is semantically consistent; Γ is verifiable]

4. Proofs and Derivations

In the following, we presuppose a given axiom system \mathbb{A} with respect to which all deductive notions are defined.

σ is a proof	$\stackrel{\text{df}}{=}$	σ is a finite sequence of formulas every item of which follows by a rule of \mathbb{A}
σ proves α	$\stackrel{\text{df}}{=}$	
σ is a proof of α	$\stackrel{\text{df}}{=}$	σ is a proof, and $\text{last}(\sigma)=\alpha$
σ derives from Γ	$\stackrel{\text{df}}{=}$	
σ is a derivation from Γ	$\stackrel{\text{df}}{=}$	σ is a finite sequence of formulas every item of which follows by a rule of \mathbb{A} , or is an element of Γ
σ derives α from Γ	$\stackrel{\text{df}}{=}$	
σ is a derivation of α from Γ	$\stackrel{\text{df}}{=}$	σ derives from Γ , and $\text{last}(\sigma)=\alpha$
α is an axiom	$\stackrel{\text{df}}{=}$	α is output by a zero-place rule

5. Theoremhood, Deductive Entailment, and Deductive Consistency

$\vdash \alpha$	\equiv_{df}	$\exists p[p \text{ is a proof of } \alpha]$	[α is a theorem/thesis]
$\Gamma \vdash \alpha$	\equiv_{df}	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	[Γ deductively entails α]
$\Gamma \vdash$	\equiv_{df}	$\forall \alpha[\Gamma \vdash \alpha]$	[Γ is deductively inconsistent]
$\Gamma \not\vdash$	\equiv_{df}	$\sim[\Gamma \vdash]$	[Γ is deductively consistent]

6. Maximal Consistent Sets

$MC[\Gamma]$	\equiv_{df}	$\Gamma \not\vdash \& \forall \Delta\{\Gamma \subset \Delta \rightarrow \Delta \vdash\}$	[Γ is maximal consistent]
$\Gamma \subset \Delta$	\equiv_{df}	$\Gamma \subseteq \Delta \& \sim[\Delta \subseteq \Gamma]$	[proper inclusion]

7. Weak Soundness, Completeness, and Mutual Consistency

In the following, \mathbb{A} is an axiom system, with respect to which \vdash is defined, and \mathbb{V} is a class of valuations, with respect to which \vDash is defined.

\mathbb{A} is sound for \mathbb{V} wrt formulas	\equiv_{df}	$\forall \alpha\{\vdash \alpha \rightarrow \vDash \alpha\}$
\mathbb{A} is complete for \mathbb{V} wrt formulas	\equiv_{df}	$\forall \alpha\{\vDash \alpha \rightarrow \vdash \alpha\}$
\mathbb{A} and \mathbb{V} are mutually consistent wrt formulas	\equiv_{df}	$\forall \alpha\{\vdash \alpha \leftrightarrow \vDash \alpha\}$

8. Strong Soundness, Completeness, and Mutual Consistency

\mathbb{A} is sound for \mathbb{V} wrt arguments	\equiv_{df}	$\forall \Gamma \forall \alpha\{\Gamma \vdash \alpha \rightarrow \Gamma \vDash \alpha\}$
\mathbb{A} is complete for \mathbb{V} wrt arguments	\equiv_{df}	$\forall \Gamma \forall \alpha\{\Gamma \vDash \alpha \rightarrow \Gamma \vdash \alpha\}$
\mathbb{A} and \mathbb{V} are mutually consistent wrt arguments	\equiv_{df}	$\forall \Gamma \forall \alpha\{\Gamma \vdash \alpha \leftrightarrow \Gamma \vDash \alpha\}$

9. The Underlying Formal Language

The underlying formal language, \mathbb{L} , is a zero-order language written in algebraic format, in which the only connectives are ‘ \sim ’ (one-place) and ‘ \rightarrow ’ (two-place).

10. The Class of Valuations – $V(\text{CSL})$

A valuation v on \mathbb{L} counts as admissible for CSL if and only if it satisfies the following for all $\alpha, \beta \in \mathbb{L}$.

$$\begin{aligned} v(\sim \alpha) &= \sim v(\alpha) \\ v(\alpha \rightarrow \beta) &= v(\alpha) \rightarrow v(\beta) \end{aligned}$$

Here, the functions on the right refer to the usual truth-functions.

11. The Axiom System – AS1

AS1 is a deductive system based on language \mathbb{L} above. It has four rules, the first three of which are zero-place rules.

- R1: $\neg \alpha \rightarrow (\beta \rightarrow \alpha)$
- R2: $\neg [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- R3: $\neg (\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)$
- R4: $\alpha \rightarrow \beta, \alpha \neg \rightarrow \beta$

Note: Any formula output by one of the zero-place rules, R1-R3, is referred to as an *axiom* of AS1. R4 is referred to as MP (modus ponens). Thus, for example, in a proof in AS1, every line is either an axiom of AS1 or follows from previous lines by MP.

2. General Theorems about Deduction

1. General Theorems about \vdash

G0: $a \vdash G \text{ } \mathbb{R} \text{ } G \vdash a$

(1)	SHOW: $\alpha \in \Gamma \rightarrow \Gamma \vdash \alpha$	CD
(2)	$\alpha \in \Gamma$	As
(3)	SHOW: $\Gamma \vdash \alpha$	Def \vdash
(4)	SHOW: $\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	5,QL
(5)	SHOW: $\langle \alpha \rangle \text{ derives } \alpha \text{ from } \Gamma$	Def derives [&D]
(6)	a:SHOW: $\text{last}(\langle \alpha \rangle) = \alpha$	ST
(7)	b:SHOW: $\forall \delta \in \langle \alpha \rangle: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	UCD
(8)	$\delta \in \langle \alpha \rangle$	As
(9)	SHOW: $\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	10,11,SL
(10)	$\delta = \alpha$	8,ST
(11)	$\delta \in \Gamma$	2,11,IL

corollary: $\{a\} \vdash a$
 $\{a,b\} \vdash a$
 $\{a,b,g\} \vdash a$
etc.

G1: $G \vdash D \text{ } \mathbb{R} . \text{ } G \vdash a \text{ } \mathbb{R} \text{ } D \vdash a$

(1)	SHOW: $\Gamma \subseteq \Delta \rightarrow \Gamma \vdash \alpha \rightarrow \Delta \vdash \alpha$	CCD
(2)	$\Gamma \subseteq \Delta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Delta \vdash \alpha$	Def \vdash
(5)	SHOW: $\exists d[d \text{ derives } \alpha \text{ from } \Delta]$	8,QL
(6)	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	3, Def \vdash
(7)	D derives α from Γ	6, $\exists O$
(8)	SHOW: D derives α from Δ	9,10,Def derives
(9)	a:SHOW: $\text{last}(D) = \alpha$	7, Def derives [a]
(10)	b:SHOW: $\forall \delta \in D: \delta \in \Delta \text{ or } \delta \text{ follows by a rule ...}$	11,12,QL
(11)	$\forall \delta \in D: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	7, Def derives [b]
(12)	$\forall x \{x \in \Gamma \rightarrow x \in \Delta\}$	2, Def \subseteq

corollary: $\mathbf{G} \vdash \mathbf{D} \circledR . \mathbf{G} \vdash \mathbf{D} \circledR \mathbf{D} \vdash$

(1)	SHOW: $\Gamma \subseteq \Delta \rightarrow \Gamma \vdash \rightarrow \Delta \vdash$	CCD
(2)	$\Gamma \subseteq \Delta$	As
(3)	$\Gamma \vdash$	As
(4)	SHOW: $\Delta \vdash$	Def $\Gamma \vdash$
(5)	SHOW: $\forall \alpha [\Delta \vdash \alpha]$	UD
(6)	SHOW: $\Delta \vdash \omega$	2,8,G1
(7)	$\forall \alpha [\Gamma \vdash \alpha]$	3,Def $\Gamma \vdash$
(8)	$\Gamma \vdash \omega$	7,QL

G2: $\vdash a \circledR \mathbf{G} \vdash a$

(1)	SHOW: $\vdash \alpha \rightarrow \Gamma \vdash \alpha$	CD
(2)	$\vdash \alpha$	As
(3)	SHOW: $\Gamma \vdash \alpha$	Def $\Gamma \vdash \alpha$
(4)	SHOW: $\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	9,QL
(5)	$\exists p[p \text{ proves } \alpha]$	2, Def $\vdash \alpha$
(6)	P proves α	5, $\exists O$
(7)	last(P)= α & $\forall \delta \in P: \delta$ follows by a rule ...	6, Def proves
(8)	last(P)= α & $\forall \delta \in P: \delta \in \Gamma$ or δ follows by a rule ...	7,QL
(9)	P derives α from Γ	8,Def derives

G3: $\vdash a \ll \mathbf{A} \vdash a$

(1)	SHOW: $\vdash \alpha \leftrightarrow \emptyset \vdash \alpha$	$\leftrightarrow D$
(2)	SHOW: $\vdash \alpha \rightarrow \emptyset \vdash \alpha$	G2
(3)	SHOW: $\emptyset \vdash \alpha \rightarrow \vdash \alpha$	CD
(4)	$\emptyset \vdash \alpha$	As
(5)	SHOW: $\vdash \alpha$	Def $\vdash \alpha$
(6)	SHOW: $\exists p[p \text{ proves } \alpha]$	QL
(7)	$\exists d[d \text{ derives } \alpha \text{ from } \emptyset]$	4, Def $\emptyset \vdash \alpha$
(8)	D derives α from \emptyset	9, $\exists O$
(9)	SHOW: D proves α	10,11,Def proves
(10)	a: SHOW: last(D)= α	8, Def derives [a]
(11)	b: SHOW: $\forall \delta \in D: \delta$ follows by a rule ...	UD
(12)	SHOW: δ follows by a rule ...	13,14,QL
(13)	$\forall \delta \in D: \delta \in \emptyset$ or δ follows by a rule ...	8, Def derives [b]
(14)	$\sim \exists x[x \in \emptyset]$	ST

G4: $\vdash a \ll " G[G \vdash a]$

(1)	SHOW: $\vdash \alpha \leftrightarrow \forall \Gamma[\Gamma \vdash \alpha]$	$\leftrightarrow D$
(2)	SHOW: \rightarrow	CD
(3)	$\vdash \alpha$	As
(4)	SHOW: $\forall \Gamma[\Gamma \vdash \alpha]$	UD
(5)	SHOW: $\Gamma \vdash \alpha$	3,G2
(6)	SHOW: \leftarrow	CD
(7)	$\forall \Gamma[\Gamma \vdash \alpha]$	As
(8)	SHOW: $\vdash \alpha$	9,G3
(9)	$\emptyset \vdash \alpha$	7,QL

G5:	$\text{Ax}[a] \circledR \vdash a$	
(1)	SHOW: $\text{Ax}[\alpha] \rightarrow \vdash \alpha$	CD
(2)	Ax[α]	As
(3)	SHOW: $\vdash \alpha$	Def \vdash
(4)	SHOW: $\exists p [p \text{ proves } \alpha]$	5,QL
(5)	SHOW: $\langle \alpha \rangle \text{ proves } \alpha$	6,7,Def proves
(6)	a:SHOW: $\text{last}(\langle \alpha \rangle) = \alpha$	ST
(7)	b:SHOW: $\forall \delta \in \langle \alpha \rangle: \delta \text{ follows by a rule ...}$	UCD
(8)	$\delta \in \langle \alpha \rangle$	As
(9)	SHOW: $\delta \text{ follows by a rule ...}$	12,QL
(10)	$\alpha \text{ follows by a zero-place rule}$	2, Def Ax
(11)	$\delta = \alpha$	8, Def $\langle \alpha \rangle$
(12)	$\delta \text{ follows by a zero-place rule}$	10,11,IL
G6:	$\Gamma \vdash a \& \Gamma \not\vdash \{a\} \vdash b . \circledR \Gamma \vdash b$	
(1)	SHOW: $\Gamma \vdash \alpha \& \Gamma \cup \{\alpha\} \vdash \beta . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \cup \{\alpha\} \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def \vdash
(5)	SHOW: $\exists d [d \text{ derives } \beta \text{ from } \Gamma]$	12,QL
(6)	$\exists d [d \text{ derives } \alpha \text{ from } \Gamma]$	2, Def \vdash
(7)	$\exists d [d \text{ derives } \beta \text{ from } \Gamma \cup \{\alpha\}]$	3, Def \vdash
(8)	$D_1 \text{ derives } \alpha \text{ from } \Gamma$	6,3O
(9)	$D_2 \text{ derives } \beta \text{ from } \Gamma \cup \{\alpha\}$	7,3O
(10)	$\exists d \{d = D_2[D_1/\alpha]\}$	ST
(11)	$D_3 = D_2[D_1/\alpha]$	10,3O
(12)	SHOW: $D_3 \text{ derives } \beta \text{ from } \Gamma$	13,15,Def derives
(13)	a:SHOW: $\text{last}(D_3) = \beta$	14,Def D_3 , ST
(14)	$\text{last}(D_2) = \beta$	9, Def derives [a]
(15)	b:SHOW: $\forall \delta \in D_3: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	UCD
(16)	$\delta \in D_3$	As
(17)	SHOW: $\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	19-30,SC/SC
(18)	$\delta \in D_2[D_1/\alpha]$	11,16,IL
(19)	$\{\delta \in D_1\} \text{ or } \{\delta \in D_2 \& \delta \neq \alpha\}$	18, Def $\sigma[\pi/\varepsilon]$
(20)	c1: $\delta \in D_1$	As
(21)	$\forall \delta \in D_1 \{\delta \in \Gamma \text{ or } \delta \text{ follows by a rule}\}$	8,Def derives [b]
(22)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	20,21,QL
(23)	c2: $\delta \in D_2 \& \delta \neq \alpha$	As
(24)	$\forall \delta \in D_2 \{\delta \in \Gamma \cup \{\alpha\} \text{ or } \delta \text{ follows by a rule}\}$	9,Def derives [b]
(25)	$\delta \in \Gamma \cup \{\alpha\} \text{ or } \delta \text{ follows by a rule}$	23,24,QL
(26)	c1: $\delta \in \Gamma \cup \{\alpha\}$	As
(27)	$\delta \in \Gamma$	23b,26,ST
(28)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	27,SL
(29)	c2: $\delta \text{ follows by a rule ...}$	As
(30)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	29,SL

G7:	$\mathbf{G \vdash a \ \& \ GE\{a\} \vdash . \circledR \ G \vdash}$	
(1)	SHOW: $\Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash . \rightarrow \Gamma \vdash$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \cup \{\alpha\} \vdash$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash b$	2,8,G6
(7)	$\forall \beta [\Gamma \cup \{\alpha\} \vdash \beta]$	3, Def \vdash
(8)	$\Gamma \cup \{\alpha\} \vdash b$	7,QL

G8:	$" d\{d \hat{I} D \circledR G \vdash d\} \ \& \ D \vdash b \ . \circledR \ G \vdash b$	
(1)	SHOW: $\forall \delta \in \Delta \rightarrow \Gamma \vdash \delta \ \& \ \Delta \vdash \beta \ . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\forall \delta \in \Delta \rightarrow \Gamma \vdash \delta$	As
(3)	$\Delta \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def \vdash
(5)	SHOW: $\exists d [d \text{ derives } \beta \text{ from } \Gamma]$	10,QL
(6)	$\exists d [d \text{ derives } \beta \text{ from } \Delta]$	3, Def \vdash
(7)	$D_0 \text{ derives } \beta \text{ from } \Delta$	6,EO
(8)	$\exists d [d = D_0[D_k/\delta_k : \delta_k \in \Delta]]$	ST
(9)	$D = D_0[D_k/\delta_k : \delta_k \in \Delta]$	8,EO
(10)	SHOW: D derives β from Γ	
(11)	unfinished	

G9:	$\mathbf{G \vdash a \ \& \ a \vdash b \ . \circledR \ G \vdash b}$	
(1)	SHOW: $\Gamma \vdash \alpha \ \& \ \alpha \vdash \beta \ . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\alpha \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	2,7,G8
(5)	$\{\alpha\} \vdash \beta$	3, Def \vdash
(6)	$\{\alpha\} \subseteq \Gamma \cup \{\alpha\}$	ST
(7)	$\Gamma \cup \{\alpha\} \vdash \beta$	5,6,G1

2. General Theorems about Maximal Consistency

G10:	$\mathbf{MC[G] \ \& \ a \hat{I} G \ . \circledR \ GE\{a\} \vdash}$	
(1)	SHOW: $MC[\Gamma] \ \& \ \alpha \notin \Gamma \ . \rightarrow \Gamma \cup \{\alpha\} \vdash$	&CD
(2)	$MC[\Gamma]$	As
(3)	$\alpha \notin \Gamma$	As
(4)	SHOW: $\Gamma \cup \{\alpha\} \vdash$	5,6,QL
(5)	$\Gamma \subset \Gamma \cup \{\alpha\}$	3,ST
(6)	$\forall \Delta \{\Gamma \subset \Delta \rightarrow \Delta \vdash\}$	2, Def MC [b]

G11: $\mathbf{MC[G] \& G \vdash a . \mathbb{R} \ a \bar{I} G}$

(1)	SHOW: $\mathbf{MC[\Gamma]} \ \& \ \Gamma \vdash \alpha . \rightarrow \alpha \in \Gamma$	&CD
(2)	$\mathbf{MC[\Gamma]}$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\alpha \in \Gamma$	ID
(5)	$a \notin \Gamma$	As
(6)	SHOW: \times	8,11,SL
(7)	$\Gamma \subset \Gamma \cup \{\alpha\}$	5,ST
(8)	$\Gamma \not\models$	2, Def MC [a]
(9)	$\forall \Delta \{\Gamma \subset \Delta \rightarrow \Delta \vdash\}$	2, Def MC [b]
(10)	$\Gamma \cup \{\alpha\} \vdash$	7,9,QL
(11)	$\Gamma \vdash$	3,10,G7

3. General Theorems about Derivations**G12:** " $d \$ n [len(d)=n]$ "

(1)	SHOW: $\forall d \exists n [len(d)=n]$	UD
(2)	SHOW: $\exists n [len(\circ)=n]$	DD
(3)	\circ is a derivation	sortal assumption
(4)	\circ is a finite sequence	3, Def derivation
(5)	$\exists n [len(\circ)=n]$	4, Def finite sequence

G13: " n " $d (len(d)=n) \mathbb{R} F[d] \mathbb{R} "d F[d]$ "

(1)	SHOW: $\forall n \forall d (len(d)=n \rightarrow F[d]) \rightarrow \forall d F[d]$	CD
(2)	$\forall n \forall d (len(d)=n \rightarrow F[d])$	As
(3)	SHOW: $\forall d F[d]$	UD
(4)	SHOW: $F[\circ]$	2,6,QL
(5)	$\exists n [len(\circ)=n]$	G12,QL
(6)	$len(\circ)=\square$	5,EO

G14: " n " d " $v_1 \dots v_k (F[d, v_1 \dots v_k] \ \& \ len(d)=n) . \mathbb{R} P$ " $\mathbb{R} "d v_1 \dots v_k (F[d, v_1 \dots v_k] \mathbb{R} P)$

(1)	SHOW: $\forall n \forall d \forall v_1 \dots v_k (F[d, v_1 \dots v_k] \ \& \ len(d)=n . \rightarrow P) \rightarrow \forall d \forall v_1 \dots v_k (F[d, v_1 \dots v_k] \rightarrow P)$	CD
(2)	$\forall n \forall d \forall v_1 \dots v_k (F[d, v_1 \dots v_k] \ \& \ len(d)=n . \rightarrow P)$	As
(3)	SHOW: $\forall d \forall v_1 \dots v_k (F[d, v_1 \dots v_k] \rightarrow P)$	UCD
(4)	$F[\circ, c_1 \dots c_k]$	As
(5)	SHOW: P	4,6,7,QL
(6)	$\exists n [len(\circ)=n]$	G12,QL
(7)	$len(\circ)=\square$	6,EO

Here, F and P are formulas, v_1, \dots, v_k are variables, and c_1, \dots, c_k are constants appropriately substituted for v_1, \dots, v_k .

4. Ordinary Mathematical Induction

SHOW: $\forall n F[n]$	MI	
SHOW: $F[0]$		[Base Case]
SHOW: $\forall n \{ F[n] \rightarrow F[n^+] \}$	UCD	[Inductive Case; optional]
$F[n]$	As	[Inductive Hypothesis]
SHOW: $F[n^+]$		[Inductive Step]

5. Strong Mathematical Induction

SHOW: $\forall n F[n]$	SMI	
SHOW: $\forall n \{ \forall k \{ k < n \rightarrow F[k] \} \rightarrow F[n] \}$	UCD	[Inductive Case; optional]
$\forall k \{ k < n \rightarrow F[k] \}$	As	[Inductive Hypothesis]
SHOW: $F[n]$		[Inductive Step]

3. Lemmas Used to Prove the Soundness Theorem

S1:	Ax[a] \mathbb{R} $G \vDash a$	
(1)	SHOW: $\text{Ax}[\delta] \rightarrow \Gamma \vDash \delta$	CD
(2)	Ax[δ]	As
(3)	SHOW: $\Gamma \vDash \delta$	Def \vDash
(4)	SHOW: $\forall v \{ v < \Gamma \rightarrow v < \delta \}$	5,QL
(5)	SHOW: $\forall v \{ v < \delta \}$	UD
(6)	SHOW: $v < \delta$	Def <
(7)	SHOW: $v(\delta) = T$	DD
(8)	δ is output of a zero-place rule	2,Def Ax
(9)	R1, R2, R3 are the zero-place rules of AS1	Def AS1
(10)	R1[δ] or R2[δ] or R3[δ]	8,9,IL
(11)	c1: R1[δ]	As
(12)	$\exists \alpha \exists \beta [\delta = \alpha \rightarrow (\beta \rightarrow \alpha)]$	11,Def R1
(13)	$\forall v \forall \alpha \forall \beta \{ v[\alpha \rightarrow (\beta \rightarrow \alpha)] = T \}$	S1.1
(14)	$v(\delta) = T$	12,13,IL
(15)	c2: R2[δ]	As
(16)	$\exists \alpha \exists \beta \exists \gamma [\delta = [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]]$	15,Def R2
(17)	$\forall v \forall \alpha \forall \beta \forall \gamma \{ v[(\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)] = T \}$	S1.2
(18)	$v(\delta) = T$	16,17,IL
(19)	c3: R3[δ]	As
(20)	$\exists \alpha \exists \beta [\delta = (\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)]$	19,Def R3
(21)	$\forall v \forall \alpha \forall \beta \{ v[(\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)] = T \}$	S1.3
(22)	$v(\delta) = T$	20,21,IL

Sub-Lemmas:

S1.1: $\forall v \forall \alpha \forall \beta \{ v[\alpha \rightarrow (\beta \rightarrow \alpha)] = T \}$

(1)	SHOW: $\forall v \forall \alpha \forall \beta \{ v[\alpha \rightarrow (\beta \rightarrow \alpha)] = T \}$	U3D
(2)	SHOW: $v[\alpha \rightarrow (\beta \rightarrow \alpha)] = T$	ID
(3)	$v[\alpha \rightarrow (\beta \rightarrow \alpha)] \neq T$	As
(4)	SHOW: \times	12,Axiom 0
(5)	$v[\alpha \rightarrow (\beta \rightarrow \alpha)] = F$	3,vt3
(6)	$v[\alpha \rightarrow (\beta \rightarrow \alpha)] = v(\alpha) \rightarrow [v(\beta) \rightarrow v(\alpha)]$	6,Def CSL val ($\times 2$)
(7)	$v(\alpha) \rightarrow [v(\beta) \rightarrow v(\alpha)] = F$	5,6,IL
(8)	$v(\alpha) = T$	7,Def \rightarrow
(9)	$v(\beta) \rightarrow v(\alpha) = F$	7,Def \rightarrow
(10)	$v(\beta) = T$	9,Def \rightarrow
(11)	$v(\alpha) = F$	9,Def \rightarrow
(12)	$T = F$	8,11,IL

S1.2: $\forall v \forall \alpha \forall \beta \forall \gamma \{ v([\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]) = T \}$

(1)	SHOW: $\forall v \forall \alpha \forall \beta \forall \gamma \{ v([\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]) = T \}$
(2)	similar to S1.1

S1.3: $\forall v \forall \alpha \forall \beta \{ v[(\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)] = T \}$

(1)	SHOW: $\forall v \forall \alpha \forall \beta \{ v[(\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)] = T \}$
(2)	similar to S1.1, but also uses Def \sim

S2: $a \hat{\in} G \quad \mathbb{R} \quad G \vDash a$

(1)	SHOW: $\alpha \in \Gamma \rightarrow \Gamma \vDash \alpha$	CD
(2)	$\alpha \in \Gamma$	As
(3)	SHOW: $\Gamma \vDash \alpha$	Def \vDash
(4)	SHOW: $\forall v \{ v < \Gamma \rightarrow v < \alpha \}$	UCD
(5)	$v < \Gamma$	As
(6)	SHOW: $v < \alpha$	2,7,QL
(7)	$\forall x \{ x \in \Gamma \rightarrow v < x \}$	5, Def $v < \Gamma$

S3:	$\mathbf{G} \vDash a @ b \ \& \ \mathbf{G} \vDash a . @ \mathbf{G} \vDash b$	
(1)	SHOW: $\Gamma \vDash \alpha \rightarrow \beta \ \& \ \Gamma \vDash \alpha . \rightarrow \ \Gamma \vDash \beta$	&CD
(2)	$\Gamma \vDash \alpha \rightarrow \beta$	As
(3)	$\Gamma \vDash \alpha$	As
(4)	SHOW: $\Gamma \vDash \beta$	Def \vDash
(5)	SHOW: $\forall v \{ v < \Gamma \rightarrow v < \beta \}$	UCD
(6)	$v < \Gamma$	As
(7)	SHOW: $v < \beta$	Def $<$
(8)	SHOW: $v(\beta) = T$	DD
(9)	$\forall v \{ v < \Gamma \rightarrow v < \alpha \rightarrow \beta \}$	2,Def \vDash
(10)	$\forall v \{ v < \Gamma \rightarrow v < \alpha \}$	3,Def \vDash
(11)	$v < \alpha \rightarrow \beta$	6,9,QL
(12)	$v < \alpha$	6,10,QL
(13)	$v(\alpha \rightarrow \beta) = T$	11,Def $<$
(14)	$v(\alpha) = T$	12,Def $<$
(15)	$v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$	Def val for CSL
(16)	$T = T \rightarrow v(\beta)$	13-15,IL
(17)	$v(\beta) = T$	16,Def \rightarrow

4. The Soundness Theorem

ST: AS1 is sound for V(CSL) with respect to arguments;
i.e.,
" G " $a\{G \vdash a\}$ \circledR $G \vdash a\}$

Note: $dD\Gamma$ \equiv_{df} d is a derivation from Γ
 $dD\alpha/\Gamma$ \equiv_{df} d is a derivation of α from Γ
 $dD\alpha/\Gamma/n$ \equiv_{df} d is a derivation of α from Γ of length n
i.e., $dD\alpha/\Gamma$ & $\text{len}(d)=n$

$\text{Ax}[\alpha]$ \equiv_{df} α is an axiom
 $\text{MP}[\alpha]$ \equiv_{df} α follows from previous lines by modus ponens
 $\langle d_i : i \leq m \rangle$ \equiv_{df} the sub-sequence of d consisting of the first m elements of d
for example, if $d = \langle a, b, c, d, e \rangle$, then $\langle d_i : i \leq 3 \rangle = \langle a, b, c \rangle$

(1)	SHOW: $\forall \Gamma \forall \alpha \{\Gamma \vdash \alpha \rightarrow \Gamma \vdash \alpha\}$	Def \vdash
(2)	SHOW: $\forall \Gamma \forall \alpha \{\exists d [dD\alpha/\Gamma] \rightarrow \Gamma \vdash \alpha\}$	3,QL
(3)	SHOW: $\forall d \forall \Gamma \forall \alpha \{dD\alpha/\Gamma \rightarrow \Gamma \vdash \alpha\}$	4+G14
(4)	SHOW: $\forall n: \forall d \forall \Gamma \forall \alpha \{dD\alpha/\Gamma/n \rightarrow \Gamma \vdash \alpha\}$	SMI
(5)	$\forall k < n: \forall d \forall \Gamma \forall \alpha \{dD\alpha/\Gamma/k \rightarrow \Gamma \vdash \alpha\}$	As [IH]
(6)	SHOW: $\forall d \forall \Gamma \forall \alpha \{dD\alpha/\Gamma/n \rightarrow \Gamma \vdash \alpha\}$	U3CD [IS]
(7)	$dD\alpha/\Gamma/n$	As
(8)	SHOW: $\Gamma \vdash \alpha$	DD
(9)	$\alpha = d_n$	7,Def derives/n
(10)	$\forall \delta \in d: \text{Ax}[\delta] \text{ or } \delta \in \Gamma \text{ or } \text{MP}[\delta]$	7,Def derives [b]
(11)	$\alpha \in d$	9,ST
(12)	$\text{Ax}[\alpha] \text{ or } \alpha \in \Gamma \text{ or } \text{MP}[\alpha]$	10,11,QL
(13)	c1: $\text{Ax}[\alpha]$	As
(14)	$\Gamma \vdash \alpha$	13,S1
(15)	c2: $\alpha \in \Gamma$	As
(16)	$\Gamma \vdash \alpha$	15,S2
(17)	c3: $\text{MP}[\alpha]$	As
(18)	$\exists j, k < n, \exists \gamma: d_j = \gamma \rightarrow \alpha \text{ & } d_k = \gamma$	9,17,Def MP[]
(19)	$j < n \text{ & } d_j = \gamma \rightarrow \alpha$	18, $\exists \& O$
(20)	$k < n \text{ & } d_k = \gamma$	18, $\exists \& O$
(21)	SHOW: $\langle d_i : i \leq j \rangle D \gamma \rightarrow \alpha / \Gamma / j$	22,23,25,Def D/n
(22)	a: SHOW: $\text{len}(\langle d_i : i \leq j \rangle) = j$	ST
(23)	b: SHOW: $\text{last}(\langle d_i : i \leq j \rangle) = \gamma \rightarrow \alpha$	19b,24,IL
(24)	$\text{last}(\langle d_i : i \leq j \rangle) = d_j$	ST
(25)	c: SHOW: $\langle d_i : i \leq j \rangle D \Gamma$	Def dD Γ
(26)	SHOW: $\forall \delta \in \langle d_i : i \leq j \rangle: \text{Ax}[\delta] \text{ or } \delta \in \Gamma \text{ or } \text{MP}[\delta]$	UCD
(27)	$\delta \in \langle d_i : i \leq j \rangle$	As
(28)	SHOW: $\text{Ax}[\delta] \text{ or } \delta \in \Gamma \text{ or } \text{MP}[\delta]$	10,29,QL
(29)	$\delta \in d$	27,ST
(30)	SHOW: $\langle d_i : i \leq k \rangle D \gamma / \Gamma / j$	Def D/n
(31)	similar to derivation of line 26	
(32)	$\Gamma \vdash \gamma \rightarrow \alpha$	19a,21,IH
(33)	$\Gamma \vdash \gamma$	20a,30,IH
(34)	$\Gamma \vdash \alpha$	32,33,S4

5. Basic Theorems About AS1

B1: $G \vdash a @ (b @ a)$

B2: $G \vdash [a @ (b @ g)] @ [(a @ b) @ (a @ g)]$

B3: $G \vdash (\sim a @ \sim b) @ (b @ a)$

These are all proved the same way. The following is the schema.

The formula ϕ is an axiom, so by G5, $\vdash \phi$, so by G2, $\Gamma \vdash \phi$.

B4: $\{a @ b, a\} \vdash b$

(1)	SHOW: $\{\alpha \rightarrow \beta, \alpha\} \vdash \beta$	Def \vdash
(2)	SHOW: $\exists d[d \text{ derives } \beta \text{ from } \{\alpha \rightarrow \beta, \alpha\}]$	3,QL
(3)	SHOW: $\langle \alpha \rightarrow \beta, \alpha, \beta \rangle$ derives β from $\{\alpha \rightarrow \beta, \alpha\}$	4,5,Def derives
(4)	a:SHOW: last($\langle \alpha \rightarrow \beta, \alpha, \beta \rangle$) = β	ST
(5)	b:SHOW: $\forall \delta \in \langle \alpha \rightarrow \beta, \alpha, \beta \rangle: Ax[\delta] \text{ or } \delta \in \{\alpha \rightarrow \beta, \alpha\} \text{ or } MP[\delta]$	UCD
(6)	$\delta \in \langle \alpha \rightarrow \beta, \alpha, \beta \rangle$	As
(7)	SHOW: $Ax[\delta] \text{ or } \delta \in \{\alpha \rightarrow \beta, \alpha\} \text{ or } MP[\delta]$	SC
(8)	$\delta = \alpha \rightarrow \beta \text{ or } \delta = \alpha \text{ or } \delta = \beta$	6, ST
(9)	c1: $\delta = \alpha \rightarrow \beta$	As
(10)	$\delta \in \{\alpha \rightarrow \beta, \alpha\}$	9,ST
(11)	$Ax[\delta] \text{ or } \delta \in \{\alpha \rightarrow \beta, \alpha\} \text{ or } MP[\delta]$	10,SL
(12)	c2: $\delta = \alpha$	As
(13)	$\delta \in \{\alpha \rightarrow \beta, \alpha\}$	12,ST
(14)	$Ax[\delta] \text{ or } \delta \in \{\alpha \rightarrow \beta, \alpha\} \text{ or } MP[\delta]$	10,SL
(15)	c3: $\delta = \beta$	As
(16)	β follows from $\alpha \rightarrow \beta$ and α by MP	Def MP
(17)	$\#(\alpha \rightarrow \beta), \#(\alpha) < \#(\beta)$	Def $\langle \alpha \rightarrow \beta, \alpha, \beta \rangle$
(18)	δ follows from $\alpha \rightarrow \beta$ and α by MP	15,16,IL
(19)	$\#(\alpha \rightarrow \beta), \#(\alpha) < \#(\delta)$	15,17,IL
(20)	MP[δ]	18,19,Def follow...
(21)	$Ax[\delta] \text{ or } \delta \in \{\alpha \rightarrow \beta, \alpha\} \text{ or } MP[\delta]$	20,SL

6. Named Principles

1. Modus Ponens Principle

MPP:	$\mathbf{G \vdash a @ b \And G \vdash a . @ G \vdash b}$	
(1)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta \And \Gamma \vdash \alpha . \rightarrow \Gamma \vdash \beta$	CD
(2)	$\Gamma \vdash \alpha \rightarrow \beta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def $\Gamma \vdash \alpha$
(5)	SHOW: $\exists d[d \text{ derives } \beta \text{ from } \Gamma]$	10, $\exists I$
(6)	$\exists d[d \text{ derives } \alpha \rightarrow \beta \text{ from } \Gamma]$	2, Def $\Gamma \vdash \alpha$
(7)	$D_1 \text{ derives } \alpha \rightarrow \beta \text{ from } \Gamma$	6, $\exists O$
(8)	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	2, Def $\Gamma \vdash \alpha$
(9)	$D_2 \text{ derives } \alpha \text{ from } \Gamma$	8, $\exists O$
(10)	SHOW: $D_1 + D_2 + \langle \beta \rangle$ derives β from Γ	11, 12, Def derives
(11)	SHOW: $\beta = \text{last}(D_1 + D_2 + \langle \beta \rangle)$	ST
(12)	SHOW: $\forall \delta \{ \alpha \in D_1 + D_2 + \langle \beta \rangle \rightarrow . Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta] \}$	UCD
(13)	$\delta \in D_1 + D_2 + \langle \beta \rangle$	As
(14)	SHOW: $Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta]$	15-31, SC
(15)	$\delta \in D_1 \text{ or } \delta \in D_2 \text{ or } \delta \in \langle \beta \rangle$	13, Def +
(16)	c1: $\delta \in D_1$	As
(17)	$\forall \delta \in D_1: Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta]$	8, Def derives [b]
(18)	$Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta]$	16, 17, QL
(19)	c2: $\delta \in D_2$	As
(20)	$\forall \delta \in D_2: Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta]$	10, Def derives [b]
(21)	$Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta]$	19, 20, QL
(22)	c3: $\delta \in \langle \beta \rangle$	As
(23)	$\delta = \beta$	22, Def $\langle \rangle$
(24)	$\delta = \text{last}(D_1 + D_2 + \langle \beta \rangle)$	23+Def +
(25)	$\alpha \rightarrow \beta = \text{last}(D_1)$	8, Def derives [a]
(26)	$\alpha = \text{last}(D_2)$	10, Def derives [a]
(27)	$\text{last}(D_1) < \delta$	Def $D_1 + D_2 + \langle \delta \rangle$
(28)	$\text{last}(D_2) < \delta$	Def $D_1 + D_2 + \langle \delta \rangle$
(29)	β follows from $\alpha \rightarrow \beta$ and α by MP	Def MP
(30)	$MP[\delta]$	23-29, IL
(31)	$Ax[\delta] \text{ or } \delta \in \Gamma \text{ or } MP[\delta]$	30, SL

This is a direct proof. MPP is also a special case of a general theorem, as seen in the following proof.

(1)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta \And \Gamma \vdash \alpha . \rightarrow \Gamma \vdash \beta$	CD
(2)	$\Gamma \vdash \alpha \rightarrow \beta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Gamma \vdash \beta$	5, 6, G8
(5)	$\{ \alpha \rightarrow \beta, \alpha \} \vdash \beta$	B4
(6)	$\forall \delta \in \{ \alpha \rightarrow \beta, \alpha \}: \Gamma \vdash \delta$	2, 3, ST

2. Prefix Principles

PRE: $\vdash b \text{ } \textcircled{R} \text{ } \vdash a @ b$

(1)	SHOW: $\vdash \beta \rightarrow \vdash \alpha \rightarrow \beta$	CD
(2)	$\vdash \beta$	As
(3)	SHOW: $\vdash \alpha \rightarrow \beta$	2,4,MPP
(4)	$\vdash \beta \rightarrow (\alpha \rightarrow \beta)$	R1

$G \vdash b \text{ } \textcircled{R} \text{ } G \vdash a @ b$

(1)	SHOW: $\Gamma \vdash \beta \rightarrow \Gamma \vdash \alpha \rightarrow \beta$	CD
(2)	$\Gamma \vdash \beta$	As
(3)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta$	2,4,MPP
(4)	$\Gamma \vdash \beta \rightarrow (\alpha \rightarrow \beta)$	R1

3. Distribution Principle

DIST: $\vdash a @ (b @ g) \text{ } \textcircled{R} \text{ } \vdash (a @ b) @ (a @ g)$

(1)	SHOW: $\vdash \alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow \vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	CD
(2)	$\vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	As
(3)	SHOW: $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	2,4,MPP
(4)	$\vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$	R2

corollary: $\vdash a @ (b @ g) \text{ } \& \text{ } \vdash a @ b \text{ } .\text{ }\textcircled{R} \text{ } \vdash a @ g$

(1)	SHOW: $\vdash \alpha \rightarrow (\beta \rightarrow \gamma) \text{ } \& \text{ } \vdash \alpha \rightarrow \beta \text{ } .\rightarrow \vdash \alpha \rightarrow \gamma$	&CD
(2)	$\vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	As
(3)	$\vdash \alpha \rightarrow \beta$	As
(4)	SHOW: $\vdash \alpha \rightarrow \gamma$	3,5,MPP
(5)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	3,DIST

corollary: $\vdash b @ g \text{ } .\text{ }\textcircled{R} \text{ } \vdash (a @ b) @ (a @ g)$

(1)	SHOW: $\vdash \beta \rightarrow \gamma \text{ } .\rightarrow \vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	CD
(2)	$\vdash \beta \rightarrow \gamma$	As
(3)	SHOW: $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	3,DIST
(4)	$\vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	2,PRE

4. Transitivity Principle

TR: $\vdash a @ b \text{ } \& \text{ } \vdash b @ g \text{ } .\text{ }\textcircled{R} \text{ } \vdash a @ g$

(1)	SHOW: $\vdash \alpha \rightarrow \beta \text{ } \& \text{ } \vdash \beta \rightarrow \gamma \text{ } .\rightarrow \vdash \alpha \rightarrow \gamma$	&CD
(2)	$\vdash \alpha \rightarrow \beta$	As
(3)	$\vdash \beta \rightarrow \gamma$	As
(4)	SHOW: $\vdash \alpha \rightarrow \gamma$	CD
(5)	$\vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	3,PRE
(6)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	5,DIST
(7)	$\vdash \alpha \rightarrow \gamma$	2,6,MPP

7. Lemmas Used to Prove the Deduction Theorem

D1: $\text{Ax[b]} \text{ } \textcircled{R} \text{ } G \vdash a \text{ } \textcircled{R} \text{ } b$

(1)	SHOW: $\text{Ax}[\beta] \rightarrow \Gamma \vdash \alpha \rightarrow \beta$	CD
(2)	$\text{Ax}[\beta]$	As
(3)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta$	DD
(4)	$\vdash \beta$	2,G5
(5)	$\vdash \alpha \rightarrow \beta$	4,PRE
(6)	$\Gamma \vdash \alpha \rightarrow \beta$	5,G2

D2: $b \text{ } \widehat{\text{I}} \text{ } G \text{ } \textcircled{R} \text{ } G \vdash a \text{ } \textcircled{R} \text{ } b$

(1)	SHOW: $\beta \in \Gamma \rightarrow \Gamma \vdash \alpha \rightarrow \beta$	CD
(2)	$\beta \in \Gamma$	As
(3)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta$	DD
(4)	$\vdash \beta$	2,G0
(5)	$\Gamma \vdash \alpha \rightarrow \beta$	4,PRE

D3: $G \vdash a \text{ } \textcircled{R} \text{ } a$

(1)	SHOW: $\Gamma \vdash \alpha \rightarrow \alpha$	DD
(2)	$\vdash \alpha \rightarrow [(\alpha \rightarrow \alpha) \rightarrow \alpha]$	B1
(3)	$\vdash [\alpha \rightarrow (\alpha \rightarrow \alpha)] \rightarrow [\alpha \rightarrow \alpha]$	2,DIST
(4)	$\vdash \alpha \rightarrow (\alpha \rightarrow \alpha)$	B1
(5)	$\vdash \alpha \rightarrow \alpha$	3,4,MPP
(6)	$\Gamma \vdash \alpha \rightarrow \alpha$	5,G2

D4: $G \vdash a \text{ } \textcircled{R} \text{ } (b \text{ } \textcircled{R} \text{ } g) \text{ } \& \text{ } G \vdash a \text{ } \textcircled{R} \text{ } b \text{ } . \text{ } \textcircled{R} \text{ } G \vdash a \text{ } \textcircled{R} \text{ } g$

(1)	SHOW: $\Gamma \vdash \alpha \rightarrow (\beta \rightarrow \gamma) \text{ } \& \text{ } \Gamma \vdash \alpha \rightarrow \beta \text{ } . \rightarrow \Gamma \vdash \alpha \rightarrow \gamma$	&CD
(2)	$\Gamma \vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	As
(3)	$\Gamma \vdash \alpha \rightarrow \beta$	As
(4)	SHOW: $\Gamma \vdash \alpha \rightarrow \gamma$	DD
(5)	$\vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$	B2
(6)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	2,5,MPP
(7)	$\Gamma \vdash \alpha \rightarrow \gamma$	3,6,MPP

8. The Deduction Theorem

Note:

$dD\Gamma$	\equiv_{df}	d is a derivation from Γ
$dD\alpha/\Gamma$	\equiv_{df}	d is a derivation of α from Γ
$dD\alpha/\Gamma/n$	\equiv_{df}	d is a derivation of α from Γ of length n i.e., $dD\alpha/\Gamma$ & $\text{len}(d)=n$

$MP[\alpha]$	\equiv_{df}	α follows from previous lines by modus ponens
$\langle d_i : i \leq m \rangle$	\equiv_{df}	the sub-sequence of d consisting of the first m elements of d for example, if $d = \langle a, b, c, d, e \rangle$, then $\langle d_i : i \leq 3 \rangle = \langle a, b, c \rangle$

DT: $G \vdash \{a\} \vdash b \quad G \vdash a \otimes b$

(1)	SHOW: $\forall \Gamma \forall \alpha \forall \beta \{ \Gamma \cup \{\alpha\} \vdash \beta \rightarrow \Gamma \vdash \alpha \rightarrow \beta \}$	Def \vdash
(2)	SHOW: $\forall \Gamma \forall \alpha \forall \beta \{ \exists d [dD\beta/\Gamma \cup \{\alpha\}] \rightarrow \Gamma \vdash \alpha \rightarrow \beta \}$	3,QL
(3)	SHOW: $\forall d \forall \Gamma \forall \alpha \forall \beta \{ dD\beta/\Gamma \cup \{\alpha\} \rightarrow \Gamma \vdash \alpha \rightarrow \beta \}$	4+G14
(4)	SHOW: $\forall n: \forall d \forall \Gamma \forall \alpha \forall \beta \{ dD\beta/\Gamma \cup \{\alpha\}/n \rightarrow \Gamma \vdash \alpha \rightarrow \beta \}$	SMI
(5)	$\left \begin{array}{l} \forall k < n: \forall d \forall \Gamma \forall \alpha \forall \beta \{ dD\beta/\Gamma \cup \{\alpha\}/k \rightarrow \Gamma \vdash \alpha \rightarrow \beta \} \\ \text{SHOW: } \forall d \forall \Gamma \forall \alpha \forall \beta \{ dD\beta/\Gamma \cup \{\alpha\}/n \rightarrow \Gamma \vdash \alpha \rightarrow \beta \} \end{array} \right.$	$\begin{array}{ll} \text{As} & [\text{IH}] \\ \text{U4CD} & [\text{IS}] \end{array}$
(6)	$\left \begin{array}{l} dD\beta/\Gamma \cup \{\alpha\}/n \\ \text{SHOW: } \Gamma \vdash \alpha \rightarrow \beta \end{array} \right.$	$\begin{array}{l} \text{As} \\ \text{SC} \end{array}$
(7)	$\beta = \text{last}(d) = d_n$	7,Def derives/n
(8)	$\forall \delta \in d: \text{Ax}[\delta] \text{ or } \delta \in \Gamma \cup \{\alpha\} \text{ or } MP[\delta]$	7,Def derives [b]
(9)	$\beta \in d$	9,ST
(10)	$\text{Ax}[\beta] \text{ or } \beta \in \Gamma \cup \{\alpha\} \text{ or } MP[\beta]$	10,11,QL
(11)	c1: $\text{Ax}[\beta]$	As
(12)	$\left \begin{array}{l} \Gamma \vdash \alpha \rightarrow \beta \\ c2: \beta \in \Gamma \cup \{\alpha\} \end{array} \right.$	13,D1
(13)	$\beta \in \Gamma$	As
(14)	$\left \begin{array}{l} \Gamma \vdash \alpha \rightarrow \beta \\ c1: \beta \in \Gamma \end{array} \right.$	15,ST
(15)	$\beta \in \Gamma \text{ or } \beta = \alpha$	As
(16)	$\left \begin{array}{l} c1: \beta \in \Gamma \\ \Gamma \vdash \alpha \rightarrow \beta \end{array} \right.$	As
(17)	$c2: \beta = \alpha$	17,D2
(18)	$\left \begin{array}{l} \Gamma \vdash \alpha \rightarrow \alpha \\ c1: \beta = \alpha \end{array} \right.$	As
(19)	$\Gamma \vdash \alpha \rightarrow \alpha$	D3
(20)	$\left \begin{array}{l} \Gamma \vdash \alpha \rightarrow \beta \\ c3: MP[\beta] \end{array} \right.$	19,20,IL
(21)	$\Gamma \vdash \alpha \rightarrow \beta$	As
(22)	c3: $MP[\beta]$	9,22,Def MP[]
(23)	$\exists j, k < n, \exists \gamma: d_j = \gamma \rightarrow \beta \text{ and } d_k = \gamma$	23, $\exists \& O$
(24)	$j < n \text{ and } d_j = \gamma \rightarrow \beta$	23, $\exists \& O$
(25)	$k < n \text{ and } d_k = \gamma$	27,28,30,Def D/n
(26)	SHOW: $\langle d_i : i \leq j \rangle D \gamma \rightarrow \beta / \Gamma \cup \{\alpha\} / j$	ST
(27)	a: SHOW: $\text{len}(\langle d_i : i \leq j \rangle) = j$	24b,29,IL
(28)	b: SHOW: $\text{last}(\langle d_i : i \leq j \rangle) = \gamma \rightarrow \beta$	ST
(29)	$\left \begin{array}{l} \text{last}(\langle d_i : i \leq j \rangle) = d_j \\ c: \text{SHOW: } \langle d_i : i \leq j \rangle D \Gamma \cup \{\alpha\} \end{array} \right.$	Def dD Γ
(30)	c: SHOW: $\langle d_i : i \leq j \rangle D \Gamma \cup \{\alpha\}$	UCD
(31)	SHOW: $\forall \delta \in \langle d_i : i \leq j \rangle: \text{Ax}[\delta] \text{ or } \delta \in \Gamma \cup \{\alpha\} \text{ or } MP[\delta]$	As
(32)	$\left \begin{array}{l} \delta \in \langle d_i : i \leq j \rangle \\ \text{SHOW: } \text{Ax}[\delta] \text{ or } \delta \in \Gamma \cup \{\alpha\} \text{ or } MP[\delta] \end{array} \right.$	10,34,QL
(33)	$\left \begin{array}{l} \delta \in \langle d_i : i \leq j \rangle \\ \delta \in d \end{array} \right.$	32,ST
(34)	SHOW: $\langle d_i : i \leq k \rangle D \gamma / \Gamma \cup \{\alpha\} / j$	Def D/n
(35)	$\left \begin{array}{l} \text{similar to derivation of line 26} \\ \Gamma \vdash \alpha \rightarrow (\gamma \rightarrow \beta) \end{array} \right.$	24a,26,IH
(36)	$\Gamma \vdash \alpha \rightarrow \gamma$	25a,35,IH
(37)	$\Gamma \vdash \alpha \rightarrow \beta$	37,38,D4

9. Lemmas Used to Prove Lindenbaum's Lemma

L1: $\vdash \sim a @ (a @ b)$

(1)	SHOW: $\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	DD
(2)	$\vdash \sim \alpha \rightarrow (\sim \beta \rightarrow \sim \alpha)$	B1
(3)	$\vdash (\sim \beta \rightarrow \sim \alpha) \rightarrow (\alpha \rightarrow \beta)$	B3
(4)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	2,3,TR

L2: $\vdash [a @ (a @ b)] @ (a @ b)$

(1)	SHOW: $\vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$	DD
(2)	$\vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow [(\alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \beta)]$	B2
(3)	$\vdash \alpha \rightarrow \alpha$	D3+G4
(4)	$\vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \alpha)$	3,PRE
(5)	$\vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$	2,4,DIST(c)

corollary: $\vdash a @ (a @ b) @ a @ b$

(1)	SHOW: $\vdash \alpha \rightarrow (\alpha \rightarrow \beta) \rightarrow \vdash \alpha \rightarrow \beta$	CD
(2)	$\vdash \alpha \rightarrow (\alpha \rightarrow \beta)$	As
(3)	SHOW: $\vdash \alpha \rightarrow \beta$	2,4,MPP
(4)	$\vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$	L2

L3: $\vdash (\sim a @ a) @ a$

(1)	SHOW: $\vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$	DD
(2)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \sim [\sim \alpha \rightarrow \alpha])$	L1
(3)	$\vdash (\sim \alpha \rightarrow \alpha) \rightarrow (\sim \alpha \rightarrow \sim [\sim \alpha \rightarrow \alpha])$	2,DIST
(4)	$\vdash (\sim \alpha \rightarrow \sim [\sim \alpha \rightarrow \alpha]) \rightarrow ([\sim \alpha \rightarrow \alpha] \rightarrow \alpha)$	B3
(5)	$\vdash (\sim \alpha \rightarrow \alpha) \rightarrow ([\sim \alpha \rightarrow \alpha] \rightarrow \alpha)$	3,4,TR
(6)	$\vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$	5,L2c

L4: $\vdash \sim \sim a @ a$

(1)	SHOW: $\vdash \sim \sim \alpha \rightarrow \alpha$	DD
(2)	$\vdash \sim \sim \alpha \rightarrow (\sim \alpha \rightarrow \sim \sim \sim \alpha)$	L1
(3)	$\vdash (\sim \alpha \rightarrow \sim \sim \sim \alpha) \rightarrow (\sim \sim \alpha \rightarrow \alpha)$	B3
(4)	$\vdash \sim \sim \alpha \rightarrow (\sim \sim \alpha \rightarrow \alpha)$	2,3,TR
(5)	$\vdash \sim \sim \alpha \rightarrow \alpha$	4,L2c

L5: $\vdash a @ \sim \sim a$

(1)	SHOW: $\vdash \alpha \rightarrow \sim \sim \alpha$	DD
(2)	$\vdash \sim \sim \alpha \rightarrow \sim \alpha$	L4
(3)	$\vdash (\sim \sim \alpha \rightarrow \sim \alpha) \rightarrow (\alpha \rightarrow \sim \sim \alpha)$	B3
(4)	$\vdash \alpha \rightarrow \sim \sim \alpha$	2,3,MPP

L6:	$\vdash (a \otimes b) \otimes (\sim b \otimes \sim a)$	
(1)	SHOW: $\vdash (\alpha \rightarrow \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	DD
(2)	$\vdash \beta \rightarrow \sim \sim \beta$	L5
(3)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \sim \sim \beta)$	2,DIST(c2)
(4)	$\vdash [\sim \sim \alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow [\sim \sim \alpha \rightarrow (\alpha \rightarrow \sim \sim \beta)]$	3,DIST(c2)
(5)	$\vdash [\sim \sim \alpha \rightarrow (\alpha \rightarrow \sim \sim \beta)] \rightarrow [(\sim \sim \alpha \rightarrow \alpha) \rightarrow (\sim \sim \alpha \rightarrow \sim \sim \beta)]$	B3
(6)	$\vdash (\alpha \rightarrow \sim \sim \beta) \rightarrow [\sim \sim \alpha \rightarrow (\alpha \rightarrow \sim \sim \beta)]$	B1
(7)	$\vdash (\alpha \rightarrow \beta) \rightarrow [(\sim \sim \alpha \rightarrow \alpha) \rightarrow (\sim \sim \alpha \rightarrow \sim \sim \beta)]$	3,4,6,TR×2
(8)	$\vdash [(\alpha \rightarrow \beta) \rightarrow (\sim \sim \alpha \rightarrow \alpha)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\sim \sim \alpha \rightarrow \sim \sim \beta)]$	7,DIST
(9)	$\vdash \sim \sim \alpha \rightarrow \alpha$	L4
(10)	$\vdash [(\alpha \rightarrow \beta) \rightarrow (\sim \sim \alpha \rightarrow \alpha)]$	9,PRE
(11)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\sim \sim \alpha \rightarrow \sim \sim \beta)$	8,10,MPP
(12)	$\vdash (\sim \sim \alpha \rightarrow \sim \sim \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	B3
(13)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	11,12,MPP

Alternative Proof

(1)	SHOW: $\vdash (\alpha \rightarrow \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	DD
(2)	$\exists \Gamma[\Gamma = \{\alpha \rightarrow \beta, \sim \sim \alpha\}]$	Def formula/set
(3)	$\Gamma = \{\alpha \rightarrow \beta, \sim \sim \alpha\}$	2,∃O
(4)	$\alpha \rightarrow \beta \in \Gamma$	2,ST
(5)	$\sim \sim \alpha \in \Gamma$	2,ST
(6)	$\Gamma \vdash \alpha \rightarrow \beta$	4,G0
(7)	$\Gamma \vdash \sim \sim \alpha$	4,G0
(8)	$\Gamma \vdash \sim \sim \alpha \rightarrow \alpha$	L4+G2
(9)	$\Gamma \vdash \alpha$	7,8,MPP
(10)	$\Gamma \vdash \beta$	6,9,MPP
(11)	$\Gamma \vdash \beta \rightarrow \sim \sim \beta$	L5+G2
(12)	$\Gamma \vdash \sim \sim \beta$	10,11,MPP
(13)	$\{\alpha \rightarrow \beta, \sim \sim \alpha\} \vdash \sim \sim \beta$	3,12,IL
(14)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\sim \sim \alpha \rightarrow \sim \sim \beta)$	13,DT×2
(15)	$\vdash (\sim \sim \alpha \rightarrow \sim \sim \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	B3
(16)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	11,12,MPP

L7: $\{a \otimes b, \sim a \otimes b\} \vdash b$

(1)	SHOW: $\{\alpha \rightarrow \beta, \sim a \rightarrow \beta\} \vdash \beta$	DD
(2)	$\exists \Gamma[\Gamma = \{\sim \beta, \alpha \rightarrow \beta, \sim a \rightarrow \beta\}]$	Def formula/set
(3)	$\Gamma = \{\sim \beta, \alpha \rightarrow \beta, \sim a \rightarrow \beta\}$	2,∃O
(4)	$\Gamma \vdash \sim \beta$	3,ST,G0
(5)	$\Gamma \vdash \alpha \rightarrow \beta$	3,ST,G0
(6)	$\Gamma \vdash \sim a \rightarrow \beta$	3,ST,G0
(7)	$\Gamma \vdash (\alpha \rightarrow \beta) \rightarrow (\sim \beta \rightarrow \sim \alpha)$	L6+G2
(8)	$\Gamma \vdash \beta$	4-7,MPP×3
(9)	$\{\alpha \rightarrow \beta, \sim a \rightarrow \beta\} \vdash \sim \beta \rightarrow \beta$	3,8,DT+ST
(10)	$\{\alpha \rightarrow \beta, \sim a \rightarrow \beta\} \vdash (\sim \beta \rightarrow \beta) \rightarrow \beta$	L3+G2
(11)	$\{\alpha \rightarrow \beta, \sim a \rightarrow \beta\} \vdash \beta$	9,10,MPP

corollary: $\mathbf{G}\bar{\wedge}\{a\} \vdash b \ \& \ \mathbf{G}\bar{\wedge}\{\sim a\} \vdash b . \textcircled{R} \quad \mathbf{G} \vdash b$

(1)	SHOW: $\Gamma \cup \{\alpha\} \vdash \beta \ \& \ \Gamma \cup \{\sim \alpha\} \vdash \beta . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \cup \{\alpha\} \vdash \beta$	As
(3)	$\Gamma \cup \{\sim \alpha\} \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	5,6,L7,G8
(5)	$\Gamma \vdash \alpha \rightarrow \beta$	2,DT
(6)	$\Gamma \vdash \sim \alpha \rightarrow \beta$	3,DT

corollary: $\mathbf{G}\bar{\wedge}\{a\} \vdash \ \& \ \mathbf{G}\bar{\wedge}\{\sim a\} \vdash . \textcircled{R} \quad \mathbf{G} \vdash$

(1)	SHOW: $\Gamma \cup \{\alpha\} \vdash \ \& \ \Gamma \cup \{\sim \alpha\} \vdash . \rightarrow \Gamma \vdash$	&CD
(2)	$\Gamma \cup \{\alpha\} \vdash$	As
(3)	$\Gamma \cup \{\sim \alpha\} \vdash$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash b$	9,10,L7c1
(7)	$\forall \beta [\Gamma \cup \{\alpha\} \vdash \beta]$	2, Def \vdash
(8)	$\forall \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta]$	3, Def \vdash
(9)	$\Gamma \cup \{\alpha\} \vdash b$	7,QL
(10)	$\Gamma \cup \{\sim \alpha\} \vdash b$	8,QL

corollary: $\mathbf{G} \not\vdash \textcircled{R} . \quad \mathbf{G}\bar{\wedge}\{a\} \not\vdash \text{ or } \mathbf{G}\bar{\wedge}\{\sim a\} \not\vdash$

(1)	SHOW: $\Gamma \not\vdash \rightarrow . \Gamma \cup \{\alpha\} \not\vdash \text{ or } \Gamma \cup \{\sim \alpha\} \not\vdash$	L7c2,QL
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L8: $a \bar{\top} G \ \& \ \sim a \bar{\top} G . \textcircled{R} \quad G \vdash$

(1)	SHOW: $\alpha \in \Gamma \ \& \ \sim \alpha \in \Gamma . \rightarrow \Gamma \vdash$	&CD
(2)	$\alpha \in \Gamma$	As
(3)	$\sim \alpha \in \Gamma$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash \beta$	7,8,9,MPP×2
(7)	$\Gamma \vdash \alpha$	2,G0
(8)	$\Gamma \vdash \sim \alpha$	3,G0
(9)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	L1
(10)	$\Gamma \vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	9,G2

L9: $G \vdash a \ \& \ G \vdash \sim a . \textcircled{R} \quad G \vdash$

(1)	SHOW: $\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha . \rightarrow \Gamma \vdash$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \vdash \sim \alpha$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash \beta$	DD
(7)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	L1
(8)	$\Gamma \vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	7+G3
(9)	$\Gamma \vdash \alpha \rightarrow \beta$	3,8,MPP
(10)	$\Gamma \vdash \beta$	2,9,MPP

L10:	$\mathbf{G} \vdash \Leftarrow \mathbf{Sa}\{\mathbf{G} \vdash a \ \& \ \mathbf{G} \vdash \sim a\}$	
(1)	SHOW: $\Gamma \vdash \leftrightarrow \exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\}$	$\leftrightarrow D$
(2)	SHOW: \rightarrow	CD
(3)	$\Gamma \vdash$	As
(4)	SHOW: $\exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\}$	7,8,QL
(5)	$\forall \alpha [\Gamma \vdash \alpha]$	3,Def $\Gamma \vdash$
(6)	P is a formula [of CSL], and so is $\sim P$	Def formula
(7)	$\Gamma \vdash P$	5,6a,QL
(8)	$\Gamma \vdash \sim P$	5,6b,QL
(9)	SHOW: \Leftarrow	CD
(10)	$\exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\}$	As
(11)	$\Gamma \vdash \alpha$	10, $\exists O + \& O$
(12)	$\Gamma \vdash \sim \alpha$	10, $\exists O + \& O$
(13)	SHOW: $\Gamma \vdash$	11,12,L9

10. Outline of Lindenbaum's Lemma

1. Statement of Theorem

For any set Γ , if Γ is deductively consistent, then there exists a set Δ such that $\Gamma \subseteq \Delta$ and Δ is maximal consistent.

$$\forall \Gamma \{\Gamma \not\models \rightarrow \exists \Delta \{\Gamma \subseteq \Delta \ \& \ MC[\Delta]\}\}$$

$$MC[\Delta] \quad =_{df} \quad \Delta \not\models \ \& \ \forall \Delta' (\Delta \subset \Delta' \rightarrow \Delta' \vdash)$$

$$\Gamma \vdash \quad =_{df} \quad \forall \alpha [\Gamma \vdash \alpha] \quad \quad \quad \text{[general definition]}$$

$$\Gamma \vdash \quad \leftrightarrow \quad \exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\} \quad \quad \quad \text{[theorem about AS1]}$$

2. Fundamental Background Lemma

The set \mathbb{S} of sentences/formulas is denumerable, so there is an enumeration of \mathbb{S} .

Let $\langle \sigma_1, \sigma_2, \dots \rangle$ be one such enumeration. In particular,

$$\forall \alpha \exists k [\alpha = \sigma_k]$$

3. The Construction of Sequence $\langle G_1, G_2, \dots \rangle$

$$\begin{array}{lll} \Gamma_1 & =_{df} & \Gamma \\ \Gamma_{n+1} & =_{df} & \begin{cases} \Gamma \cup \{\sigma_n\} & \text{if } \Gamma \cup \{\sigma_n\} \not\models \\ \Gamma \cup \{\sim \sigma_n\} & \text{if } \Gamma \cup \{\sigma_n\} \models \end{cases} \end{array}$$

4. Every G_k is Consistent

Proven by simple math induction.

Key Lemma: $\Gamma \cup \{\alpha\} \vdash \& \Gamma \cup \{\sim \alpha\} \vdash . \rightarrow \Gamma \vdash$

Key Sub-Lemma: $\{\alpha \rightarrow \beta, \sim \alpha \rightarrow \beta\} \vdash \beta$

5. Construction of Maximal Set W

$$\Omega = \bigcup \{\Gamma_1, \Gamma_2, \dots\}$$

6. G is included in W

By set theory, $\forall k[\Gamma_k \subseteq \bigcup \{\Gamma_1, \Gamma_2, \dots\}]$. By hypothesis, $\Gamma = \Gamma_1$.

7. W is “Maximal” (i.e.:)

$$\forall \Omega' (\Omega \subset \Omega' \rightarrow \Omega' \vdash)$$

Suppose $\Omega \subset \Omega'$. Then $\delta \in \Omega' \& \delta \notin \Omega$ (some δ). $\delta = \sigma_k$ (some k). $\Gamma_{k+1} = \Gamma \cup \{\sigma_k\}$ or $\Gamma_{k+1} = \Gamma \cup \{\sim \sigma_k\}$, so $\Gamma_{k+1} = \Gamma \cup \{\delta\}$ or $\Gamma_{k+1} = \Gamma \cup \{\sim \delta\}$. Also, $\Gamma_{k+1} \subseteq \Omega$. In case 1, $\delta \in \Omega$, and hence $\delta \in \Omega'$, which is contradictory. In case 2, $\sim \delta \in \Omega$, and hence $\sim \delta \in \Omega'$. Then $\delta, \sim \delta \in \Omega'$, which means that Ω' is inconsistent.

8. W is Consistent

Reductio: Suppose $\Omega \vdash$. Then $\Omega \vdash \alpha \& \Omega \vdash \sim \alpha$ (some α). Let D_1 derive α from Ω ; let D_2 derive $\sim \alpha$ from Ω . Let $U_1 = D_1 \cap \Omega$; let $U_2 = D_2 \cap \Omega$; let $U = U_1 \cup U_2$. Since D_1, D_2 are both finite, U_1, U_2 are finite, so U is finite. Claim: $U \vdash \alpha \& U \vdash \sim \alpha$. So $U \vdash$. Also, $\langle \sigma_1, \sigma_2, \dots \rangle \cap U$ is finite, so $\{k: \sigma_k \in U\}$ has a largest element, call it m . Claim: $U \subseteq \Gamma_{m+1}$. But by item 4, Γ_{m+1} is consistent, so U is consistent; i.e., $U \not\vdash$.

11. Lindenbaum's Lemma: Formal Proof

LL:	" $G \models \mathbb{R} \ \& \ SD\{\Gamma \vdash D \ \& \ MC[D]\}$ "	
(1)	SHOW: $\forall \Gamma \{ \Gamma \vdash \rightarrow \exists \Delta \{ \Gamma \subseteq \Delta \ \& \ MC[\Delta] \} \}$	UCD
(2)	$\Gamma \vdash$	As
(3)	SHOW: $\exists \Delta \{ \Gamma \subseteq \Delta \ \& \ MC[\Delta] \}$	10,13,QL
(4)	$\exists \sigma \{ \sigma \text{ enumerates the class of formulas} \}$	Lemma E (unproven)
(5)	$\sigma \text{ enumerates the class of formulas}$	4,EO
(6)	$\sigma = \langle \sigma_1, \sigma_2, \dots \rangle \ \& \ \forall \alpha \exists k [\alpha = \sigma_k]$	5,ST
(7)	$\exists \langle \Gamma_1, \Gamma_2, \dots \rangle:$ $\Gamma_1 = \Gamma \ \& \ \forall n$ $\{\Gamma \cup \{\sigma_n\} \vdash \rightarrow [\Gamma_{n+1} = \Gamma \cup \{\sigma_n\}] \ .\&$ $\Gamma \cup \{\sigma_n\} \vdash \rightarrow [\Gamma_{n+1} = \Gamma \cup \{\sim \sigma_n\}] \}$	ST
(8)	let $\langle \Gamma_1, \Gamma_2, \dots \rangle$ be such a sequence	7,EO
(9)	let $\Omega = \cup \langle \Gamma_1, \Gamma_2, \dots \rangle$	ST,EO
(10)	SHOW: $\Gamma \subseteq \Omega$	9,11,12,IL
(11)	$\Gamma_1 \subseteq \cup \langle \Gamma_1, \Gamma_2, \dots \rangle$	ST
(12)	$\Gamma = \Gamma_1$	8,QL
(13)	SHOW: $MC[\Omega]$	Def MC
(14)	SHOW: $\Omega \vdash \ \& \ \forall \Omega' \{ \Omega \subset \Omega' \rightarrow \Omega' \vdash \}$	&D
(15)	b: SHOW: $\forall \Omega' \{ \Omega \subset \Omega' \rightarrow \Omega' \vdash \}$	UCD
(16)	$\Omega \subset \Omega'$	As
(17)	SHOW: $\Omega' \vdash$	23-31,SC
(18)	$\exists \delta \{ \delta \in \Omega' \ \& \ \delta \notin \Omega \}$	16,ST
(19)	$\delta \in \Omega' \ \& \ \delta \notin \Omega$	18,EO
(20)	$\exists k \{ \delta = \sigma_k \}$	5,Def enumerates
(21)	$\delta = \sigma_k$	20,EO
(22)	$\Gamma_{k+1} = \Gamma_k \cup \{\sigma_k\}$ or $\Gamma_{k+1} = \Gamma_k \cup \{\sim \sigma_k\}$	8,QL
(23)	$\Gamma_{k+1} = \Gamma_k \cup \{\delta\}$ or $\Gamma_{k+1} = \Gamma_k \cup \{\sim \delta\}$	21,22,IL
(24)	c1: $\Gamma_{k+1} = \Gamma_k \cup \{\delta\}$	As
(25)	$\delta \in \Omega$	9,24,ST
(26)	\times	19b,25,SL
(27)	c2: $\Gamma_{k+1} = \Gamma_k \cup \{\sim \delta\}$	As
(28)	$\sim \delta \in \Omega$	9,27,ST
(29)	$\sim \delta \in \Omega'$	16,28,ST
(30)	$\delta, \sim \delta \in \Omega'$	19a,29,SL
(31)	$\Omega' \vdash$	30,L8

(32)	a:SHOW: $\Omega \vdash$	ID
(33)	$\Omega \vdash$	As
(34)	SHOW: \times	132,133,SL
(35)	$\exists \delta [\Omega \vdash \delta \text{ & } \Omega \vdash \neg \delta]$	33,L10
(36)	$\Omega \vdash \delta$	35, $\exists \& O$
(37)	$\Omega \vdash \neg \delta$	35, $\exists \& O$
(38)	$\exists d [d \text{ derives } \delta \text{ from } \Omega]$	36, Def \vdash
(39)	$\exists d [d \text{ derives } \neg \delta \text{ from } \Omega]$	37, Def \vdash
(40)	$D_1 \text{ derives } \delta \text{ from } \Omega$	38, $\exists O$
(41)	$D_1 \text{ is a finite sequence}$	40, Def derives
(42)	$D_2 \text{ derives } \neg \delta \text{ from } \Omega$	39, $\exists O$
(43)	$D_2 \text{ is a finite sequence}$	42, Def derives
(44)	let $U_1 = \{x: x \in \Delta \text{ & } x \in D_1\}$	ST, $\exists O$
(45)	let $U_2 = \{x: x \in \Delta \text{ & } x \in D_2\}$	ST, $\exists O$
(46)	$U_1 \text{ is finite}$	41,44,ST
(47)	$U_2 \text{ is finite}$	43,45,ST
(48)	let $U = U_1 \cup U_2$	ST, $\exists O$
(49)	$U \text{ is finite}$	46,47,48,ST
(50)	$U \subseteq \Omega$	44,45,48,ST
(51)	$D_1 \text{ derives } \delta \text{ from } U$	40,48,G1
(52)	$\exists d [d \text{ derives } \delta \text{ from } U]$	51,QL
(53)	$U \vdash \delta$	52, Def \vdash
(54)	$D_2 \text{ derives } \neg \delta \text{ from } U$	42,48,G1
(55)	$\exists d [d \text{ derives } \neg \delta \text{ from } U]$	54,QL
(56)	$U \vdash \neg \delta$	55,Def \vdash
(57)	$U \vdash$	53,56,L9
(58)	$\{k: \sigma_k \in U\} \text{ is finite}$	5,49,ST
(59)	$\exists m [m = \max \{k: \sigma_k \in U\}]$ (largest element)	58,ST
(60)	$m = \max \{k: \sigma_k \in U\}$	59, $\exists O$
(61)	SHOW: $\forall n [\Gamma_n \vdash]$	MI[n ≥ 1]
(62)	SHOW: $\Gamma_1 \vdash$	2,12,IL
(63)	SHOW: $\forall n \{\Gamma_n \vdash \rightarrow \Gamma_{n+1} \vdash\}$	UCD [IC]
(64)	$\Gamma_n \vdash$	As [IH]
(65)	SHOW: $\Gamma_{n+1} \vdash$	SC [IS]
(66)	$\Gamma_n \cup \{\sigma_n\} \vdash \text{ or } \Gamma_n \cup \{\neg \sigma_n\} \vdash$	SL
(67)	c1: $\Gamma_n \cup \{\sigma_n\} \vdash$	As
(68)	$\Gamma_{n+1} = \Gamma_n \cup \{\sigma_n\}$	8,67,QL
(69)	$\Gamma_{n+1} \vdash$	67,68,IL
(70)	c2: $\Gamma_n \cup \{\neg \sigma_n\} \vdash$	As
(71)	$\Gamma_{n+1} = \Gamma_n \cup \{\neg \sigma_n\}$	8,70,QL
(72)	$\Gamma_n \vdash \rightarrow. \Gamma_n \cup \{\sigma_n\} \vdash \text{ or } \Gamma_n \cup \{\neg \sigma_n\} \vdash$	L7c
(73)	$\Gamma_n \cup \{\neg \sigma_n\} \vdash$	64,71,72,SL
(74)	$\Gamma_{n+1} \vdash$	71,73,IL

(75)	SHOW: $U \subseteq \Gamma_{m+1}$	UCD, Def \subseteq
(76)	$e \in U$	As
(77)	SHOW: $e \in \Gamma_{m+1}$	82-92,SC
(78)	$\exists k \{k \leq m \ \& \ e = \sigma_k\}$	5,60,62,ST
(79)	$k \leq m$	78, $\exists \& O$
(80)	$e = \sigma_k$	78, $\exists \& O$
(81)	$\Gamma_{k+1} \vdash$	61,QL
(82)	$\Gamma_k \cup \{\sigma_k\} \vdash$ or $\Gamma_k \cup \{\sigma_k\} \vdash$	SL
(83)	c1: $\Gamma_k \cup \{\sigma_k\} \vdash$	As
(84)	$\Gamma_{k+1} = \Gamma_k \cup \{\sigma_k\}$	8,83,QL
(85)	$e \in \Gamma_{k+1}$	80,84,ST
(86)	c2: $\Gamma_k \cup \{\sigma_k\} \vdash$	As
(87)	$\Gamma_{k+1} = \Gamma_k \cup \{\sim \sigma_k\}$	8,70,QL
(88)	$\Gamma_{k+1} \subseteq \Delta$	9,ST
(89)	$\sim \sigma_k \in \Delta$	87,88,ST
(90)	$\sim e \in \Delta$	80,89,IL
(91)	$e \in \Delta$	50,76,ST
(92)	\times	90,91,93,QL
(93)	SHOW: $\forall \alpha \{ \alpha \in \Omega \rightarrow \sim \alpha \notin \Omega \}$	UCD
(94)	$\alpha \in \Omega$	As
(95)	SHOW: $\sim \alpha \notin \Omega$	ID
(96)	$\sim \alpha \in \Omega$	As
(97)	SHOW: \times	105,106
(98)	$\exists k \{ \alpha \in \Gamma_k \}$	8,94,ST
(99)	$\exists k \{ \sim \alpha \in \Gamma_k \}$	8,96,ST
(100)	$\alpha \in \Gamma_{k1}$	98, $\exists O$
(101)	$\sim \alpha \in \Gamma_{k2}$	99, $\exists O$
(102)	$k_1 \leq k_2$	As [WLOG]
(103)	$\Gamma_{k1} \subseteq \Gamma_{k2}$	102,107
(104)	$\alpha \in \Gamma_{k2}$	100,103,ST
(105)	$\Gamma_{k2} \vdash$	101,104,L8
(106)	$\Gamma_{k2} \vdash$	61,QL

(107)		SHOW: $\forall n \forall m \{m \leq n \rightarrow \Gamma_m \subseteq \Gamma_n\}$	MI[n ≥ 1]
(108)		SHOW: $\forall m \{m \leq 1 \rightarrow \Gamma_m \subseteq \Gamma_1\}$	UCD[m ≥ 1]
(109)		$m \geq 1$	As
(110)		$m \leq 1$	As
(111)		SHOW: $\Gamma_m \subseteq \Gamma_1$	113,ST
(112)		$m = 1$	109,110,Arith
(113)		$\Gamma_m = \Gamma_1$	112,IL
(114)		SHOW: IC	CD
(115)		$\forall m \{m \leq n \rightarrow \Gamma_m \subseteq \Gamma_n\}$	As
(116)		SHOW: $\forall m \{m \leq n+1 \rightarrow \Gamma_m \subseteq \Gamma_{n+1}\}$	UCD
(117)		$m \leq n+1$	As
(118)		SHOW: $\Gamma_m \subseteq \Gamma_{n+1}$	SC
(119)		$m = n+1$ or $m \leq n$	117,Arith
(120)		c1: $m = n+1$	As
(121)		$\Gamma_m = \Gamma_{m+1}$	120,IL
(122)		$\Gamma_m \subseteq \Gamma_{n+1}$	121,ST
(123)		c2: $m \leq n$	As
(124)		$\Gamma_m \subseteq \Gamma_n$	115,123,QL
(125)		$\Gamma_n \cup \{\sigma_n\} \not\vdash$ or $\Gamma_n \cup \{\sigma_n\} \vdash$	SL
(126)		c1: $\Gamma_n \cup \{\sigma_n\} \not\vdash$	As
(127)		$\Gamma_{n+1} = \Gamma_n \cup \{\sigma_n\}$	8,126,QL
(128)		$\Gamma_m \subseteq \Gamma_{n+1}$	124,127,ST
(129)		c2: $\Gamma_n \cup \{\sigma_n\} \vdash$	As
(130)		$\Gamma_{n+1} = \Gamma_n \cup \{\sim \sigma_n\}$	8,129,QL
(131)		$\Gamma_m \subseteq \Gamma_{n+1}$	124,130,ST
(132)		$\Gamma_{m+1} \vdash$	75,G1c
(133)		$\Gamma_{m+1} \not\vdash$	61,QL

12. Lemmas Used to Prove the Verifiability Theorem

V1:	MC[G] \circledR " $a\{\alpha \in G \text{ or } \neg \alpha \in G\}$"	
(1)	SHOW: $MC[\Gamma] \rightarrow \forall \alpha \{\alpha \in \Gamma \text{ or } \neg \alpha \in \Gamma\}$	CD
(2)	MC[Γ]	As
(3)	SHOW: $\forall \alpha \{\alpha \in \Gamma \text{ or } \neg \alpha \in \Gamma\}$	UD
(4)	SHOW: $\alpha \in \Gamma \text{ or } \neg \alpha \in \Gamma$	\vee ID
(5)	$\alpha \notin \Gamma$	As
(6)	$\neg \alpha \in \Gamma$	As
(7)	SHOW: \times	13,14,SL
(8)	$\Gamma \subset \Gamma \cup \{\alpha\}$	5,ST
(9)	$\Gamma \subset \Gamma \cup \{\neg \alpha\}$	6,ST
(10)	$\forall \Delta \{\Gamma \subset \Delta \rightarrow \Delta \vdash\}$	2, Def MC [b]
(11)	$\Gamma \cup \{\alpha\} \vdash$	8,10,QL
(12)	$\Gamma \cup \{\neg \alpha\} \vdash$	9,10,QL
(13)	$\Gamma \vdash$	11,12,L7c2
(14)	$\Gamma \not\vdash$	2, Def MC [a]
V2:	MC[G] \circledR " $a\{\alpha \in G \text{ & } \neg \alpha \in G\}$"	
(1)	SHOW: $MC[\Gamma] \rightarrow \forall \alpha \{\alpha \in \Gamma \rightarrow \neg \alpha \notin \Gamma\}$	CD
(2)	MC[Γ]	As
(3)	SHOW: $\forall \alpha \{\alpha \in \Gamma \rightarrow \neg \alpha \notin \Gamma\}$	UCD
(4)	$\alpha \in \Gamma$	As
(5)	SHOW: $\neg \alpha \notin \Gamma$	ID
(6)	$\neg \alpha \in \Gamma$	As
(7)	SHOW: \times	10,11,SL
(8)	$\Gamma \vdash \alpha$	4,G0
(9)	$\Gamma \vdash \neg \alpha$	6,G0
(10)	$\Gamma \vdash$	8,9,L10
(11)	$\Gamma \not\vdash$	2, Def MC [a]
V3:	MC[G] \circledR {$a\{\alpha \in G \text{ & } \alpha \rightarrow \beta \in G\}$, $b\{\beta \in G\}$}	
(1)	SHOW: $MC[\Gamma] \rightarrow \{\alpha \in \Gamma \text{ & } \alpha \rightarrow \beta \in \Gamma \rightarrow \beta \in \Gamma\}$	C&CD
(2)	MC[Γ]	As
(3)	$\alpha \in \Gamma$	As
(4)	$\alpha \rightarrow \beta \in \Gamma$	As
(5)	SHOW: $\beta \in \Gamma$	8,G11
(6)	$\Gamma \vdash \alpha$	3,G0
(7)	$\Gamma \vdash \alpha \rightarrow \beta$	4,G0
(8)	$\Gamma \vdash \beta$	6,7,MPP
V4:	MC[G] \circledR . $b\{\beta \in G\}$ \circledR $a\{\alpha \rightarrow \beta \in G\}$	
(1)	SHOW: $MC[\Gamma] \rightarrow . \beta \in \Gamma \rightarrow \alpha \rightarrow \beta \in \Gamma$	CCD
(2)	MC[Γ]	As
(3)	$\beta \in \Gamma$	As
(4)	SHOW: $\alpha \rightarrow \beta \in \Gamma$	7,G0
(5)	$\Gamma \vdash \beta$	3,G0
(6)	Ax[$\beta \rightarrow (\alpha \rightarrow \beta)$]	R1
(7)	$\Gamma \vdash \beta \rightarrow (\alpha \rightarrow \beta)$	6,G5
(8)	$\Gamma \vdash \alpha \rightarrow \beta$	5,6,MPP
V5:	MC[G] \circledR . $a\{\alpha \in G \text{ & } \alpha \rightarrow \beta \in G\}$ \circledR $b\{\beta \in G\}$	
(1)	SHOW: $MC[\Gamma] \rightarrow . \alpha \in \Gamma \rightarrow \alpha \rightarrow \beta \in \Gamma$	CCD
(2)	MC[Γ]	As
(3)	$\alpha \in \Gamma$	As
(4)	SHOW: $\alpha \rightarrow \beta \in \Gamma$	8,G11
(5)	$\neg \alpha \in \Gamma$	3,V1
(6)	$\Gamma \vdash \neg \alpha$	5,G0
(7)	$\Gamma \vdash \neg \alpha \rightarrow (\alpha \rightarrow \beta)$	L1+G2
(8)	$\Gamma \vdash \alpha \rightarrow \beta$	6,7,MPP

13. The Verifiability Theorem

VT:	$\text{MC}[\Gamma] \text{ } \textcircled{R} \text{ } \Gamma\#$	
(1)	SHOW: $\text{MC}[\Gamma] \rightarrow \Gamma\#$	CD
(2)	MC[Γ]	As
(3)	SHOW: $\Gamma\#$	Def $\#$
(4)	SHOW: $\exists v[v < \Gamma]$	7,QL
(5)	$\exists f: \forall \alpha \{\alpha \in \Gamma \rightarrow f(\alpha)=T \text{ } \& \text{ } \alpha \notin \Gamma \rightarrow f(\alpha)=F\}$	ST
(6)	$\forall \alpha \{\alpha \in \Gamma \rightarrow v(\alpha)=T \text{ } \& \text{ } \alpha \notin \Gamma \rightarrow v(\alpha)=F\}$	5, $\exists \& O$
(7)	SHOW: v is a CSL-admissible valuation	Def val for CSL [&D]
(8)	a:SHOW: v is a valuation	Def val
(9)	SHOW: $\forall \alpha [v(\alpha) \in \{T,F\}]$	UD
(10)	SHOW: $v(\alpha) \in \{T,F\}$	11,ST
(11)	SHOW: $v(\alpha)=T$ or $v(\alpha)=F$	6,QL
(12)	b:SHOW: $\forall \alpha [v(\sim \alpha) = \sim v(\alpha)]$	UD
(13)	SHOW: $v(\sim \alpha) = \sim v(\alpha)$	14-26,SC
(14)	$\alpha \in \Gamma$ or $\alpha \notin \Gamma$	SL
(15)	c1: $\alpha \in \Gamma$	As
(16)	$v(\alpha)=T$	6,15,QL
(17)	$\sim \alpha \notin \Gamma$	2,15,V2
(18)	$v(\sim \alpha) = F$	6,17,QL
(19)	$F = \sim T$	Def \sim
(20)	$v(\sim \alpha) = \sim v(\alpha)$	16,18,19,IL
(21)	c2: $\alpha \notin \Gamma$	As
(22)	$v(\alpha)=F$	6,21,QL
(23)	$\sim \alpha \in \Gamma$	2,21,V1
(24)	$v(\sim \alpha) = T$	6,23,QL
(25)	$T = \sim F$	Def \sim
(26)	$v(\sim \alpha) = \sim v(\alpha)$	22,24,25,IL
(27)	c:SHOW: $\forall \alpha \forall \beta [v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)]$	UD
(28)	SHOW: $v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$	29-49,SC
(29)	$\alpha \in \Gamma$ or $\alpha \notin \Gamma$	SL
(30)	c1: $\alpha \in \Gamma$	As
(31)	$v(\alpha)=T$	6,30,QL
(32)	$\alpha \rightarrow \beta \in \Gamma$ or $\alpha \rightarrow \beta \notin \Gamma$	SL
(33)	c1: $\alpha \rightarrow \beta \in \Gamma$	As
(34)	$v(\alpha \rightarrow \beta) = T$	6,33,QL
(35)	$\beta \in \Gamma$	2,30,33,V3
(36)	$v(\beta) = T$	2,35,QL
(37)	$T = T \rightarrow T$	Def \rightarrow
(38)	$v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$	31,34,36,IL
(39)	c2: $\alpha \rightarrow \beta \notin \Gamma$	As
(40)	$\sim \beta \notin \Gamma$	2,39,V4
(41)	$\sim \beta \in \Gamma$	2,40,V1
(42)	\times	2,35,41,V2
(43)	c2: $\alpha \notin \Gamma$	As
(44)	$v(\alpha)=F$	6,43,QL
(45)	$\alpha \rightarrow \beta \in \Gamma$	2,43,V5
(46)	$v(\alpha \rightarrow \beta) = T$	6,45,QL
(47)	$\forall x[T = F \rightarrow x]$	Def \rightarrow
(48)	$T = F \rightarrow v(\beta)$	47,QL
(49)	$v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$	44,46,48,IL

14. Further Lemmas Used to Prove the Completeness Theorem

C0: $\sim \$v\$a\{v < a \& v < \sim a\}$

(1)	SHOW: $\sim \exists v \exists \alpha \{ v < \alpha \& v < \sim \alpha \}$	$\sim \exists 2 \& D$
(2)	$v < \alpha$	As
(3)	$v < \sim \alpha$	As
(4)	SHOW: \times	9,10,SL
(5)	$v(\alpha) = T$	2,Def <
(6)	$v(\sim \alpha) = T$	3,Def <
(7)	$v(\sim \alpha) = \sim v(\alpha)$	Def CSL valuation
(8)	$T = \sim T$	5-7,IL
(9)	$T = F$	8,Def \sim ,IL
(10)	$T \neq F$	Axiom 0

C1: $G \vdash a \Leftarrow G \dot{\vdash} \{\sim a\} \vdash$

(1)	SHOW: $\Gamma \vdash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vdash$	$\leftrightarrow D$
(2)	SHOW: \rightarrow	CD
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Gamma \cup \{\sim \alpha\} \vdash$	5,L10
(5)	SHOW: $\exists \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta \& \Gamma \cup \{\sim \alpha\} \vdash \sim \beta]$	7,9,QL
(6)	$\sim \alpha \in \Gamma \cup \{\sim \alpha\}$	ST
(7)	$\Gamma \cup \{\sim \alpha\} \vdash \sim \alpha$	6,G0
(8)	$\Gamma \subseteq \Gamma \cup \{\sim \alpha\}$	ST
(9)	$\Gamma \cup \{\sim \alpha\} \vdash \alpha$	3,8,G1
(10)	SHOW: \leftarrow	CD
(11)	$\Gamma \cup \{\sim \alpha\} \vdash$	As
(12)	SHOW: $\Gamma \vdash \alpha$	15,16,MPP
(13)	$\forall \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta]$	11, Def \vdash
(14)	$\Gamma \cup \{\sim \alpha\} \vdash \alpha$	13,QL
(15)	$\Gamma \vdash \sim \alpha \rightarrow \alpha$	14,DT
(16)	$\Gamma \vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$	L3+G2

C2: $G \vdash D \& D \# . \textcircled{R} \quad G \#$

(1)	SHOW: $\Gamma \subseteq \Delta \& \Delta \# . \rightarrow \Gamma \#$	&CD
(2)	$\Gamma \subseteq \Delta$	As
(3)	$\Delta \#$	As
(4)	SHOW: $\Gamma \#$	Def $\#$
(5)	SHOW: $\exists v [v < \Gamma]$	11,QL
(6)	$\exists v [v < \Delta]$	3, Def $\#$
(7)	$v < \Delta$	6,EO
(8)	$\forall x \{x \in \Delta \rightarrow v < x\}$	7, Def $v < \Gamma$
(9)	$\forall x \{x \in \Gamma \rightarrow x \in \Delta\}$	2,ST
(10)	$\forall x \{x \in \Gamma \rightarrow v < x\}$	8,9,QL
(11)	$v < \Gamma$	10, Def $v < \Gamma$

C3:	$\mathbf{G} \models a \Leftarrow \mathbf{G} \bar{E}\{\sim a\} \models$	
(1)	SHOW: $\Gamma \models \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \models$	$\leftrightarrow D$
(2)	SHOW: \rightarrow	CD
(3)	$\Gamma \models \alpha$	As
(4)	SHOW: $\Gamma \cup \{\sim \alpha\} \models$	Def $\Gamma \models$
(5)	SHOW: $\sim \exists v[v < \Gamma \cup \{\sim \alpha\}]$	$\sim \exists D$
(6)	$v < \Gamma \cup \{\sim \alpha\}$	As
(7)	SHOW: \times	9,11,C0
(8)	$v < \Gamma$	6,ST,C2
(9)	$v < \sim \alpha$	6,ST,C2
(10)	$\forall v\{v < \Gamma \rightarrow v < \alpha\}$	3, Def $\Gamma \models \alpha$
(11)	$v < \alpha$	8,10,QL
(12)	SHOW: \leftarrow	CD
(13)	$\Gamma \cup \{\sim \alpha\} \models$	As
(14)	SHOW: $\Gamma \models \alpha$	Def $\Gamma \models \alpha$
(15)	SHOW: $\forall v\{v < \Gamma \rightarrow v < \alpha\}$	UCD
(16)	$v < \Gamma$	As
(17)	SHOW: $v < \alpha$	ID
(18)	$\sim [v < \alpha]$	As
(19)	SHOW: \times	26,27,QL
(20)	$v(\alpha) \neq T$	18,Def $<$
(21)	$v(\alpha) = F$	20,Def valuation
(22)	$v(\sim \alpha) = \sim v(a)$	Def CSL-valuation
(23)	$v(\sim \alpha) = \sim F$	21,22,IL
(24)	$v(\sim \alpha) = T$	23,Def \sim
(25)	$v < \sim \alpha$	24,Def $<$
(26)	$v < \Gamma \cup \{\sim \alpha\}$	16,25,ST,Def $<$
(27)	$\sim \exists v[v < \Gamma \cup \{\sim \alpha\}]$	13, Def \models

15. The Completeness Theorem

AS1 is complete for V(CSL) with respect to arguments;
 i.e., for any set Γ of formulas, and for any formula α ,
 if Γ semantically entails α wrt V(CSL),
 then Γ deductively entails α wrt AS1.

CT: " \mathbf{G} " $\mathbf{a}\{\mathbf{G}\vDash\mathbf{a}\}$ \mathbb{R} $\mathbf{G}\vdash\mathbf{a}\}$

(1)	SHOW: $\Gamma\vDash\alpha \rightarrow \Gamma\vdash\alpha$	2,SL
(2)	SHOW: $\Gamma\not\vDash\alpha \rightarrow \Gamma\not\vdash\alpha$	CD
(3)	$\Gamma\not\vDash\alpha$	As
(4)	SHOW: $\Gamma\not\vdash\alpha$	DD
(5)	$\Gamma\cup\{\sim\alpha\}\not\vdash$	3,C1
(6)	$\exists\Delta\{\Gamma\cup\{\sim\alpha\}\subseteq\Delta \ \& \ \text{MC}[\Delta]\}$	5,LL
(7)	$\Gamma\cup\{\sim\alpha\}\subseteq\Delta$	6, $\exists\&O$
(8)	$\text{MC}[\Delta]$	6, $\exists\&O$
(9)	$\Delta\not\vdash$	8,VT
(10)	$\Gamma\cup\{\sim\alpha\}\not\vdash$	7,9,C2
(11)	$\Gamma\not\vdash\alpha$	10,C3

16. Study Questions for Chapters 8-12

1. The Semantic Characterization of Logic

Define the following:

- (1) Semantic entailment ($\Gamma \vDash \alpha$)
- (2) Semantic validity ($\models \alpha$)
- (3) Semantic (in)consistency [(un)verifiability] ($\Gamma \vDash$)

2. The Deductive Characterization of Logic

Define the following:

- (1) Deductive system
- (2) Axiom system
- (3) Derivation
- (4) Proof
- (5) Deductive entailment ($\Gamma \vdash \alpha$)
- (6) Deductive validity (theoremhood) ($\vdash \alpha$)
- (7) Deductive (in)consistency ($\Gamma \vdash$)
- (8) Maximal consistent set ($MC[\Gamma]$)

3. The Relation between Semantic and Deductive Characterizations of Logic

Define the following:

- (1) Soundness (wrt formulas/arguments)
- (2) Completeness (wrt formulas/arguments)
- (3) Mutual Consistency (wrt formulas/arguments)

4. General Theorems about Deductive/Axiom Systems

Prove the following:

- (1) $\vdash \alpha \leftrightarrow \emptyset \vdash \alpha$
- (2) α is an axiom $\rightarrow \vdash \alpha$
- (3) $\alpha \in \Gamma \rightarrow \Gamma \vdash \alpha$
- (4) $\Gamma \vdash \alpha \ \& \ \Gamma \subseteq \Delta \ . \rightarrow \Delta \vdash \alpha$
- (5) $\Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \beta \ . \rightarrow \ \Gamma \vdash \beta$
- (6) $\forall \alpha (\alpha \in \Delta \rightarrow \Gamma \vdash \alpha) \ \& \ \Delta \vdash \beta \ . \rightarrow \ \Gamma \vdash \beta$
- (7) $\alpha \vdash \beta \ \& \ \beta \vdash \alpha \ . \rightarrow \ \Gamma \cup \{\alpha\} \vdash \gamma \leftrightarrow \Gamma \cup \{\beta\} \vdash \gamma$
- (8) $MC[\Gamma] \ \& \ \alpha \notin \Gamma \ . \rightarrow \ \Gamma \cup \{\alpha\} \vdash$
- (9) $MC[\Gamma] \rightarrow \forall \alpha (\Gamma \vdash \alpha \rightarrow \alpha \in \Gamma)$

5. Mathematical Induction

- (a) Describe the general form of Weak/Strong Induction.
- (b) Describe the general form of a (strong) inductive proof about derivations.

6. Axiom System AS1

- (a) State the rules of AS1.
- (b) Prove the following in AS1, including line numbers and annotation:
 - (1) $P \rightarrow P$
 - (2) $\sim P \rightarrow (P \rightarrow Q)$
 - (3) $Q \rightarrow (P \rightarrow P)$
 - (4) $[P \rightarrow (P \rightarrow Q)] \rightarrow [P \rightarrow Q]$

7. “Small” Theorems about AS1

Prove the following:

- (1) $\vdash \alpha \ \& \ \vdash \alpha \rightarrow \beta . \rightarrow \vdash \beta$
- (2) $\vdash \alpha \rightarrow \vdash \beta \rightarrow \alpha$
- (3) $\vdash \alpha \rightarrow \sim \sim \alpha$
- (4) $\{\alpha \rightarrow \beta, \beta \rightarrow \gamma, \alpha\} \vdash \gamma$
- (5) $\vdash (\alpha \rightarrow \sim \alpha) \rightarrow \sim \alpha$

8. “Medium Size” Theorems about AS1

Prove the following:

- (1) $\Gamma \vdash \alpha \rightarrow \beta \rightarrow \Gamma \cup \{\alpha\} \vdash \beta$
- (2) $\Gamma \cup \{\alpha\} \vdash \beta \ \& \ \Gamma \cup \{\sim \alpha\} \vdash \beta . \rightarrow \Gamma \vdash \beta$
- (3) $\Gamma \cup \{\alpha\} \vdash \ \& \ \Gamma \cup \{\sim \alpha\} \vdash . \rightarrow \Gamma \vdash$
- (4) $MC[\Gamma] \rightarrow \forall \alpha (\alpha \in \Gamma \vee \sim \alpha \in \Gamma)$
- (5) $\Gamma \vdash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vdash$

9. “Big” Theorems about AS1

Prove the following:

- (1) The Deduction Theorem
- (2) The Soundness Theorem (weak and strong)
- (3) Every maximal consistent set is verifiable
- (4) Lindenbaum’s Lemma
- (5) Γ is deductively consistent \leftrightarrow Γ is semantically consistent.
- (6) The Completeness Theorem (weak and strong)

2. Appendix — Generalization of Lindenbaum’s Lemma

1. Lindenbaum’s Lemma

Lindenbaum’s Lemma is proved for a particular axiom system — AS1. In particular, we prove the following theorem.

$$(LL) \quad \Gamma \not\models \rightarrow \exists \Delta \{ \Gamma \subseteq \Delta \ \& \ \Gamma \not\models \ \& \ \forall \Delta (\Gamma \subset \Delta \rightarrow \Delta \vdash) \} \quad \text{for deductive system AS1.}$$

Recall that the following are the relevant definitions.

- (d1) $\Gamma \vdash \alpha \equiv_{df} \text{there is a derivation of } \alpha \text{ from } \Gamma \text{ (the in relevant system)}$
- (d2) $\Gamma \vdash \equiv_{df} \forall \alpha [\Gamma \vdash \alpha]$
- (d3) $\Gamma \not\models \equiv_{df} \sim [\Gamma \vdash]$

2. The Obvious Generalization of LL is Not True!

The obvious generalization of LL goes as follows.

$$(LL?) \quad \Gamma \not\models \rightarrow \exists \Delta \{ \Gamma \subseteq \Delta \ \& \ \Gamma \not\models \ \& \ \forall \Delta (\Gamma \subset \Delta \rightarrow \Delta \vdash) \} \quad \text{for any deductive system } \Sigma$$

To see that (LL?) is false, we merely need to construct a counterexample — which is to say, an axiom system for which (LL?) is not true. Consider the following (admittedly bizarre) axiom system. First the underlying language consists of a denumerable sequence $\langle \alpha_1, \alpha_2, \dots \rangle$ of atomic formulas, and nothing else. There is exactly one rule, given as follows.

$$(R) \quad \alpha_{k+1} \rightarrow \alpha_k \quad \text{for any } k$$

In other words, one is entitled to infer any formula from its successor.

Notice that every finite set is consistent, its logical closure being obtained by adding all “earlier” formulas. We want to show that at least one consistent set cannot be extended to a maximal consistent set. Consider the empty set \emptyset , which is evidently consistent. Consider applying the Lindenbaum construction beginning at \emptyset . At every stage in the construction, the formula considered at that stage is consistent with all the formulas already added, so it is added as well. If we do not add it, then the resulting set will not be maximal. So, when the process is over, every formula has been added. Although the resulting set is maximal, it is not consistent.

The associated semantics for this logic is also peculiar. In particular, for each index k , define v_k so that $v_k(\alpha_i) = T$ iff $i \leq k$. Notice, as a result, that there is no maximal valuation. No matter what valuation you examine, there is another valuation that verifies more formulas. For the same reason there are no maximal consistent sets of formulas.

3. A Useful Generalization of LL

The reason that LL is useful in the completeness proof for classical sentential logic is that we have the following supporting theorems.

$$(t1) \quad \Gamma \vdash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vdash \quad [\text{AS1}]$$

$$(t2) \quad \Gamma \vDash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vDash \quad [\text{CSL}]$$

Although these are true of classical logic, they are not generally true. We cannot even presume that the logic under consideration has a negation sign. This is not completely bizarre; we might be interested in the *implicative fragment* of CSL, or the *positive fragment* of CSL. In proving completeness for these systems, there is no negation sign, so a proof cannot rely on (t1) and (t2).

Remember what we want to prove — completeness — which amounts to the following.

$$(c) \quad \Gamma \vDash \alpha \rightarrow \Gamma \vdash \alpha$$

By contraposition, this is equivalent to.

$$(c') \quad \Gamma \not\vDash \alpha \rightarrow \Gamma \not\vdash \alpha$$

With this in mind, we propose the following generalization of LL.

LL+

Let Σ be an axiom system underwritten by a denumerable language S . Define the deductive notions for Σ in the usual manner. Then for any Γ, δ ,

$$\Gamma \not\vDash \delta \rightarrow \exists \Delta \{ \Gamma \subseteq \Delta \ \& \ \Delta \not\vDash \delta \ \& \ \forall \Delta' (\Delta \subset \Delta' \rightarrow \Delta' \vdash \delta) \}$$

4. Proof of LL+

Proof: Suppose $\Gamma \not\vDash \delta$ to show $\exists \Delta \{ \Gamma \subseteq \Delta \ \& \ \Delta \not\vDash \delta \ \& \ \forall \Delta' (\Delta \subset \Delta' \rightarrow \Delta' \vdash \delta) \}$. It is sufficient to construct a set for which this is true. Consider an enumeration $\langle \varepsilon_1, \varepsilon_2, \dots \rangle$ of S . Based on this enumeration, construct the infinite sequence of sets $\langle \Gamma_1, \Gamma_2, \dots \rangle$, inductively defined as follows.

$$\begin{aligned} \Gamma_1 &= \Gamma; \\ \Gamma_{n+1} &= \Gamma_n \cup \{\varepsilon_n\} \quad \text{if} \quad \Gamma_n \cup \{\varepsilon_n\} \not\vDash \beta \\ &= \Gamma_n \quad \text{otherwise.} \end{aligned}$$

$$\text{Let } \Omega = \cup \{ \Gamma_n : n = 1, 2, 3, \dots \}$$

- Claim:**
- (1) $\Gamma \subseteq \Omega$,
 - (2) $\Omega \not\vDash \delta$,
 - (3) $\forall \Omega' (\Omega \subset \Omega' \rightarrow \Omega' \vdash \delta)$

$$(1) \quad \Gamma \subseteq \Omega$$

This follows from the fact that $\Gamma = \Gamma_1$, and $\Gamma_1 \subseteq \Omega$. The latter follows from the definition of Ω , together with set theory [including the theorem: $X \in \Delta \rightarrow X \subseteq \cup \Delta$].

$$(2) \quad \Omega \not\vdash \delta$$

Suppose $\Omega \vdash \delta$, to show a contradiction. Then there is a derivation D of δ from Ω . By definition of derivation, D is a finite sequence; accordingly, at most finitely-many premises (elements of Ω) are used in D. Let U be the finite subset of used premises. Notice first that $U \vdash \delta$. Also, since U is finite, by (2.1), for some m, $U \subseteq \Gamma_m$, so $\Gamma_m \vdash \delta$. But this contradicts (2.2).

$$(2.1) \quad \text{if } \Delta \text{ is finite, and } \Delta \subseteq \Omega, \text{ then } \Delta \subseteq \Gamma_n, \text{ for some } n$$

Suppose Δ is finite, and $\Delta \subseteq \Omega$. Then the set $\{n : \varepsilon_n \in \Delta\}$ has a largest element, call it g. Claim: $\Delta \subseteq \Gamma_{g+1}$. Suppose $\alpha \in \Delta$; then $\alpha = \varepsilon_k$ for some $k \leq g$. Also, since $\Delta \subseteq \Omega$, $\alpha \in \Omega$, so $\varepsilon_k \in \Omega$. So, by (2.1.1), $\varepsilon_k \in \Gamma_{k+1}$, so $\alpha \in \Gamma_{k+1}$. But by (2.1.2), $\Gamma_{k+1} \subseteq \Gamma_{g+1}$, so $\alpha \in \Gamma_{g+1}$.

$$(2.1.1) \quad \varepsilon_n \in \Omega \rightarrow \varepsilon_n \in \Gamma_{n+1}$$

Suppose $\varepsilon_n \in \Omega$. Given the definition of Ω , and the fact that the natural numbers are well-ordered, there is an m such that, $\varepsilon_n \in \Gamma_m \& \sim \exists k(k < m \& \varepsilon_n \in \Gamma_k)$. Notice that Γ_m properly includes all its predecessors. Now, $m=1$ or $m>1$. Case 1: $m=1$. $\varepsilon_n \in \Gamma_1$, from which it follows (2.1.2) that $\varepsilon_n \in \Gamma_{n+1}$. Case 2: $m>1$. Then $\Gamma_m = \Gamma_{m-1} \cup \{\varepsilon_{m-1}\}$ or $\Gamma_m = \Gamma_{m-1}$. The latter is ruled out, since Γ_m properly includes all its predecessors. So $\varepsilon_n \in \Gamma_{m-1} \cup \{\varepsilon_{m-1}\}$. But since $m-1 < m$, $\varepsilon_n \notin \Gamma_{m-1}$, so $\varepsilon_n = \varepsilon_{m-1}$. Given that the enumeration is 1-1 [every formula appears exactly once], it follows that $n = m-1$. So $n+1 = m$. So $\varepsilon_n \in \Gamma_{n+1}$.

$$(2.1.2) \quad \forall n \forall j(j \leq n \rightarrow \Gamma_j \subseteq \Gamma_n)$$

Shown by induction on n [exercise].

$$(2.2) \quad \forall n[\Gamma_n \not\vdash \delta]$$

(by weak induction):

$$(BC) \quad \text{show: } \Gamma_1 \not\vdash \delta$$

By hypothesis $\Gamma \not\vdash \delta$, and by definition, $\Gamma_1 = \Gamma$.

$$(IH) \quad \text{assume: } \Gamma_n \not\vdash \delta$$

$$(IS) \quad \text{show: } \Gamma_{n+1} \not\vdash \delta$$

By definition, $\Gamma_{n+1} = \Gamma \cup \{\varepsilon_n\}$ if $\Gamma \cup \{\varepsilon_n\} \not\vdash \beta$, and $\Gamma_{n+1} = \Gamma$ otherwise. In either case $\Gamma_{n+1} \not\vdash \delta$.

$$(3) \quad \forall \Omega'(\Omega \subseteq \Omega' \rightarrow \Omega' \vdash \delta) \quad \text{which is ST-equivalent to: } \forall \varepsilon[\varepsilon \notin \Omega \rightarrow \Omega \cup \{\varepsilon\} \vdash \delta]$$

Suppose $\Omega \subseteq \Omega'$. $\alpha \in \Omega' \& \alpha \notin \Omega$, for some α . Given the definition of Ω , $\forall n[\alpha \notin \Gamma_n]$. On the other hand, α appears exactly once in the enumeration $\langle \varepsilon_1, \varepsilon_2, \dots \rangle$; let $\alpha = \varepsilon_k$. By definition, $\Gamma_{k+1} = \Gamma_k \cup \{\alpha\}$ if $\Gamma_k \cup \{\alpha\} \not\vdash \delta$. But since $\forall n[\alpha \notin \Gamma_n]$, $\alpha \notin \Gamma_{k+1}$, so $\Gamma_k \cup \{\alpha\} \vdash \delta$. But $\Gamma_k \subseteq \Omega$, and $\alpha \in \Omega'$, so $\Gamma_k \cup \{\alpha\} \subseteq \Omega'$, so $\Omega' \vdash \delta$.

5. Further Theorems about W

Before continuing, we prove the following very useful theorem about the set Ω constructed in the proof of LL+ [Section 4].

Th

Let Σ be an axiom system underwritten by a denumerable language S . Define the deductive notions for Σ in the usual manner. Suppose $\Gamma \not\vdash \delta$. Let Ω be the maximal set constructed in the proof of LL+. Then

- (1) $\forall \alpha \{\Omega \vdash \alpha \rightarrow \alpha \in \Omega\}$
- (2) $\Omega \not\vdash$

In other words, Ω is (1) logically closed, and (2) consistent.

Proof:

- (1) Suppose $\Omega \vdash \alpha$, but $\alpha \notin \Omega$. But according to LL+ Clause 3, $\Omega \cup \{\alpha\} \vdash \delta$. But we also have the general theorem that, if $\Omega \vdash \alpha$, and $\Omega \cup \{\alpha\} \vdash \delta$, then $\Omega \vdash \delta$. But according to LL+ Clause 2, $\Omega \not\vdash \delta$.
- (2) Suppose Ω is inconsistent [$\Omega \vdash \bot$]. Then, by definition, $\forall \alpha [\Omega \vdash \alpha]$. But according to LL+ Clause 2, $\Omega \not\vdash \delta$.

6. Characteristic Valuations and A General Completeness Result

With the generalized Lindenbaum-Lemma LL+ in hand, we are now in a position to prove a general result about completeness. First, we define the general set-theoretic notion of *characteristic function*.

Def

Let S be a set of formulas of a formal language, and let Δ be a subset of S . Then the *characteristic valuation* of Δ (relative to S) is defined to be the function v_Δ from S into $\{T, F\}$ defined so that:

$$\begin{aligned} v_S(\varepsilon) &= T && \text{if } \varepsilon \in \Delta \\ v_S(\varepsilon) &= F && \text{otherwise} \end{aligned}$$

In other words, v_Δ assigns T to formulas in Δ , and it assigns F to formulas not in Δ . With this notion in hand, we can now state and prove the first of our general theorems.

Th.

Let Σ be an axiom system, and let V be a class admissible valuations, both defined for the same formal language. Consider an arbitrary Γ and δ such that

$\Gamma \not\vdash \delta$, and suppose Ω is the maximal set constructed in the proof of LL+.

Consider the *characteristic valuation* associated with Ω .

Suppose that, for each such Ω , $v_\Omega \in V$.

Then Σ is complete for V .

Proof: We wish to show $\Gamma \vDash \alpha \rightarrow \Gamma \vdash \alpha$. We argue contrapositively. Suppose $\Gamma \not\vdash \alpha$, to show $\Gamma \not\models \alpha$. Given LL+, we know that there is a set Ω such that $\Gamma \subseteq \Omega$, $\Omega \not\vdash \alpha$, and $\forall \Omega' \{\Omega \subset \Omega' \rightarrow \Omega' \vdash \alpha\}$. By hypothesis, the characteristic valuation v_Ω is an element of V . So $v_\Omega(\phi) = T$ iff $\phi \in \Omega$. Since $\Gamma \subseteq \Omega$, $v_\Omega(\gamma) = T$ for every $\gamma \in \Gamma$. Also, since $\Omega \not\vdash \alpha$, $\alpha \notin \Omega$, so $v_\Omega(\alpha) = F$. Thus, v_Ω verifies Γ , but falsifies α , so $\Gamma \not\models \alpha$.

The previous theorem gives a sufficient condition for completeness. The following theorem provides the natural converse.

Th.

Let Σ be an axiom system, and let V be a class admissible valuations, both defined for the same formal language. Suppose that Σ is both complete and sound for V , which is to say that the following holds, for all Γ and α .

$$\Gamma \vdash \alpha \leftrightarrow \Gamma \vDash \alpha$$

Suppose $\Gamma \not\vdash \delta$. Suppose Ω is any maximal set constructed by the technique prescribed in the proof of LL+.

Then the characteristic valuation v_Ω must be an element of V .

In other words, in order for Σ to be complete and sound for V , every maximal set constructed in the manner prescribed in the proof of LL+ must give rise to a V -admissible valuation.

Proof: Suppose $\Gamma \not\vdash \delta$, and suppose Ω is one of the associated maximal sets. Then $\Omega \not\vdash \delta$. So, by completeness, we have $\Omega \not\models \delta$. This means that there is a valuation v in V that verifies Ω but falsifies δ . Claim: the v in question is in fact the characteristic valuation v_Ω . Given the extensionality of functions, it is sufficient to prove that $v(\phi) = T$ iff $\phi \in \Omega$. The if-half is obvious, so we consider the only-if-half. Suppose $v(\phi) = T$, but $\phi \notin \Omega$. Then, v verifies $\Omega \cup \{\phi\}$ but falsifies δ , so $\Omega \cup \{\phi\} \not\models \delta$. So, by soundness, $\Omega \cup \{\phi\} \not\vdash \delta$. But, $\Omega \subset \Omega \cup \{\phi\}$, so by LL+ Clause 3, $\Omega \cup \{\phi\} \vdash \delta$.

7. How the General Completeness Theorem Works in the Case of CSL

We now have a general theorem about axiom systems (with denumerable languages). How can we use this theorem in the proof of completeness for a given axiom system with respect to a given semantic system? In particular, how do we use it in connection with AS1 and CSL? In this section, we answer that question.

We want to prove the completeness of AS1 wrt CSL. That means we want to show the following, where \vDash is defined relative to CSL, and \vdash is defined relative to AS1.

$$\Gamma \vDash \alpha \rightarrow \Gamma \vdash \alpha$$

Proof: We argue contrapositively. Suppose $\Gamma \not\vDash \alpha$, to show $\Gamma \not\models \alpha$. Given LL+, we know that there is a set Ω such that $\Gamma \subseteq \Omega$, $\Omega \not\models \alpha$, and $\forall \Omega' \{\Omega \subset \Omega' \rightarrow \Omega' \vdash \alpha\}$. Let v be the characteristic valuation of Ω ; $v = v_\Omega$. As argued before, v verifies Γ but falsifies α , so the question then is whether v is CSL-admissible. This amounts to whether the following hold.

- (1) $v(\sim \alpha) = \sim v(\alpha)$
- (2) $v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$

(1) $\alpha \in \Omega$ or $\alpha \notin \Omega$.

c1: $\alpha \in \Omega$. Then $v(\alpha) = T$, so $\sim v(\alpha) = F$, so we need to show that $v(\sim \alpha) = F$. But this is true iff $\sim \alpha \notin \Omega$. But this follows from LΩ1. [Note: supporting lemmas are proven later.]

c2: $\alpha \notin \Omega$. Then $v(\alpha) = F$, so $\sim v(\alpha) = T$, so we need to show that $v(\sim \alpha) = T$, which is true iff $\sim \alpha \in \Omega$. But this follows from LΩ2.

(2) $\alpha \notin \Omega$, or $\beta \in \Omega$, or $\alpha \in \Omega \& \beta \notin \Omega$

c1: $\alpha \notin \Omega$. Then $v(\alpha) = F$, so $v(\alpha) \rightarrow v(\beta) = F \rightarrow v(\beta) = T$. The question then is whether $v(\alpha \rightarrow \beta) = T$, which is the question whether $\alpha \rightarrow \beta \in \Omega$. Since $\alpha \notin \Omega$, by L2, $\sim \alpha \in \Omega$. But by L1, $\sim \alpha \vdash \alpha \rightarrow \beta$; therefore, since Ω is logically closed [Section 5], $\alpha \rightarrow \beta \in \Omega$.

c2: $\beta \in \Omega$. Then $v(\beta) = T$, so $v(\alpha) \rightarrow v(\beta) = v(\alpha) \rightarrow T = T$. So the question is whether $v(\alpha \rightarrow \beta) = T$, which is the question whether $\alpha \rightarrow \beta \in \Omega$. By L2, $\beta \vdash \alpha \rightarrow \beta$. Therefore, since Ω is logically closed, so $\alpha \rightarrow \beta \in \Omega$, so $v(\alpha \rightarrow \beta) = T$.

c3: $\alpha \in \Omega$ and $\beta \notin \Omega$. Then $v(\alpha) = T$, and $v(\beta) = F$, so $v(\alpha) \rightarrow v(\beta) = T \rightarrow F = F$. So the question is whether $v(\alpha \rightarrow \beta) = F$; this is true iff $\alpha \rightarrow \beta \notin \Omega$. Suppose $\alpha \rightarrow \beta \in \Omega$. By L3, $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$. Therefore, since Ω is logically closed, $\beta \in \Omega$, which contradicts our initial hypothesis.

8. Supporting Lemmas

We have now reduced the completeness theorem to showing the lemmas alluded to in the previous section. First, we prove the two lemmas about the set Ω constructed in the proof of LL+.

$$(L\Omega 1) \quad \forall \alpha \{ \alpha \in \Omega \rightarrow \sim \alpha \notin \Omega \}$$

| Proof: Suppose $\alpha \in \Omega$, and $\sim \alpha \in \Omega$. By L2, $\sim \alpha \vdash \alpha \rightarrow \beta$, and by L3, $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$. Therefore since Ω is logically closed, $\beta \in \Omega$, but this contradicts the result that $\Omega \not\models \beta$.

$$(L\Omega 2) \quad \forall \alpha \{ \alpha \notin \Omega \rightarrow \sim \alpha \in \Omega \}$$

Proof: Suppose $\alpha \notin \Omega$, and $\sim\alpha \notin \Omega$. Then by LL+ Clause 3, $\Omega \cup \{\alpha\} \vdash \delta$, and $\Omega \cup \{\sim\alpha\} \vdash \delta$, so by L4, $\Omega \vdash \delta$, which contradicts LL+ Clause 2.

That leaves proving the following lemmas about system AS1.

- (L1) $\sim\alpha \vdash \alpha \rightarrow \beta$
- (L2) $\beta \vdash \alpha \rightarrow \beta$
- (L3) $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$
- (L4) $\Gamma \cup \{\alpha\} \vdash \beta \text{ & } \Gamma \cup \{\sim\alpha\} \vdash \beta \rightarrow \Gamma \vdash \beta$

At this point, we have reduced the completeness theorem to its bare essentials. If an axiom system yields the above theorems, then it is complete for CSL.

At this point, the simplest thing to do is simply postulate whatever rules we need to achieve the above theorems. Consider starting with the following rules.

- (r1) $\sim\alpha \rightarrow \alpha \rightarrow \beta$
- (r2) $\beta \rightarrow \alpha \rightarrow \beta$
- (r3) $\alpha ; \alpha \rightarrow \beta \rightarrow \beta$

These clearly will yield (L1)-(L3).

But (L4) is more complicated. What single rule will produce this result. The following is a reasonable first approximation.

- (r4) $\alpha \rightarrow \beta ; \sim\alpha \rightarrow \beta \rightarrow \beta$

But this is not quite enough. Suppose we try to prove (L4). Suppose $\Gamma \cup \{\alpha\} \vdash \beta$ and $\Gamma \cup \{\sim\alpha\} \vdash \beta$, to show $\Gamma \vdash \beta$. Given (r4) and some general theorems about deduction, what we need is $\Gamma \vdash \alpha \rightarrow \beta$, and $\Gamma \vdash \sim\alpha \rightarrow \beta$. Obtaining these, however, involves the Deduction Theorem (DT), so the issue comes down to whether we can prove DT. But, as shown earlier, given that modus ponens is a rule, DT holds if and only if we have the following theorems.

- (1) $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$
- (2) $\vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$

The easiest way to obtain these theorems is to make the formulas in question axiom schemata. That is exactly what we do in formulating AS1.

- (R1) $\rightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2) $\rightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$

Adding these to our earlier rules, and renumbering, we have the following.

- (R1) $\rightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2) $\rightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3) $\sim\alpha \rightarrow \alpha \rightarrow \beta$
- (R4) $\alpha \rightarrow \beta ; \sim\alpha \rightarrow \beta \rightarrow \beta$
- (R5) $\alpha ; \alpha \rightarrow \beta \rightarrow \beta$

Notice that our old rule (r2) is absorbed into (R1) and (R5).

If we insist on having only one non-axiomatic rule, modus ponens, then (R3) and (R4) can be replaced by the following.

- (R3) $\vdash \sim\alpha \rightarrow (\alpha \rightarrow \beta)$
- (R4) $\vdash (\sim\alpha \rightarrow \beta) \rightarrow [(\alpha \rightarrow \beta) \rightarrow \beta]$

9. Review

We began with a proof of a general Lindenbaum Lemma – called LL+. We then showed how to prove completeness for any axiom system based on \sim and \rightarrow . In particular, proving completeness reduces to proving a number of lemmas — L1–L4. The easiest way to obtain these lemmas is to build them into our axiom system. We must also prove the Deduction Theorem along the way.

The result is an axiom system that allows the most efficient proof of the completeness theorem.

AS*

- (R1) $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2) $\vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3) $\vdash \sim\alpha \rightarrow (\alpha \rightarrow \beta)$
- (R4) $\vdash (\sim\alpha \rightarrow \beta) \rightarrow [(\alpha \rightarrow \beta) \rightarrow \beta]$
- (R5) $\alpha ; \alpha \rightarrow b \vdash \beta$

All that is missing is a proof of DT and proofs of (L1)–(L4). Given the presence of (R1), (R2), and (R5), the proof of DT is exactly the same as earlier. Given the presence of (R1), (R4), and (R5), the proofs of (L1)–(L3) are very simple. What remains is (L4), which we prove now.

- (L4) $\Gamma \cup \{\alpha\} \vdash \beta \text{ & } \Gamma \cup \{\sim\alpha\} \vdash \beta \rightarrow \Gamma \vdash \beta$

Proof: Suppose $\Gamma \cup \{\alpha\} \vdash \beta$, and $\Gamma \cup \{\sim\alpha\} \vdash \beta$. By DT, we have $\Gamma \vdash \alpha \rightarrow \beta$, and $\Gamma \vdash \sim\alpha \rightarrow \beta$. Given (R4), we have $\vdash (\sim\alpha \rightarrow \beta) \rightarrow [(\alpha \rightarrow \beta) \rightarrow \beta]$, so by a general theorem (monotonicity), we have $\Gamma \vdash (\sim\alpha \rightarrow \beta) \rightarrow [(\alpha \rightarrow \beta) \rightarrow \beta]$. So by two applications of the MP-principle, we have $\Gamma \vdash \beta$.

3. Appendix 2—Alternative Proof of Completeness for CSL

1. Supporting Lemmas

Recall that the following lemmas are used in proving the completeness theorem.

- | | | |
|--------------------|--|-------------------------|
| (VT) | $\text{MC}[\Gamma] \rightarrow \exists v \{v \text{ is CSL admissible and } v \text{ verifies } \Gamma\}$ | [Verifiability Theorem] |
| (LL) | $\Gamma \not\models \rightarrow \exists \Delta \{\Gamma \subseteq \Delta \text{ & } \text{MC}[\Gamma] \models\}$ | [Lindenbaum's Lemma] |
| (NT ₁) | $\Gamma \vdash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vdash$ | [Negation Theorem 1] |
| (NT ₂) | $\Gamma \models \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \models$ | [Negation Theorem 2] |

The following are the relevant definitions.

- | | |
|------|--|
| (d4) | $\Gamma \vdash \alpha =_{df} \text{ there is a derivation of } \alpha \text{ from } \Gamma \text{ (the in relevant system)}$ |
| (d5) | $\Gamma \vdash =_{df} \forall \alpha [\Gamma \vdash \alpha]$ |
| (d6) | $\Gamma \not\models =_{df} \sim [\Gamma \vdash]$ |
| (d7) | $\text{MC}[\Gamma] =_{df} \Gamma \not\models \& \forall \Delta (\Gamma \subseteq \Delta \rightarrow \Delta \vdash)$ |

2. What is Needed to Prove NT₁?

We consider the lemmas one at a time, working backwards. Notice, first, that (NT₂) is a purely semantic theorem about CSL, which we can presuppose here. Next, consider proving (NT₁), which we divide into two parts. The starred items are subordinate lemmas that we must antecedently prove about our axiom system.

(1)	$[\rightarrow]$	CD
(2)	$\Gamma \vdash \alpha$	As
(3)	SHOW: $\Gamma \cup \{\sim \alpha\} \vdash$	Def(\vdash)
(4)	SHOW: $\forall \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta]$	UD
(5)	SHOW: $\Gamma \cup \{\sim \alpha\} \vdash \beta$	DD
(6)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	L1*
(7)	$\Gamma \vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	6, monotonicity
(8)	$\Gamma \cup \{\sim \alpha\} \vdash \alpha$	2, monotonicity
(9)	$\Gamma \cup \{\sim \alpha\} \vdash \sim \alpha$	GenTh about \vdash
(10)	$\Gamma \cup \{\sim \alpha\} \vdash \beta$	6,8,9, MPP*

(1)	$[\leftarrow]$	CD
(2)	$\Gamma \cup \{\sim \alpha\} \vdash$	As
(3)	SHOW: $\Gamma \vdash \alpha$	DD
(4)	$\left \begin{array}{l} \forall \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta] \\ \Gamma \cup \{\sim \alpha\} \vdash \alpha \end{array} \right.$	2, Def(\vdash)
(5)	$\Gamma \vdash \sim \alpha \rightarrow \alpha$	4, QL
(6)	$\vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$	5, DT*
(7)	$\Gamma \vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$	Lemma 2*
(8)	$\Gamma \vdash \alpha$	7, monotonicity
(9)		6,8,MPP*

So, what we need to prove are the following theorems about whatever axiom system we are considering.

$$(MPP) \quad \Gamma \vdash \alpha \ \& \ \Gamma \vdash \alpha \rightarrow \beta \rightarrow \Gamma \vdash \beta$$

$$(DT) \quad \Gamma \cup \{\alpha\} \vdash \beta \rightarrow \Gamma \vdash \alpha \rightarrow \beta$$

$$(L1) \quad \vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$$

$$(L2) \quad \vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$$

The proof of the *modus ponens* principle can be straightforwardly proven using general facts about derivation assuming the derivation system admits the rule *modus ponens*. The easiest way to achieve this result is simply to add the rule *modus ponens*.

$$(mp) \quad \alpha ; \alpha \rightarrow \beta \rightsquigarrow \beta$$

As seen in earlier sections, a system that admits *modus ponens* satisfies the deduction theorem (DT) if and only if the following are true.

$$(d1) \quad \vdash \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$(d2) \quad \vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$$

The easiest way to achieve these results is to add the corresponding axiom schemata to the system.

$$(r1) \quad \rightsquigarrow \alpha \rightarrow (\beta \rightarrow \alpha)$$

$$(r2) \quad \rightsquigarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$$

Similarly, the easiest way to achieve (L1) and (L2) is to add the corresponding axiom schemata to the system.

$$(r3) \quad \rightsquigarrow \sim \alpha \rightarrow (\alpha \rightarrow \beta)$$

$$(r4) \quad \rightsquigarrow (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$$

3. What is Needed to Prove LL?

The proof of LL involves a number of subordinate lemmas. Given general theorems about deduction, the following suffice.

$$(L3) \quad \forall \alpha \{\alpha \in \Gamma \text{ or } \sim \alpha \in \Gamma\} \rightarrow. \Gamma \not\models \rightarrow MC[\Gamma]$$

In other words,

if a consistent set Γ contains every formula or its negation,
then Γ is maximal consistent

Proof

(1)	$\forall \alpha \{\alpha \in \Gamma \text{ or } \sim \alpha \in \Gamma\}$	As
(2)	$\Gamma \not\models$	As
(3)	SHOW: $MC[\Gamma]$	2,4,Def(MC)
(4)	SHOW: $\forall \alpha \{\alpha \notin \Gamma \rightarrow \Gamma \cup \{\alpha\} \vdash\}$	UCD
(5)	$\alpha \notin \Gamma$	As
(6)	SHOW: $\Gamma \cup \{\alpha\} \vdash$	Def(\vdash)
(7)	SHOW: $\forall \beta [\Gamma \cup \{\alpha\} \vdash \beta]$	UD
(8)	SHOW: $\Gamma \cup \{\alpha\} \vdash \beta$	
(9)	$\sim \alpha \in \Gamma$	1,5,QL
(10)	$\Gamma \vdash \sim \alpha$	9,Gen Th about \vdash
(11)	$\Gamma \cup \{\alpha\} \vdash \sim \alpha$	10, mono
(12)	$\Gamma \cup \{\alpha\} \vdash \alpha$	Gen th about \vdash
(13)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	L1*
(14)	$\Gamma \cup \{\alpha\} \vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	13, mono
(15)	$\Gamma \cup \{\alpha\} \vdash \beta$	11,12,14,MPP*(twice)

Notice that the needed lemmas have already been proven.

So in order to prove that the set Ω constructed in the proof of LL is maximal consistent, we need merely show that it is consistent, and that it contains every formula or its negation. Given a general theorem about deduction, in order to show that Ω is consistent, we need merely show that every set in the sequence $\langle \Gamma_1, \Gamma_2, \dots \rangle$ is consistent. This is shown by weak induction. The key lemma in proving this result is the following.

$$(L4) \quad \Gamma \not\models \rightarrow. \Gamma \cup \{\alpha\} \not\models \text{ or } \Gamma \cup \{\sim \alpha\} \not\models$$

This is of course equivalent to:

$$(L4') \quad \Gamma \cup \{\alpha\} \vdash \& \Gamma \cup \{\sim \alpha\} \vdash . \rightarrow \Gamma \vdash$$

whose proof is quite simple, given what we already have.

(1)	$\Gamma \cup \{\alpha\} \vdash$	As
(2)	$\Gamma \cup \{\sim \alpha\} \vdash$	As
(3)	SHOW: $\Gamma \vdash$	Def(\vdash)
(4)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(5)	SHOW: $\Gamma \vdash \beta_0$	
(6)	$\left \begin{array}{l} \Gamma \vdash \alpha \\ \forall \beta [\Gamma \cup \{\alpha\} \vdash \beta] \end{array} \right.$	2, NT ₁
(7)	$\left \begin{array}{l} \forall \beta [\Gamma \cup \{\alpha\} \vdash \beta] \\ \Gamma \vdash \beta_0 \end{array} \right.$	1, Def(\vdash)
(8)	$\left \begin{array}{l} \Gamma \vdash \beta_0 \\ \Gamma \vdash \alpha \end{array} \right.$	7, QL
(9)	$\Gamma \vdash \beta_0$	6,7, Gen Th about \vdash

4. What is Needed to Prove VT?

The Verification Theorem states that every maximal consistent set is verifiable. The proof depends upon the following lemmas.

- (v1) $\Gamma \cup \{\alpha\} \vdash \& \Gamma \cup \{\sim \alpha\} \vdash \rightarrow \Gamma \vdash$
- (v2) $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$
- (v3) $\alpha \vdash \beta \rightarrow \alpha$
- (v4) $\sim \alpha \vdash \alpha \rightarrow \beta$

We have already proven (v1) in the previous section. (v2) is true, given that *modus ponens* is a rule. Finally, (v3) and (v4) are true given the following rules.

- (r1) $\dashv \beta \rightarrow (\alpha \rightarrow \beta)$
- (r3) $\dashv \sim \alpha \rightarrow (\alpha \rightarrow \beta)$

5. Review

The result is a fairly simple axiom system that allows an efficient proof of the completeness theorem.

AS*

- (R1) $\dashv \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2) $\dashv [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3) $\dashv \sim \alpha \rightarrow (\alpha \rightarrow \beta)$
- (R4) $\dashv (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$
- (R5) $\alpha ; \alpha \rightarrow b \dashv \beta$

4. Appendix

1. Semantic Entailment and Semantic Consistency

$$\begin{array}{lll} \Gamma \vDash \alpha & =_{\text{df}} & \forall v \{ v < \Gamma \rightarrow v < \alpha \} \\ \Gamma \vDash & =_{\text{df}} & \sim \exists v [v < \Gamma] \\ \Gamma \# & =_{\text{df}} & \sim [\Gamma \vDash] \end{array}$$

Γ semantically entails α
 Γ is semantically inconsistent
 Γ is semantically consistent

2. Theorem about CSL

$$\Gamma \vDash \alpha \iff \Gamma \cup \{\sim \alpha\} \vDash$$

3. Deductive Entailment and Deductive Consistency

$$\begin{array}{lll} \Gamma \vdash \alpha & =_{\text{df}} & \exists d [d \text{ is a derivation of } \alpha \text{ from } \Gamma] \\ \Gamma \vdash & =_{\text{df}} & \forall \alpha [\Gamma \vdash \alpha] \\ \Gamma \# & =_{\text{df}} & \sim [\Gamma \vdash] \end{array}$$

Γ deductively entails α
 Γ is deductively inconsistent
 Γ is deductively consistent

4. Theorems about AS1

$$\begin{array}{lll} \Gamma \vdash & \iff & \exists \alpha \{ \Gamma \vdash \alpha \text{ & } \Gamma \vdash \sim \alpha \} \\ \Gamma \vdash \alpha & \iff & \Gamma \cup \{\sim \alpha\} \vdash \end{array}$$

5. Soundness, Completeness, and Mutual Consistency

$$\begin{array}{lll} \mathbb{A} \text{ is sound for } \mathbb{V} & =_{\text{df}} & \forall \Gamma \forall \alpha \{ \Gamma \vdash \alpha \rightarrow \Gamma \vDash \alpha \} \\ \mathbb{A} \text{ is complete for } \mathbb{V} & =_{\text{df}} & \forall \Gamma \forall \alpha \{ \Gamma \vDash \alpha \rightarrow \Gamma \vdash \alpha \} \\ \mathbb{A} \text{ and } \mathbb{V} \text{ are} \\ \text{mutually consistent} & =_{\text{df}} & \forall \Gamma \forall \alpha \{ \Gamma \vdash \alpha \leftrightarrow \Gamma \vDash \alpha \} \end{array}$$

6. Lindenbaum's Lemma

Every d-consistent set can be extended to a maximal d-consistent set.

7. Key Semantic Lemma

Every maximal d-consistent set is s-consistent (i.e., verifiable).

8. Completeness

Suppose $\Gamma \vDash \alpha$, to show $\Gamma \vdash \alpha$; contrapositively, suppose $\Gamma \not\vdash \alpha$, to show $\Gamma \# \alpha$. Then by Lemma 1, $\Gamma \cup \{\sim \alpha\} \vdash$. So by LL, $\exists \Delta \{ \Gamma \cup \{\sim \alpha\} \subseteq \Delta \text{ & } \text{MC}[\Delta] \}$. By KSL, Δ is verifiable, so $\exists v [v < \Delta]$. Since v verifies every element of Δ , it verifies every element of $\Gamma \cup \{\sim \alpha\}$, so $\Gamma \cup \{\sim \alpha\}$ is verifiable, so $\Gamma \cup \{\sim \alpha\} \#$, so by Lemma 2, $\Gamma \# \alpha$.

9. Supporting Lemmas

- (L1) $\Gamma \vdash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vdash$
- (L2) $\Gamma \vDash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vDash$
- (LL) Every d-consistent set can be extended to a maximal d-consistent set.
- (KSL) Every maximal d-consistent set is s-consistent (i.e., verifiable).

5. Appendix – General Theorems about Deduction

1. $\text{G} \vdash D \text{ } \textcircled{R} \text{ . } \text{G} \vdash a \text{ } \textcircled{R} \text{ } D \vdash a$

(1)	SHOW: $\Gamma \subseteq \Delta \rightarrow \Gamma \vdash \alpha \rightarrow \Delta \vdash \alpha$	CCD
(2)	$\Gamma \subseteq \Delta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Delta \vdash \alpha$	Def \vdash
(5)	SHOW: $\exists d[d \text{ derives } \alpha \text{ from } \Delta]$	8,QL
(6)	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	3, Def \vdash
(7)	D derives α from Γ	6, $\exists O$
(8)	SHOW: D derives α from Δ	Def derives [&D]
(9)	SHOW: $\text{last}(D)=\alpha$	7, Def derives [a]
(10)	SHOW: $\forall \delta \in D: \delta \in \Delta \text{ or } \delta \text{ follows by a rule ...}$	11,12,QL
(11)	$\forall \delta \in D: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	7, Def derives [b]
(12)	$\forall x\{x \in \Gamma \rightarrow x \in \Delta\}$	2, Def \subseteq

2. $\vdash a \text{ } \textcircled{R} \text{ } \text{G} \vdash a$

(1)	SHOW: $\vdash \alpha \rightarrow \Gamma \vdash \alpha$	CD
(2)	$\vdash \alpha$	As
(3)	SHOW: $\Gamma \vdash \alpha$	Def $\Gamma \vdash \alpha$
(4)	SHOW: $\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	Def derives
(5)	SHOW: $\forall \delta \in P: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	8,QL
(6)	$\exists p[p \text{ proves } \alpha]$	2, Def $\vdash \alpha$
(7)	P proves α	6, $\exists O$
(8)	$\forall \delta \in P: \delta \text{ follows by a rule ...}$	7, Def proves

3. $\vdash a \Leftarrow A \vdash a$

(1)	SHOW: $\vdash \alpha \leftrightarrow \emptyset \vdash \alpha$	$\leftrightarrow D$
(2)	SHOW: $\vdash \alpha \rightarrow \emptyset \vdash \alpha$	GT2
(3)	SHOW: $\emptyset \vdash \alpha \rightarrow \vdash \alpha$	CD
(4)	$\emptyset \vdash \alpha$	As
(5)	SHOW: $\vdash \alpha$	Def $\vdash \alpha$
(6)	SHOW: $\exists p[p \text{ proves } \alpha]$	QL
(7)	$\exists d[d \text{ derives } \alpha \text{ from } \emptyset]$	4, Def $\emptyset \vdash \alpha$
(8)	D derives α from \emptyset	9, $\exists O$
(9)	SHOW: D proves α	Def proves [&D]
(10)	SHOW: $\text{last}(D)=\alpha$	8, Def derives [a]
(11)	SHOW: $\forall \delta \in D: \delta \text{ follows by a rule ...}$	UD
(12)	SHOW: $\delta \text{ follows by a rule ...}$	13,14,QL
(13)	$\forall \delta \in D: \delta \in \emptyset \text{ or } \delta \text{ follows by a rule ...}$	8, Def derives [b]
(14)	$\sim \exists x[x \in \emptyset]$	ST

4. $\vdash a \Leftarrow "G[G \vdash a]$

(1)	SHOW: $\vdash \alpha \leftrightarrow \forall \Gamma[\Gamma \vdash \alpha]$	$\leftrightarrow D$
(2)	SHOW: \rightarrow	CD
(3)	$\vdash \alpha$	As
(4)	SHOW: $\forall \Gamma[\Gamma \vdash \alpha]$	UD
(5)	SHOW: $\Gamma \vdash \alpha$	3,GT2
(6)	SHOW: \leftarrow	CD
(7)	$\forall \Gamma[\Gamma \vdash \alpha]$	As
(8)	SHOW: $\vdash \alpha$	9,GT3
(9)	$\emptyset \vdash \alpha$	7,QL

5. $a \hat{\in} Ax \circledR \vdash a$

(1)	SHOW: $\alpha \in Ax \rightarrow \vdash \alpha$	CD
(2)	$\alpha \in Ax$	As
(3)	SHOW: $\vdash \alpha$	Def \vdash
(4)	SHOW: $\exists p[p \text{ proves } \alpha]$	5,QL
(5)	SHOW: $\langle \alpha \rangle \text{ proves } \alpha$	Def proves [&D]
(6)	SHOW: $\text{last}(\langle \alpha \rangle) = \alpha$	ST
(7)	SHOW: $\forall \delta \in \langle \alpha \rangle: \delta \text{ follows by a rule ...}$	UCD
(8)	$\delta \in \langle \alpha \rangle$	As
(9)	SHOW: $\delta \text{ follows by a rule ...}$	12,QL
(10)	$\alpha \text{ follows by a zero-place rule}$	2, Def Ax
(11)	$\delta = \alpha$	8, Def $\langle \alpha \rangle$
(12)	$\delta \text{ follows by a zero-place rule}$	10,11,IL

6. $\mathbf{G} \vdash a \& \mathbf{G} \dot{\exists} \{a\} \vdash b . \mathbb{R} \quad \mathbf{G} \vdash b$

(1)	SHOW: $\Gamma \vdash \alpha \& \Gamma \cup \{\alpha\} \vdash \beta . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \cup \{\alpha\} \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def \vdash
(5)	SHOW: $\exists d[d \text{ derives } \beta \text{ from } \Gamma]$	12,QL
(6)	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	2, Def \vdash
(7)	$\exists d[d \text{ derives } \beta \text{ from } \Gamma \cup \{\alpha\}]$	3, Def \vdash
(8)	$D_1 \text{ derives } \alpha \text{ from } \Gamma$	6,EO
(9)	$D_2 \text{ derives } \beta \text{ from } \Gamma \cup \{\alpha\}$	7,EO
(10)	$\exists d[d = D_2[D_1/\alpha]]$	ST
(11)	$D_3 = D_2[D_1/\alpha]$	10,EO
(12)	SHOW: $D_3 \text{ derives } \beta \text{ from } \Gamma$	Def derives [&D]
(13)	SHOW: $\text{last}(D_3) = \beta$	14,Def D_3 , ST
(14)	$\text{last}(D_2) = \beta$	9, Def derives [a]
(15)	SHOW: $\forall \delta \in D_3: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	UCD
(16)	$\delta \in D_3$	As
(17)	SHOW: $\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	19-30,SC/SC
(18)	$\delta \in D_2[D_1/\alpha]$	11,16,IL
(19)	$\{\delta \in D_1\} \text{ or } \{\delta \in D_2 \& \delta \neq \alpha\}$	18, Def $\sigma[\pi/\varepsilon]$
(20)	c1: $\delta \in D_1$	As
(21)	$\forall \delta \in D_1 \{\delta \in \Gamma \text{ or } \delta \text{ follows by a rule}\}$	8,Def derives [b]
(22)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	20,21,QL
(23)	c2: $\delta \in D_2 \& \delta \neq \alpha$	As
(24)	$\forall \delta \in D_2 \{\delta \in \Gamma \cup \{\alpha\} \text{ or } \delta \text{ follows by a rule}\}$	9,Def derives [b]
(25)	$\delta \in \Gamma \cup \{\alpha\} \text{ or } \delta \text{ follows by a rule}$	23,24,QL
(26)	c1: $\delta \in \Gamma \cup \{\alpha\}$	As
(27)	$\delta \in \Gamma$	23b,26,ST
(28)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	27,SL
(29)	c2: $\delta \text{ follows by a rule ...}$	As
(30)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	29,SL

7. $\mathbf{G} \vdash a \& \mathbf{G} \dot{\exists} \{a\} \vdash . \mathbb{R} \quad \mathbf{G} \vdash$

(1)	SHOW: $\Gamma \vdash \alpha \& \Gamma \cup \{\alpha\} \vdash . \rightarrow \Gamma \vdash$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \cup \{\alpha\} \vdash$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta[\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash b$	2,8,GT6
(7)	$\forall \beta[\Gamma \cup \{\alpha\} \vdash \beta]$	3, Def \vdash
(8)	$\Gamma \cup \{\alpha\} \vdash b$	7,QL

8. " $d \{ d \hat{I} D \text{ } \textcircled{R} \text{ } G \vdash d \} \& D \vdash b . \textcircled{R} \text{ } G \vdash b$

(1)	SHOW: $\forall \delta \{ \delta \in \Delta \rightarrow \Gamma \vdash \delta \} \& \Delta \vdash \beta . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\forall \delta \{ \delta \in \Delta \rightarrow \Gamma \vdash \delta \}$	As
(3)	$\Delta \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def \vdash
(5)	SHOW: $\exists d [d \text{ derives } \beta \text{ from } \Gamma]$	10,QL
(6)	$\exists d [d \text{ derives } \beta \text{ from } \Delta]$	3, Def \vdash
(7)	$D_0 \text{ derives } \beta \text{ from } \Delta$	6, $\exists O$
(8)	$\exists d \{ d = D_0[D_k/\delta_k : \delta_k \in \Delta] \}$	ST
(9)	$D = D_0[D_k/\delta_k : \delta_k \in \Delta]$	8, $\exists O$
(10)	SHOW: $D \text{ derives } \beta \text{ from } \Gamma$	
(11)	unfinished	

9. $G \vdash a \& a \vdash b . \textcircled{R} \text{ } G \vdash b$

(1)	SHOW: $\Gamma \vdash \alpha \& \alpha \vdash \beta . \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\alpha \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	2,7,GT6
(5)	$\{ \alpha \} \vdash \beta$	3, Def \vdash
(6)	$\{ \alpha \} \subseteq \Gamma \cup \{ \alpha \}$	ST
(7)	$\Gamma \cup \{ \alpha \} \vdash \beta$	6,GT1

6. Lemmas About AS1 Used in Completeness Proof

1. Transitivity Principle (TR)

$$\vdash a @ b \And \vdash b @ g . \textcircled{R} \quad \vdash a @ g$$

(1)	SHOW: $\vdash \alpha \rightarrow \beta \And \vdash \beta \rightarrow \gamma . \rightarrow \vdash \alpha \rightarrow \gamma$	CD
(2)	$\vdash \alpha \rightarrow \beta$	As +&O
(3)	$\vdash \beta \rightarrow \gamma$	As +&O
(4)	SHOW: $\vdash \alpha \rightarrow \gamma$	CD
(5)	$\vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	3,PRE
(6)	$\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	5,DIST
(7)	$\vdash \alpha \rightarrow \gamma$	2,6,MPP

2. Prefix Principle (PRE)

$$\vdash b \textcircled{R} \quad \vdash a @ b$$

(1)	SHOW: $\vdash \beta \rightarrow \vdash \alpha \rightarrow \beta$	CD
(2)	$\vdash \beta$	As
(3)	SHOW: $\vdash \alpha \rightarrow \beta$	2,4,MPP
(4)	$\vdash \beta \rightarrow (\alpha \rightarrow \beta)$	R1

3. Distribution Principle (DIST)

$$\vdash a @ (b @ g) \textcircled{R} \quad \vdash (a @ b) @ (a @ g)$$

(1)	SHOW: $\vdash \alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow \vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	CD
(2)	$\vdash \alpha \rightarrow (\beta \rightarrow \gamma)$	As
(3)	SHOW: $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)$	2,4,MPP
(4)	$\vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$	R2

4. Modus Ponens Principle (MPP)

$\mathbf{G \vdash a \textcircled{R} b \quad \& \quad G \vdash a . \textcircled{R} \quad G \vdash b}$

(1)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta \quad \& \quad \Gamma \vdash \alpha . \rightarrow \quad \Gamma \vdash \beta$	CD
(2)	$\Gamma \vdash \alpha \rightarrow \beta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def $\Gamma \vdash \alpha$
(5)	SHOW: $\exists d[d \text{ derives } \beta \text{ from } \Gamma]$	10, $\exists I$
(6)	$\exists d[d \text{ derives } \alpha \rightarrow \beta \text{ from } \Gamma]$	2, Def $\Gamma \vdash \alpha$
(7)	$D_1 \text{ derives } \alpha \rightarrow \beta \text{ from } \Gamma$	6, $\exists O$
(8)	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	2, Def $\Gamma \vdash \alpha$
(9)	$D_2 \text{ derives } \alpha \text{ from } \Gamma$	8, $\exists O$
(10)	SHOW: $D_1+D_2+\langle \beta \rangle$ derives β from Γ	Def derives [&D]
(11)	SHOW: $\beta = \text{last}(D_1+D_2+\langle \beta \rangle)$	ST
(12)	SHOW: $\forall \delta \{ \alpha \in D_1+D_2+\langle \beta \rangle \rightarrow .$ $\delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \delta \text{ follows by MP from previous lines} \}$	UCD
(13)	$\delta \in D_1+D_2+\langle \beta \rangle$	As
(14)	SHOW: $\delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \delta \text{ follows by MP...}$	15-31, SC
(15)	$\delta \in D_1 \text{ or } \delta \in D_2 \text{ or } \delta \in \langle \beta \rangle$	13, Def +
(16)	c1: $\delta \in D_1$	As
(17)	$\forall \delta \in D_1: \delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \delta \text{ follows by MP...}$	8, Def derives [b]
(18)	$\delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \alpha \text{ follows by MP...}$	16, 17, QL
(19)	c2: $\delta \in D_2$	As
(20)	$\forall \delta \in D_2: \delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \delta \text{ follows by MP...}$	10, Def derives [b]
(21)	$\delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \delta \text{ follows by MP...}$	19, 110, QL
(22)	c3: $\delta \in \langle \beta \rangle$	As
(23)	$\delta = \beta$	22, Def ⟨ ⟩
(24)	$\delta = \text{last}(D_1+D_2+\langle \beta \rangle)$	23+Def +
(25)	$\alpha \rightarrow \beta = \text{last}(D_1)$	8, Def derives [a]
(26)	$\alpha = \text{last}(D_2)$	10, Def derives [a]
(27)	$\text{last}(D_1) < \delta$	Def $D_1+D_2+\langle \delta \rangle$
(28)	$\text{last}(D_2) < \delta$	Def $D_1+D_2+\langle \delta \rangle$
(29)	β follows from $\alpha \rightarrow \beta$ and α by MP	Def MP
(30)	δ follows by MP from previous lines	23-29, IL
(31)	$\delta \in Ax \text{ or } \delta \in \Gamma \text{ or } \delta \text{ follows by MP...}$	30, SL

This is a direct proof. MPP is also a special case of a general theorem, as seen in the following proof.

(1)	SHOW: $\Gamma \vdash \alpha \rightarrow \beta \quad \& \quad \Gamma \vdash \alpha . \rightarrow \quad \Gamma \vdash \beta$	CD
(2)	$\Gamma \vdash \alpha \rightarrow \beta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Gamma \vdash \beta$	5, 6, GT8
(5)	$\{ \alpha \rightarrow \beta, \alpha \} \vdash \beta$	Lemma 0
(6)	$\forall \delta \in \{ \alpha \rightarrow \beta, \alpha \}: \Gamma \vdash \delta$	2, 3, ST

5. Lemma 0

$\{a @ b, a\} \vdash b$

(1)	SHOW: $\{\alpha \rightarrow \beta, \alpha\} \vdash \beta$	Def \vdash
(2)	SHOW: $\exists d[d \text{ derives } \beta \text{ from } \{\alpha \rightarrow \beta, \alpha\}]$	3,QL
(3)	SHOW: $\langle \alpha \rightarrow \beta, \alpha, \beta \rangle$ derives β from $\{\alpha \rightarrow \beta, \alpha\}$	Def derives [&D]
(4)	SHOW: $\text{last}(\alpha \rightarrow \beta, \alpha, \beta) = \beta$	ST
(5)	SHOW: $\forall \delta \in \langle \alpha \rightarrow \beta, \alpha, \beta \rangle:$	
	$\delta \in Ax$ or $\delta \in \{\alpha \rightarrow \beta, \alpha\}$ or δ follows by MP ...	UCD
(6)	$\delta \in \langle \alpha \rightarrow \beta, \alpha, \beta \rangle$	As
(7)	SHOW: $\delta \in Ax$ or $\delta \in \{\alpha \rightarrow \beta, \alpha\}$ or δ follows by MP ...	SC
(8)	$\delta = \alpha \rightarrow \beta$ or $\delta = \alpha$ or $\delta = \beta$	6, ST
(9)	c1: $\delta = \alpha \rightarrow \beta$	As
(10)	$\delta \in \{\alpha \rightarrow \beta, \alpha\}$	9,ST
(11)	$\delta \in Ax$ or $\delta \in \{\alpha \rightarrow \beta, \alpha\}$ or δ follows by MP ...	10,SL
(12)	c2: $\delta = \alpha$	As
(13)	$\delta \in \{\alpha \rightarrow \beta, \alpha\}$	12,ST
(14)	$\delta \in Ax$ or $\delta \in \{\alpha \rightarrow \beta, \alpha\}$ or δ follows by MP ...	10,SL
(15)	c3: $\delta = \beta$	As
(16)	β follows from $\alpha \rightarrow \beta$ and α by MP	Def MP
(17)	$\#(\alpha \rightarrow \beta), \#(\alpha) < \#(\beta)$	Def <-($\alpha \rightarrow \beta, \alpha, \beta$)
(18)	δ follows from $\alpha \rightarrow \beta$ and α by MP	15,16,IL
(19)	$\#(\alpha \rightarrow \beta), \#(\alpha) < \#(\delta)$	15,17,IL
(20)	δ follows by MP ...	18,19,Def follow...
(21)	$\delta \in Ax$ or $\delta \in \{\alpha \rightarrow \beta, \alpha\}$ or δ follows by MP ...	20,SL

6. Lemma 1

$\vdash \sim a @ (a @ b)$

(1)	SHOW: $\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	DD
(2)	$\vdash (\sim \beta \rightarrow \sim \alpha) \rightarrow (\alpha \rightarrow \beta)$	R3
(3)	$\vdash \sim \alpha \rightarrow (\sim \beta \rightarrow \sim \alpha)$	R1
(4)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	2,3,TR

7. Lemma 2

$G \vdash a \text{ & } G \vdash \sim a . @ . G \vdash b$

(1)	SHOW: $G \vdash \alpha \text{ & } G \vdash \sim \alpha . \rightarrow . G \vdash \beta$	CD
(2)	$G \vdash \alpha$	As +&O
(3)	$G \vdash \sim \alpha$	As +&O
(4)	SHOW: $G \vdash \beta$	DD
(5)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	Lemma 1
(6)	$\vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	5+GT3
(7)	$\vdash \alpha \rightarrow \beta$	3,6,MPP
(8)	$\vdash \beta$	2,7,MPP

7. Maximal Consistent Sets

1. Deductive Consistency

$$\begin{array}{lll} \Gamma \vdash & \stackrel{\text{def}}{=} & \forall \alpha [\Gamma \vdash \alpha] \\ \Gamma \not\vdash & \stackrel{\text{def}}{=} & \sim[\Gamma \vdash] \end{array} \quad \begin{array}{l} \text{deductive inconsistency} \\ \text{deductive consistency} \end{array}$$

2. Maximal Consistency

$$\begin{array}{lll} \text{MC}[\Gamma] & \stackrel{\text{def}}{=} & \Gamma \not\vdash \& \forall \Delta \{ \Gamma \subset \Delta \rightarrow \Delta \vdash \} \\ \Gamma \subset \Delta & \stackrel{\text{def}}{=} & \Gamma \subseteq \Delta \& \sim[\Delta \subseteq \Gamma] \end{array} \quad \begin{array}{l} \\ \\ \text{[proper inclusion]} \end{array}$$

3. General Theorems about MC

MC[G] & a ⊢ G . ⊥ G ⊢ {a} ⊢

(1)	SHOW: MC[Γ] & α ∉ Γ . → Γ ∪ {α} ⊢	&CD
(2)	MC[Γ]	As
(3)	α ∉ Γ	As
(4)	SHOW: Γ ∪ {α} ⊢	5,6,QL
(5)	Γ ⊂ Γ ∪ {α}	3,ST
(6)	∀ Δ {Γ ⊂ Δ → Δ ⊢ }	2, Def MC [b]

MC[G] & G ⊢ a . ⊥ a ⊢ G

(1)	SHOW: MC[Γ] & Γ ⊢ α . → α ∈ Γ	&CD
(2)	MC[Γ]	As
(3)	Γ ⊢ α	As
(4)	SHOW: α ∈ Γ	ID
(5)	α ∉ Γ	As
(6)	SHOW: ×	8,11,SL
(7)	Γ ⊂ Γ ∪ {α}	5,ST
(8)	Γ ⊬	2, Def MC [a]
(9)	∀ Δ {Γ ⊂ Δ → Δ ⊢ }	2, Def MC [b]
(10)	Γ ∪ {α} ⊢	7,9,QL
(11)	Γ ⊢	3,10,GT7

4. Theorems about CSL-AS1

T01: $\mathbf{G} \vdash \ll \mathbf{\$a}\{\mathbf{G} \vdash a \ \& \ \mathbf{G} \vdash \sim a\}$

(1)	SHOW: $\Gamma \vdash \leftrightarrow \exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\}$	$\leftrightarrow D$
(2)	SHOW: \rightarrow	CD
(3)	$\Gamma \vdash$	As
(4)	SHOW: $\exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\}$	7,8,QL
(5)	$\forall \alpha [\Gamma \vdash \alpha]$	3,Def $\Gamma \vdash$
(6)	P is a formula [of CSL]	Def formula
(7)	$\Gamma \vdash P$	5,6,QL
(8)	$\Gamma \vdash \sim P$	5,6,QL
(9)	SHOW: \leftarrow	CD
(10)	$\exists \alpha \{\Gamma \vdash \alpha \ \& \ \Gamma \vdash \sim \alpha\}$	As
(11)	$\Gamma \vdash \alpha$	10, $\exists O + \& O$
(12)	$\Gamma \vdash \sim \alpha$	10, $\exists O + \& O$
(13)	SHOW: $\Gamma \vdash$	Def $\Gamma \vdash$
(14)	SHOW: $\forall \alpha [\Gamma \vdash \alpha]$	UD
(15)	SHOW: $\Gamma \vdash \beta$	11,12,Lemma 2

T02: $\mathbf{G} \dot{\vDash} \{a\} \vdash b \ \& \ \mathbf{G} \dot{\vDash} \{\sim a\} \vdash b . \mathbb{R} \mathbf{G} \vdash b$

(1)	SHOW: $\Gamma \cup \{\alpha\} \vdash \beta \ \& \ \Gamma \cup \{\sim \alpha\} \vdash \beta . \rightarrow \Gamma \vdash \beta$	$\& CD$
(2)	$\Gamma \cup \{\alpha\} \vdash \beta$	As
(3)	$\Gamma \cup \{\sim \alpha\} \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	5,6,GT??
(5)	$\Gamma \vdash \alpha \rightarrow \beta$	2,DT
(6)	$\Gamma \vdash \sim \alpha \rightarrow \beta$	3,DT
(7)	$\{\alpha \rightarrow \beta, \sim \alpha \rightarrow \beta\} \vdash \beta$	Lemma ??

T03: $\mathbf{G} \dot{\vDash} \{a\} \vdash \& \mathbf{G} \dot{\vDash} \{\sim a\} \vdash . \mathbb{R} \mathbf{G} \vdash$

(1)	SHOW: $\Gamma \cup \{\alpha\} \vdash \& \ \Gamma \cup \{\sim \alpha\} \vdash . \rightarrow \Gamma \vdash$	$\& CD$
(2)	$\Gamma \cup \{\alpha\} \vdash$	As
(3)	$\Gamma \cup \{\sim \alpha\} \vdash$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash b$	9,10,T02
(7)	$\forall \beta [\Gamma \cup \{\alpha\} \vdash \beta]$	2, Def \vdash
(8)	$\forall \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta]$	3, Def \vdash
(9)	$\Gamma \cup \{\alpha\} \vdash b$	7,QL
(10)	$\Gamma \cup \{\sim \alpha\} \vdash b$	8,QL

T03c: $\mathbf{G} \nvDash \mathbb{R} . \mathbf{G} \dot{\vDash} \{a\} \nvDash \text{ or } \mathbf{G} \dot{\vDash} \{\sim a\} \nvDash$

(1)	SHOW: $\Gamma \nvDash \rightarrow . \Gamma \cup \{\alpha\} \nvDash \text{ or } \Gamma \cup \{\sim \alpha\} \nvDash$	T03,QL
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T04.1: MC[G] ® " a{aÎ G or ~aÎ G}

(1)	SHOW: MC[Γ] → ∀α{α ∈ Γ or ~α ∈ Γ}	CD
(2)	MC[Γ]	As
(3)	SHOW: ∀α{α ∈ Γ or ~α ∈ Γ}	UD
(4)	SHOW: α ∈ Γ or ~α ∈ Γ	∨ID
(5)	α ∉ Γ	As
(6)	~α ∉ Γ	As
(7)	SHOW: ×	13,14,SL
(8)	Γ ⊂ Γ ∪ {α}	5,ST
(9)	Γ ⊂ Γ ∪ {~α}	6,ST
(10)	∀Δ{Γ ⊂ Δ → Δ ⊢ }	2, Def MC [b]
(11)	Γ ∪ {α} ⊢	8,10,QL
(12)	Γ ∪ {~α} ⊢	9,10,QL
(13)	Γ ⊢	11,12,T02
(14)	Γ ⊨	2, Def MC [a]

T04.2: MC[G] ® " a{aÎ G ® ~aÎ G}

(1)	SHOW: MC[Γ] → ∀α{α ∈ Γ → ~α ∉ Γ}	CD
(2)	MC[Γ]	As
(3)	SHOW: ∀α{α ∈ Γ → ~α ∉ Γ}	UCD
(4)	α ∈ Γ	As
(5)	SHOW: ~α ∉ Γ	ID
(6)	~α ∈ Γ	As
(7)	SHOW: ×	10,11
(8)	Γ ⊢ α	4,GT??
(9)	Γ ⊢ ~α	6,GT??
(10)	Γ ⊢	8,9,T01
(11)	Γ ⊨	2, Def MC [a]

T05.1: MC[G] ® {aÎ G & a® bÎ G .® bÎ G}

(1)	SHOW: MC[Γ] → {α ∈ Γ & α → β ∈ Γ . → β ∈ Γ}	C&CD
(2)	MC[Γ]	As
(3)	α ∈ Γ	As
(4)	α → β ∈ Γ	As
(5)	SHOW: β ∈ Γ	8,GT??
(6)	Γ ⊢ α	3,GT??
(7)	Γ ⊢ α → β	4,GT??
(8)	Γ ⊢ β	6,7,MPP

T05.2: MC[G] ® . bÎ G ® a® bÎ G

(1)	SHOW: MC[Γ] → . β ∈ Γ → α → β ∈ Γ	CCD
(2)	MC[Γ]	As
(3)	β ∈ Γ	As
(4)	SHOW: α → β ∈ Γ	7,GT??
(5)	Γ ⊢ β	3,GT??
(6)	β → (α → β) ∈ Ax	R1
(7)	Γ ⊢ β → (α → β)	6,GT??
(8)	Γ ⊢ α → β	5,6,MPP

T05.3: **MC[G] ⊗ . a ⊢ G ⊗ a ⊗ b ⊢ G**

(1)	SHOW: MC[Γ] \rightarrow . $\alpha \notin \Gamma \rightarrow \alpha \rightarrow \beta \in \Gamma$	CCD
(2)	MC[Γ]	As
(3)	$\alpha \notin \Gamma$	As
(4)	SHOW: $\alpha \rightarrow \beta \in \Gamma$	8,GT??
(5)	$\sim \alpha \in \Gamma$	3,T04.1
(6)	$\Gamma \vdash \sim \alpha$	5,GT??
(7)	$\Gamma \vdash \sim \alpha \rightarrow (\alpha \rightarrow \beta)$	Lemma??
(8)	$\Gamma \vdash \alpha \rightarrow \beta$	6,7,MPP

T06: **G ⊢ a ≪ G ⊢ {~a} ⊢**

(1)	SHOW: $\Gamma \vdash \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \vdash$	\leftrightarrow D
(2)	SHOW: \rightarrow	CD
(3)	$\Gamma \vdash \alpha$	
(4)	SHOW: $\Gamma \cup \{\sim \alpha\} \vdash$	5,T01
(5)	SHOW: $\exists \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta \text{ & } \Gamma \cup \{\sim \alpha\} \vdash \sim \beta]$	7,9,QL
(6)	$\sim \alpha \in \Gamma \cup \{\sim \alpha\}$	ST
(7)	$\Gamma \cup \{\sim \alpha\} \vdash \sim \alpha$	6,GT??
(8)	$\Gamma \subseteq \Gamma \cup \{\sim \alpha\}$	ST
(9)	$\Gamma \cup \{\sim \alpha\} \vdash \alpha$	8,GT??
(10)	SHOW: \leftarrow	CD
(11)	$\Gamma \cup \{\sim \alpha\} \vdash$	As
(12)	SHOW: $\Gamma \vdash \alpha$	15,16,MPP
(13)	$\forall \beta [\Gamma \cup \{\sim \alpha\} \vdash \beta]$	11, Def \vdash
(14)	$\Gamma \cup \{\sim \alpha\} \vdash \alpha$	13,QL
(15)	$\Gamma \vdash \sim \alpha \rightarrow \alpha$	14,DT
(16)	$\Gamma \vdash (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$	Lemma??

8. Lindenbaum's Lemma**9. Key Semantic Lemma**