11 The Completeness Theorem for System AS1

1. Introduction

Having proved the soundness of system AS1 relative to the usual truth-functional semantics for CSL, we now turn to the converse problem of completeness.

Unlike the way we handled soundness, we do not warm up by proving a weak completeness theorem; rather, we first prove the strong completeness theorem, then prove the weak completeness theorem as a special case. As usual, the supporting lemmas are included in the appendix.

2. Previous Definitions

First, let us recall the relevant definitions.

$\vDash\alpha$	$=_{df}$	$\forall v\{v\in V \longrightarrow v(\alpha)=T\}$
$\Gamma\vDash\alpha$	$=_{df}$	$\forall v\{v\in V \longrightarrow. \forall x(x\in\Gamma \longrightarrow v(x)=T) \longrightarrow v(\alpha)=T\}$
$\vdash\alpha$	$=_{df}$	there is a proof of α in $\mathbb{A}$
$\Gamma\vdash\alpha$	$=_{df}$	there is a derivation of α from Γ in $\mathbb{A}$

Def

Let V be a class of valuations, and let $\mathbb{A}$ be an axiom system, both defined over a common language. Define $\vdash$ and $\vDash$ in the customary way. Then:

$\mathbb{A}$ is *complete* relative to V wrt formulas $=_{df}$ $\forall\alpha\{\vDash\alpha \longrightarrow \vdash\alpha\}$.

$\mathbb{A}$ is *complete* relative to V wrt arguments $=_{df}$ $\forall\Gamma\forall\alpha\{\Gamma\vDash\alpha \longrightarrow \Gamma\vdash\alpha\}$.

3. Deductive Consistency

Recall from a previous chapter the definition of deductive inconsistency, which is followed by the obvious correlated definition of deductive consistency.

Def

Γ is deductively <u>in</u>consistent $=_{df}$ $\forall\alpha[\Gamma\vdash\alpha]$

Γ is deductively consistent $=_{df}$ Γ is <u>not</u> deductively <u>in</u>consistent,

$\Gamma\vdash$ $=_{df}$ Γ is deductively <u>in</u>consistent

$\Gamma\nvdash$ $=_{df}$ Γ is deductively consistent

In regard to AS1, we can prove the following theorems about deductive (in)consistency.

[Note carefully: since we use ‘~’ ambiguously, one must be careful in reading formulas. For example, ‘$\sim\alpha\in\Gamma$’ means that the object language formula ¸ ~α ¹is an element of Γ, whereas ‘$\sim[\alpha\in\Gamma]$’ means that α is not an element of Γ; the latter is more often expressed by the formula ‘$\alpha\notin\Gamma$’. In this connection, recall that two-place infix predicates officially include outer brackets.]

(T38) Γ is deductively inconsistent $\leftrightarrow$ there is a formula α such that $\Gamma\vdash\alpha$ & $\Gamma\vdash\sim\alpha$.

$$\forall\Gamma\{\Gamma\vdash\ \leftrightarrow\ \exists\alpha(\Gamma\vdash\alpha\ \&\ \Gamma\vdash\sim\alpha)\}$$

(C) Γ is deductively consistent $\leftrightarrow$ there is no formula α such that $\Gamma\vdash\alpha$ & $\Gamma\vdash\sim\alpha$.

$$\forall\Gamma\{\Gamma\nvdash\ \leftrightarrow\ \sim\exists\alpha(\Gamma\vdash\alpha\ \&\ \Gamma\vdash\sim\alpha)\}$$

Proof:
[$\rightarrow$] Suppose $\Gamma\vdash$, to show $\exists\alpha(\Gamma\vdash\alpha\ \&\ \Gamma\vdash\sim\alpha)$. By the definition of inconsistency, $\forall\alpha[\Gamma\vdash\alpha]$, so by the relevant rules of formation, and QL, $\exists\alpha(\Gamma\vdash\alpha\ \&\ \Gamma\vdash\sim\alpha)$.
[$\leftarrow$] Suppose $\exists\alpha(\Gamma\vdash\alpha\ \&\ \Gamma\vdash\sim\alpha)$, to show: $\Gamma\vdash$, which amounts to showing: $\forall\beta[\Gamma\vdash\beta]$. Consider any formula β. By T8+T10, $\Gamma\vdash\sim\alpha\rightarrow(\alpha\rightarrow\beta)$, so by T14 (twice), $\Gamma\vdash\beta$.

Deductive consistency is contrasted with semantic consistency, which we have called verifiability; Γ is semantically consistent (verifiable) in V iff there is a valuation v in V that verifies every formula in Γ. As noted earlier, deductive consistency is not the exact counterpart of semantic consistency, at least in general. However, in special circumstances, including classical logic, they are very close counterparts, in the sense that the following is a theorem.

[Note: In what follows, unless otherwise specified, we presume the usual truth-functional semantics for CSL, relative to which the various instances of ‘$\vDash$’ are defined, and we presume axiom system AS1, relative to which the various instances of ‘$\vdash$’ are defined; as usual, we also presume the usual quantifier and sortal conventions.]

(T) Γ is deductively consistent $\leftrightarrow$ Γ is semantically consistent.

We first prove the “if” half of this theorem, by appealing to the Soundness Theorem (ST), and T38, above. The “only if” half, which is considerably harder, is proven later.

(T39) Γ is semantically consistent $\rightarrow$ Γ is deductively consistent (i.e.) Every semantically consistent set is deductively consistent.

Proof: Suppose Γ is semantically consistent, to show Γ is deductively consistent. Suppose, to the contrary, that Γ is deductively inconsistent, to show a contradiction. By T38, there is a formula, call it α, such that $\Gamma\vdash\alpha$ and $\Gamma\vdash\sim\alpha$. So, by the Soundness Theorem, $\Gamma\vDash\alpha$ and $\Gamma\vDash\sim\alpha$. By hypothesis, Γ is semantically consistent, so there is a valuation, call it v, that verifies every formula in Γ. But Γ semantically entails both α and $\sim\alpha$, so v verifies both α and $\sim\alpha$, which contradicts Lemma 1.

Next, we prove two theorems, very similar to one another, about the relation between deductive consistency and deductive entailment (derivability).

(T40) $\Gamma\cup\{\sim\alpha\}\vdash \;\leftrightarrow\; \Gamma\vdash\alpha$

Proof:
[$\rightarrow$] Suppose $\Gamma\cup\{\sim\alpha\}\vdash$, to show $\Gamma\vdash\alpha$. Then there is a formula, call it β, such that $\Gamma\cup\{\sim\alpha\}\vdash\beta$, and $\Gamma\cup\{\sim\alpha\}\vdash\sim\beta$. So by DT, $\Gamma\vdash\sim\alpha\rightarrow\beta$, and $\Gamma\vdash\sim\alpha\rightarrow\sim\beta$. But by T36, $\{\sim\alpha\rightarrow\beta, \sim\alpha\rightarrow\sim\beta\} \vdash \alpha$, so by Lemma 2, $\Gamma\vdash\alpha$.
[$\leftarrow$] Suppose $\Gamma\vdash\alpha$, to show $\Gamma\cup\{\sim\alpha\}\vdash$, which amounts to showing $\forall\beta[\Gamma\cup\{\sim\alpha\}\vdash\beta]$. Since $\Gamma\vdash\alpha$, by T18, $\Gamma\cup\{\sim\alpha\}\vdash\alpha$. By T8+T10, $\Gamma\vdash\sim\alpha\rightarrow(\alpha\rightarrow\beta)$, so by T17 (CDT), $\Gamma\cup\{\sim\alpha\}\vdash\alpha\rightarrow\beta$, so by T14, $\Gamma\cup\{\sim\alpha\}\vdash\beta$.

(T42) $\Gamma\cup\{\alpha\}\vdash \;\leftrightarrow\; \Gamma\vdash\sim\alpha$.

Proof: This can be proved in a very similar manner to the previous one (exercise). However, for a little variety, we do it differently. In virtue of T40, $\Gamma\cup\{\sim\sim\alpha\}\vdash$ iff $\Gamma\vdash\sim\alpha$, so in order to prove our theorem, we need merely prove the following lemma.

(L) $\Gamma\cup\{\sim\sim\alpha\}\vdash \;\leftrightarrow\; \Gamma\cup\{\alpha\}\vdash$

Proof: Given the definition of '$\vdash$' (third use), showing this amounts to showing:

(1) $\forall\beta[\Gamma\cup\{\sim\sim\alpha\}\vdash\beta] \;\leftrightarrow\; \forall\beta[\Gamma\cup\{\alpha\}\vdash\beta]$

But, given QL, in order to show (1), it is sufficient to show:

(2) $\forall\beta\{\Gamma\cup\{\sim\sim\alpha\}\vdash\beta \;\leftrightarrow\; \Gamma\cup\{\alpha\}\vdash\beta\}$

This follows from T20(c),T21(c), and the following lemma, which we now prove.

(T41) $\{\alpha\}\vdash\beta$ & $\{\beta\}\vdash\alpha$ $.\!\rightarrow\!.$ $\Gamma\cup\{\alpha\}\vdash\gamma \leftrightarrow \Gamma\cup\{\beta\}\vdash\gamma$

Proof: Suppose $\{\alpha\}\vdash\beta$ and $\{\beta\}\vdash\alpha$, to show: $\Gamma\cup\{\alpha\}\vdash\gamma$ iff $\Gamma\cup\{\beta\}\vdash\gamma$. [$\rightarrow$] Suppose $\Gamma\cup\{\alpha\}\vdash\gamma$, to show: $\Gamma\cup\{\beta\}\vdash\gamma$. By hyp, $\{\beta\}\vdash\alpha$, so by T18, $\Gamma\cup\{\beta\}\vdash\alpha$. Also, by hyp, $\Gamma\cup\{\alpha\}\vdash\gamma$, so by T18, $\Gamma\cup\{\beta\}\cup\{\alpha\}\vdash\gamma$; so, by T19, $\Gamma\cup\{\beta\}\vdash\gamma$. The converse [$\leftarrow$] argument is obtained by symmetry.

4. Maximal Consistent Sets

In what follows, for the sake of brevity, we use 'consistent' as shorthand for 'deductively consistent'.

An important step in proving the Completeness Theorem involves proving a lemma, known as Lindenbaum's Lemma, which claims that every (deductively) consistent set can be "extended" to a maximal (deductively) consistent set. In order to understand this lemma, we must understand the term 'maximal', which is a general set theoretic term, defined as follows.

> Def
>
> Let K a collection of sets. Then a *K-maximal set* is, by definition, a set M satisfying the following conditions.
>
> (1) $M \in K$
>
> (2) $\forall X\{X \in K \longrightarrow \sim[M \subset X]\}$
>
> i.e.,
>
> (1*) M is a K-set
>
> (2*) M is not properly included in any K-set.

The application of this concept to any particular situation involves identifying the relevant class K. In our particular case, K is the class of consistent sets of formulas, in which case we obtain the following instance.

> Def
>
> Let Γ be a set of formulas. Then Γ is a *maximal consistent set* if and only if:
>
> (1) Γ is consistent
>
> (2) Γ is not properly included in any consistent set.
>
> $MC[\Gamma] \quad =_{df} \quad \Gamma$ is a maximal consistent set

One way to think of maximal consistent sets is as follows; if Γ is maximal consistent, then "adding" any formula α to Γ results in an inconsistent set. This is the content of the next theorem.

(T43) $MC[\Gamma]$ & $\alpha \notin \Gamma$ $.\!\!\longrightarrow$ $\Gamma \cup \{\alpha\} \vdash$

Proof: Suppose $MC[\Gamma]$ and $\alpha \notin \Gamma$, to show $\Gamma \cup \{\alpha\} \vdash$. Since $\alpha \notin \Gamma$, $\Gamma \supseteq \Gamma \cup \{\alpha\}$. But Γ is maximal consistent, so no proper superset of Γ is consistent, so $\Gamma \cup \{\alpha\}$ is inconsistent (i.e. $\Gamma \cup \{\alpha\} \vdash$).

We next prove a important theorem, followed by an equally important corollary.

(T44) $\Gamma \cup \{\alpha\} \vdash \beta$ & $\Gamma \cup \{\sim\alpha\} \vdash \beta$ $.\!\!\longrightarrow$ $\Gamma \vdash \beta$

Proof: Suppose $\Gamma\cup\{\alpha\}\vdash\beta$ and $\Gamma\cup\{\sim\alpha\}\vdash\beta$, to show: $\Gamma\vdash\beta$. By a1, $\Gamma\cup\{\alpha\}\vdash\beta$, so by DT, $\Gamma\vdash\alpha\rightarrow\beta$. By a2, $\Gamma\cup\{\sim\alpha\}\vdash\beta$, so by DT, $\Gamma\vdash\sim\alpha\rightarrow\beta$. By T30+T10, $\Gamma\vdash(\alpha\rightarrow\beta)\rightarrow[(\sim\alpha\rightarrow\beta)\rightarrow\beta]$, so by T14 twice, $\Gamma\vdash\beta$.

(C) $\Gamma\cup\{\alpha\}\vdash$ & $\Gamma\cup\{\sim\alpha\}\vdash$ $.\longrightarrow$ $\Gamma\vdash$

(C) $\Gamma\nvdash$ $\longrightarrow.$ $\Gamma\cup\{\alpha\}\nvdash$ or $\Gamma\cup\{\sim\alpha\}\nvdash$

Proof: This follows from definition of '$\vdash$' and T44 by QL.

With this lemma (and corollary) in hand, we can prove the following theorem.

(T45) $MC[\Gamma] \longrightarrow \forall\alpha(\alpha\notin\Gamma \longleftrightarrow \sim\alpha\in\Gamma)$

(i.e.) $MC[\Gamma] \longrightarrow. \forall\alpha(\alpha\in\Gamma$ <u>xor</u> $\sim\alpha\in\Gamma)$

Proof: Suppose Γ is MC.
[$\rightarrow$] Suppose $\alpha\notin\Gamma$, to show $\sim\alpha\in\Gamma$. Suppose to the contrary that $\sim\alpha\notin\Gamma$. Then, by T43, $\Gamma\cup\{\alpha\}\vdash$, and $\Gamma\cup\{\sim\alpha\}\vdash$, so by T44(c), $\Gamma\vdash$, which contradicts the assumption that Γ is consistent.
[$\leftarrow$] Suppose $\sim\alpha\in\Gamma$, to show $\alpha\notin\Gamma$. Suppose to the contrary that $\alpha\in\Gamma$. Then by T12, $\Gamma\vdash\alpha$, and $\Gamma\vdash\sim\alpha$, which by T38 entails that Γ is inconsistent, which contradicts the assumption that Γ is consistent.

(T46) $MC[\Gamma] \longrightarrow \forall\alpha(\Gamma\vdash\alpha \longrightarrow \alpha\in\Gamma)$

Proof: Suppose $MC[\Gamma]$, and suppose $\Gamma\vdash\alpha$, to show $\alpha\in\Gamma$. Suppose to the contrary that $\alpha\notin\Gamma$. Then, by T45, $\sim\alpha\in\Gamma$, and so by T12, $\Gamma\vdash\sim\alpha$. But, by T8+T10, $\Gamma\vdash\sim\alpha\rightarrow(\alpha\rightarrow\beta)$, for any β, so by T14 twice, $\Gamma\vdash\beta$, for any β, which contradicts the assumption that Γ is consistent.

5. Lindenbaum's Lemma

In proving the Completeness Theorem, an important step involves proving a theorem known as Lindenbaum's Lemma, which is stated as follows.

(T47) Every consistent set is a subset of a maximal consistent set.

$$\forall\Gamma\{\Gamma \text{ is consistent } \longrightarrow \exists\Delta(\Gamma\subseteq\Delta \;\&\; MC[\Delta])\}$$

Proof: Suppose Γ is consistent, to show that there is a maximal consistent set Δ such that $\Gamma\subseteq\Delta$. There are denumerably many formulas in the underlying language (Lemma 3), so the formulas can be enumerated. Let $\langle\alpha_1, \alpha_2, \alpha_3, ...\rangle$ be one such enumeration. Given this enumeration, form the infinite sequence of sets $\langle\Gamma_1, \Gamma_2, ...\rangle$ ($\langle\Gamma_i\rangle$ for short), inductively defined as follows.

$$\begin{array}{lcll} \Gamma_1 & = & \Gamma; & \\ \Gamma_{n+1} & = & \Gamma_n \cup \{\alpha_n\} & \text{, if } \Gamma_n \cup \{\alpha_n\} \text{ is consistent;} \\ & = & \Gamma_n \cup \{\sim\alpha_n\} & \text{, otherwise.} \end{array}$$

Let $\Omega = \bigcup\{\Gamma_n : n =1,2,3,...\}$

Claim: $\Gamma \subseteq \Omega$, and Ω is maximal consistent.
Proof of claim:

(1) show: $\Gamma \subseteq \Omega$; this follows from the fact that $\Gamma = \Gamma_1$, and $\Gamma_1 \subseteq \Omega$. The latter follows from the definition of Ω, together with set theory [including the ST theorem: $X \in \Delta \rightarrow X \subseteq \bigcup \Delta$].

(2) show: Ω is maximal consistent; given (s2) [see below for subproofs], Ω is consistent, so all we have to show is that every proper superset of Ω is inconsistent. Suppose $\Omega \subset \Omega^*$; since Ω^* properly includes Ω, there is a formula, call it α, such that $\alpha \in \Omega^*$, but $\alpha \notin \Omega$. By (s3), since $\alpha \notin \Omega$, $\sim\alpha \in \Omega$, so $\sim\alpha \in \Omega^*$, so by T12, $\Omega^* \vdash \sim\alpha$. Therefore, $\Omega^* \vdash \alpha$, and $\Omega^* \vdash \sim\alpha$, which by T38 entails that Ω^* is inconsistent.

Subproofs of (2):

(s0) show: $\forall n \forall k (k \leqslant n \longrightarrow \Gamma_k \subseteq \Gamma_n)$; by induction [exercise].

(s1) show: $\forall n [\Gamma_n$ is consistent]; (by weak induction):
(BC) show: Γ_1 is consistent; by hypothesis Γ is consistent, and by definition of $\langle \Gamma_i \rangle$, $\Gamma_1 = \Gamma$; (IH) assume: Γ_n is consistent; (IS) show: Γ_{n+1} is consistent. By definition, $\Gamma_{n+1} = \Gamma \cup \{\alpha_n\}$ if the latter set is consistent, and $\Gamma_{n+1} = \Gamma \cup \{\sim\alpha_n\}$, otherwise. Suppose Γ_{n+1} is inconsistent (i.e., $\Gamma_{n+1} \vdash$. Given the definition of Γ_{n+1}, if $\Gamma_{n+1} \vdash$, then $\Gamma \cup \{\alpha_n\} \vdash$ and $\Gamma \cup \{\sim\alpha_n\} \vdash$, but these, together with T44(c), entail $\Gamma \vdash$. The latter, however, contradicts IH.

(s2) show: Ω is consistent; suppose Ω is inconsistent; then by T38, there is a formula, call it α, such that $\Omega \vdash \alpha$, and $\Omega \vdash \sim\alpha$. Then there is a derivation, call it D_1, of α from Ω, and there is a derivation, call it D_2, of $\sim\alpha$ from Ω. By definition, every derivation is a finite sequence; accordingly, at most finitely many premises (elements of Ω) are used in each derivation; call the two finite sets of "used" premises U_1 and U_2, respectively. Notice that $U_1 \vdash \alpha$, and $U_2 \vdash \sim\alpha$. Let $U = U_1 \cup U_2$; the union of any two finite sets is finite, so U is also finite. Also, by T18, $U \vdash \alpha$, and $U \vdash \sim\alpha$. Since U is finite, the set $\{n: \alpha_n \in U\}$ has a largest element; call it m. By (s2a), $U \subseteq \Gamma_{m+1}$, so by T18, $\Gamma_{m+1} \vdash \alpha$, and $\Gamma_{m+1} \vdash \sim\alpha$, which, together with T38, entail that Γ_{m+1} is inconsistent, which contradicts (s1).

(s2a) show: $U \subseteq \Gamma_{m+1}$; suppose $\alpha \in U$; in virtue of the definition of m, $\alpha = \alpha_k$ for some $k \leqslant m$, so $\alpha_k \in U$ for some $k \leqslant m$; it is sufficient to show $\alpha_k \in \Gamma_{k+1}$, and to show $\Gamma_{k+1} \subseteq \Gamma_{m+1}$. The latter follows from (s0) and arithmetic [$k \leqslant m \rightarrow k+1 \leqslant m+1$]. So that leaves showing $\alpha_k \in \Gamma_{k+1}$. By definition of $\langle \Gamma_i \rangle$, $\Gamma_{k+1} = \Gamma_k \cup \{\alpha_k\}$, if the latter is consistent; $\Gamma_{k+1} = \Gamma_k \cup \{\sim\alpha_k\}$, otherwise. In the first case, clearly $\alpha_k \in \Gamma_{k+1}$. In the second case, $\Gamma_{k+1} = \Gamma_k \cup \{\sim\alpha_k\}$, so $\sim\alpha_k \in \Omega$, so $\sim\alpha \in \Omega$. But by hyp, $\alpha \in \Omega$, so by (s2b), $\sim\alpha \notin \Omega$, and we have a contradiction.

(s2b) show: $\alpha \in \Omega \rightarrow \sim\alpha \notin \Omega$; suppose $\alpha \in \Omega$, and $\sim\alpha \in \Omega$, to show a contradiction. Given the definition of Ω [$=\bigcup\{\Gamma_1, \Gamma_2, ...\}$], $\alpha \in \Gamma_i$, and $\sim\alpha \in \Gamma_k$, for some i,k. Without loss of generality, we may assume $i \leqslant k$, so by (s0), $\alpha \in \Gamma_k$, but then $\Gamma_k \vdash \alpha$ and $\Gamma_n \vdash \sim\alpha$, which by T38 entail that Γ_n is inconsistent, which contradicts (s1).

(s3) show: $\forall\alpha[\alpha\in\Omega \vee \sim\alpha\in\Omega]$; Consider an arbitrary formula α, $\alpha = \alpha_k$, for some k. By definition of $\langle\Gamma_i\rangle$, $\Gamma_{k+1} = \Gamma_k\cup\{\alpha_k\}$, or $\Gamma_{k+1} = \Gamma_k\cup\{\sim\alpha_k\}$. In the first case, $\alpha\in\Gamma_{k+1}$; in the second case $\sim\alpha\in\Gamma_{k+1}$. But $\Gamma_{k+1}\subseteq\Omega$, so in either case, $\alpha\in\Omega \vee \sim\alpha\in\Omega$.

6. Every Maximal Consistent Set is Verifiable

Having proven Lindenbaum's Lemma, we next prove that every maximal consistent set is verifiable.

(T48) If Γ is maximal consistent, then Γ is verifiable.

Proof:
Suppose Γ is maximal consistent, to show that Γ is verifiable, which is to say that there is a valuation v in V that verifies every formula in Γ. Define v as follows. $v(\alpha)$=T if $\alpha\in\Gamma$; $v(\alpha)$=F, otherwise. Claim: v is a valuation, and v verifies Γ. Given the definition of v, clearly $v(\alpha)$=T for every α in Γ, so the remaining question is whether v is indeed a valuation. This amounts to the claim that v satisfies the following truth-functional requirements:

(1) $v(\sim\alpha) = \sim v(\alpha)$
(2) $v(\alpha\rightarrow\beta) = v(\alpha)\rightarrow v(\beta)$

(1) case 1: $\alpha\in\Gamma$, in which case $v(\alpha)$=T. Also, by T45, $\sim\alpha\notin\Gamma$, so $v(\sim\alpha)=F=\sim T=\sim v(\alpha)$; case 2: $\alpha\notin\Gamma$, in which case $v(\alpha)$=F; Also, by T45, $\sim\alpha\in\Gamma$, so $v(\sim\alpha)=T=\sim F=\sim v(\alpha)$.

(2) case 1: $\alpha\notin\Gamma$, in which case $v(\alpha)$=F. Also, by T45, $\sim\alpha\in\Gamma$, so by T46, $\Gamma\vdash\sim\alpha$, but by T8+T10, $\Gamma\vdash\sim\alpha\rightarrow(\alpha\rightarrow\beta)$, so by T14, $\Gamma\vdash\alpha\rightarrow\beta$, so by T46, $\alpha\rightarrow\beta \in \Gamma$, so $v(\alpha\rightarrow\beta)=T = v(\alpha)\rightarrow v(\beta)$; case 2: $\alpha\in\Gamma$, in which case $v(\alpha)$=T; c1: $\beta\in\Gamma$, in which case $v(\beta)$=T; also, by T46, $\Gamma\vdash\beta$; but, by T1+T10, $\Gamma\vdash\beta\rightarrow(\alpha\rightarrow\beta)$, so by T14, $\Gamma\vdash\alpha\rightarrow\beta$, so by T46, $\alpha\rightarrow\beta \in \Gamma$, so $v(\alpha\rightarrow\beta)=T = v(\alpha)\rightarrow v(\beta)$; c2: $\beta\notin\Gamma$, in which case $v(\beta)$=F; also, by T45, $\sim\beta\in\Gamma$, so by T46, $\Gamma\vdash\sim\beta$, and also $\Gamma\vdash\alpha$; but by T25+T10, $\Gamma\vdash\alpha\rightarrow(\sim\beta\rightarrow\sim(\alpha\rightarrow\beta))$, so by T14 (twice), $\Gamma\vdash\sim(\alpha\rightarrow\beta)$, so by T46, $\sim(\alpha\rightarrow\beta)\in\Gamma$, so by T45, $\alpha\rightarrow\beta\notin\Gamma$, so $v(\alpha\rightarrow\beta)=F = v(\alpha)\rightarrow v(\beta)$.

7. Every Deductively Consistent Set is Semantically Consistent

Having proven Lindenbaum's Lemma, and having proven that every maximal consistent set is verifiable, we are now in a position to prove our earlier claim that every deductively consistent set is semantically consistent.

T49

Γ is deductively consistent $\longrightarrow$ Γ is semantically consistent.

(i.e.) Every deductively consistent set is semantically consistent.

Proof: Suppose Γ is deductively consistent, to show that Γ is semantically consistent, which is to say that there is a valuation v in V that verifies every formula in Γ. By Lindenbaum's Lemma, Γ is included in a maximal consistent set, call it Γ^*. By T48, Γ^* is verifiable. But, every subset of a verifiable set is itself verifiable, so Γ is verifiable, which is to say that Γ is semantically consistent.

8. The Completeness Theorems

First, we prove the strong completeness theorem, after which we prove the weak completeness theorem, as a special case.

(T50) $\Gamma \vDash \alpha \longrightarrow \Gamma \vdash \alpha$

Proof: Suppose $\sim[\Gamma \vdash \alpha]$, to show: $\sim[\Gamma \vDash \alpha]$. Then, by T40, $\Gamma \cup \{\sim\alpha)$ is consistent, so by T48, $\Gamma \cup \{\sim\alpha\}$ is verifiable, so by Lemma 4, $\sim[\Gamma \vDash \alpha]$.

(C) $\vDash \alpha \longrightarrow \vdash \alpha$

Proof: Suppose $\vDash \alpha$, to show: $\vdash \alpha$. By Lemma 5, $\varnothing \vDash \alpha$, so by T50, $\varnothing \vdash \alpha$, so by Lemma 6, $\vdash \alpha$.

9. Appendix: Supporting Lemmas

1. Earlier Theorems

(T1) $\vdash \alpha\rightarrow(\beta\rightarrow\alpha)$
(T7) $\vdash \alpha\rightarrow\alpha$
(T8) $\vdash \sim\alpha\rightarrow(\alpha\rightarrow\beta)$
(T10) $\vdash\alpha \longrightarrow \Gamma\vdash\alpha$
(T12) $\alpha\in\Gamma \longrightarrow \Gamma\vdash\alpha$
(T14) $\Gamma\vdash\alpha$ & $\Gamma\vdash\alpha\rightarrow\beta$ $.\!\!\longrightarrow \Gamma\vdash\beta$
(T16) $\Gamma\cup\{\alpha\}\vdash\beta \longrightarrow \Gamma\vdash\alpha\rightarrow\beta$ (a.k.a. DT)
(T17) $\Gamma\vdash\alpha\rightarrow\beta \longrightarrow \Gamma\cup\{\alpha\} \vdash \beta$
(T18) $\Gamma\vdash\alpha$ & $\Gamma\subseteq\Delta$ $.\!\!\longrightarrow \Delta\vdash\alpha$
(T19) $\Gamma\vdash\alpha$ & $\Gamma\cup\{\alpha\}\vdash\beta$ $.\!\!\longrightarrow \Gamma\vdash\beta$
(T20) $\vdash \sim\sim\alpha\rightarrow\alpha$
(c) $\{\sim\sim\alpha\} \vdash \alpha$
(T21) $\vdash\alpha\rightarrow\sim\sim\alpha$
(c) $\{\alpha\} \vdash \sim\sim\alpha$
(T25) $\vdash\alpha\rightarrow(\sim\beta\rightarrow\sim(\alpha\rightarrow\beta))$
(T30) $\vdash(\alpha\rightarrow\beta)\rightarrow[(\sim\alpha\rightarrow\beta)\rightarrow\beta]$
(T35) $\{\alpha\rightarrow\beta, \alpha\rightarrow\sim\beta\} \vdash \sim\alpha$
(c) $\vdash(\alpha\rightarrow\beta)\rightarrow[(\alpha\rightarrow\sim\beta)\rightarrow\sim\alpha]$
(T36) $\{\sim\alpha\rightarrow\beta, \sim\alpha\rightarrow\sim\beta\} \vdash \alpha$
(c) $\vdash(\sim\alpha\rightarrow\beta)\rightarrow[(\sim\alpha\rightarrow\sim\beta)\rightarrow\alpha]$
(T37) $\Gamma\vdash\alpha \longrightarrow \Gamma\vDash\alpha$ (a.k.a. ST)
(c) $\vdash\alpha \longrightarrow \vDash\alpha$

2. New Theorems

(T38) $\forall\Gamma\{\Gamma\vdash \longleftrightarrow \exists\alpha(\Gamma\vdash\alpha\ \&\ \Gamma\vdash\sim\alpha)\}$
(T39) Γ is semantically consistent $\longrightarrow$ Γ is deductively consistent
(T40) $\Gamma\cup\{\sim\alpha\}\vdash \longleftrightarrow \Gamma\vdash\alpha$
(T41) $\{\alpha\}\vdash\beta$ & $\{\beta\}\vdash\alpha$ $.\!\!\longrightarrow\!\!.$ $\Gamma\cup\{\alpha\}\vdash\gamma \longleftrightarrow \Gamma\cup\{\beta\}\vdash\gamma$
(T42) $\Gamma\cup\{\alpha\}\vdash \longleftrightarrow \Gamma\vdash\sim\alpha$
(T43) MC[Γ] & $\alpha\notin\Gamma$ $.\!\!\longrightarrow \Gamma\cup\{\alpha\}\vdash$
(T44) $\Gamma\cup\{\alpha\}\vdash\beta$ & $\Gamma\cup\{\sim\alpha\}\vdash\beta$ $.\!\!\longrightarrow \Gamma\vdash\beta$
(c) $\Gamma\cup\{\alpha\}\vdash$ & $\Gamma\cup\{\sim\alpha\}\vdash$ $.\!\!\longrightarrow \Gamma\vdash$
(T45) MC[Γ] $\longrightarrow \forall\alpha(\alpha\in\Gamma \vee \sim\alpha\in\Gamma)$
(T46) MC[Γ] $\longrightarrow \forall\alpha(\Gamma\vdash\alpha \longrightarrow \alpha\in\Gamma)$
(T47) $\forall\Gamma\{\Gamma$ is consistent $\longrightarrow \exists\Delta(\Gamma\subseteq\Delta$ & MC[Δ])} (a.k.a. LL)
(T48) Γ is maximal consistent $\longrightarrow$ Γ is verifiable
(T49) Γ is deductively consistent $\longrightarrow$ Γ is semantically consistent
(T50) $\Gamma\vDash\alpha \longrightarrow \Gamma\vdash\alpha$ (a.k.a. CT)
(c) $\vDash\alpha \longrightarrow \vdash\alpha$

3. Additional Lemmas

(L1) $\sim\exists v\{v(\alpha)=T\ \&\ v(\sim\alpha)=T\}$
(L2) $\forall\alpha(\alpha\in\Delta \longrightarrow \Gamma\vdash\alpha)$ & $\Delta\vdash\beta$ $.\!\!\longrightarrow \Gamma\vdash\beta$
(L3) there are denumerably-many formulas
(L4) $\Gamma\vDash\alpha \longleftrightarrow \Gamma\cup\{\sim\alpha\}$ is unverifiable
(L5) $\vDash\alpha \longleftrightarrow \varnothing\vDash\alpha$
(L6) $\vdash\alpha \longleftrightarrow \varnothing\vdash\alpha$