10 The Soundness Theorem for System AS1

1. Introduction

In the current chapter, we begin examining the relation between the semantic (model-theoretic) characterization of classical sentential logic and the axiomatic (deductive, proof-theoretic) characterization.

So far, we have defined argument/formula validity both semantically and axiomatically. Recall the official definitions.

$\vDash\alpha$	$=_{df}$	$\forall v\{v\in V \longrightarrow v(\alpha)=T\}$
$\Gamma\vDash\alpha$	$=_{df}$	$\forall v\{v\in V \longrightarrow\!.\ \forall x(x\in\Gamma \longrightarrow v(x)=T) \longrightarrow v(\alpha)=T\}$
$\vdash\alpha$	$=_{df}$	there is a proof of α in $\mathbb{A}$
$\Gamma\vdash\alpha$	$=_{df}$	there is a derivation of α from Γ in $\mathbb{A}$

As usual, in order to avoid clutter, we drop reference to the set V of valuations and the axiom system $\mathbb{A}$, in the *definiendum* (i.e., left side of the definition). If we were to write these more carefully, we might write them as follows.

V: $\vDash\alpha$	$=_{df}$	$\forall v\{v\in V \longrightarrow v(\alpha)=T\}$
V: $\Gamma\vDash\alpha$	$=_{df}$	$\forall v\{v\in V \longrightarrow\!.\ \forall x(x\in\Gamma \longrightarrow v(x)=T) \longrightarrow v(\alpha)=T\}$
$\mathbb{A}$: $\vdash\alpha$	$=_{df}$	there is a proof of α in $\mathbb{A}$
$\mathbb{A}$: $\Gamma\vdash\alpha$	$=_{df}$	there is a derivation of α from Γ in $\mathbb{A}$

These are generic definitions. In our particular case, the relevant class V of valuations is given by the usual truth tables, and the relevant axiom system $\mathbb{A}$ is AS1.

2. Soundness, Completeness, and Mutual Consistency

The obvious remaining question is whether these two characterizations are *mutually consistent*. Given that we have defined both formula validity and argument validity, mutual consistency can be defined both for formulas and arguments, as follows.

Def

Let V be a class of valuations, and let $\mathbb{A}$ be an axiom system, both defined over a common language. Define $\vdash$ and $\vDash$ as above. Then:

V and $\mathbb{A}$ are *mutually consistent wrt formulas* $=_{df}$ $\forall\alpha\{\vDash\alpha \longleftrightarrow \vdash\alpha\}$.

V and $\mathbb{A}$ are *mutually consistent wrt arguments* $=_{df}$ $\forall\Gamma\forall\alpha\{\Gamma\vDash\alpha \longleftrightarrow \Gamma\vdash\alpha\}$.

[Note: 'wrt' is short for 'with respect to'] Concerning the relation between the two modes of consistency, recall that we have the following theorems about $\vdash$ and $\vDash$ respectively.

(1) $\vdash\alpha \leftrightarrow \varnothing\vdash\alpha$
(2) $\vDash\alpha \leftrightarrow \varnothing\vDash\alpha$

Given (1) and (2), it follows that, if V and $\mathbb{A}$ are mutually consistent wrt arguments, then they are automatically mutually consistent wrt formulas. The converse, however, is not true; an axiom system may agree with a semantic system with regard to formula-validity without agreeing with regard to the argument-validity. Of course, logic is ultimately concerned with argument-validity, so anything less than mutual consistency wrt arguments is not entirely satisfactory, although it might be abstractly interesting.

Mutual consistency naturally divides into two parts, which are called *soundness* and *completeness*, defined as follows.

Def

Let V be a class of valuations, and let $\mathbb{A}$ be an axiom system, both defined over a common language. Define $\vdash$ and $\vDash$ in the customary way. Then:

$\mathbb{A}$ is *sound* relative to V wrt <u>formulas</u> $=_{df}$ $\forall\alpha\{\vdash\alpha \rightarrow \vDash\alpha\}$.

$\mathbb{A}$ is *complete* relative to V wrt <u>formulas</u> $=_{df}$ $\forall\alpha\{\vDash\alpha \rightarrow \vdash\alpha\}$.

$\mathbb{A}$ is *sound* relative to V wrt <u>arguments</u> $=_{df}$ $\forall\Gamma\forall\alpha\{\Gamma\vdash\alpha \rightarrow \Gamma\vDash\alpha\}$.

$\mathbb{A}$ is *complete* relative to V wrt <u>arguments</u> $=_{df}$ $\forall\Gamma\forall\alpha\{\Gamma\vDash\alpha \rightarrow \Gamma\vdash\alpha\}$.

In other words:

formula-sound	$=_{df}$	every deductively-valid formula is semantically-valid
formula-complete	$=_{df}$	every semantically-valid formula is deductively-valid
argument-sound	$=_{df}$	every deductively-valid argument is semantically-valid
argument-complete	$=_{df}$	every semantically-valid argument is deductively-valid

Note once again that, given the relation between argument-validity and formula-validity, soundness (completeness) wrt arguments entails soundness (completeness) wrt formulas. Also note the following immediate theorems.

Th

$\mathbb{A}$ and V are mutually consistent wrt formulas (arguments) if and only if $\mathbb{A}$ is both sound and complete relative to V wrt formulas (arguments).

3. The Weak Soundness Theorem

In the next two sections, we prove that AS1 is sound relative to the usual truth functional semantics for CSL. First, we prove a *weak soundness theorem*, which states that AS1 is sound wrt formulas. In a later section, we prove a *strong soundness theorem*, which states that AS1 is sound wrt arguments. The following is the official statement of the weak soundness theorem.

Weak Soundness Theorem

Let V be the usual truth-functional semantics for CSL, and let $\mathbb{A}$ be AS1, as in previous sections; define $\vDash$ and $\vdash$ in the customary manner. Then for any formula α,

$$\vdash\alpha \longrightarrow \vDash\alpha$$

There are a number of ways to prove the Weak Soundness Theorem, all using some form of induction. We employ strong induction. Supporting lemmas are provided in a later section.

Proof:
First, in virtue of the definition of '$\vdash$', what we want to prove amounts to the following.

$$\forall\alpha\{\exists p[p \text{ is a proof of } \alpha] \longrightarrow \vDash\alpha\}$$

The latter is equivalent by QL to:

$$\forall\alpha\forall p\{p \text{ is a proof of } \alpha \longrightarrow \vDash\alpha\}$$

A proof is a finite sequence, so every proof has a length, which is a natural number. I.e.,

$$\forall p\exists n[n = \text{len}(p)]$$

So, in virtue of QL, it is sufficient to prove the following.

$$\forall n\,\forall\alpha\forall p\{p \text{ is a proof of } \alpha \text{ of length } n \longrightarrow \vDash\alpha\}$$

The latter can be proven by strong induction, as follows.

Inductive Case:
Where n is any number, suppose (IH) to show (IS), given as follows.

(IH) $\forall m\{m<n \rightarrow \forall\alpha\forall p\{p \text{ is a proof of } \alpha \text{ of length } m \longrightarrow \vDash\alpha\}\}$,
(IS) $\forall\alpha\forall p\{p \text{ is a proof of } \alpha \text{ of length } n \longrightarrow \vDash\alpha\}$.

Suppose P is a proof of α of length n, to show $\vDash\alpha$. Every line of P, including its last line (i.e., α) must either be (1) an axiom, or (2) follow from previous lines by MP.
Case 1: α is an axiom. So, in order to show $\vDash\alpha$, it suffices to show that every axiom is semantically valid. This is the content of Lemma 1.
Case 2: α follows from previous lines by MP. In virtue of the form of MP, one of the previous lines – call it i – is a conditional whose antecedent is the other line – call it j – and whose consequent is α. Thus, $P(i) = \gamma\rightarrow\alpha$, and $P(j) = \gamma$. Those lines up to and including P(i) constitute an i-long proof of $\gamma\rightarrow\alpha$. Similarly those lines up to and including P(j) constitute a j-long proof of γ. But $i, j < n$, so we can apply the inductive hypothesis (which is universally quantified over α) to these two derivations to obtain, respectively, (i) $\vDash\gamma\rightarrow\alpha$, and (j) $\vDash\gamma$. From these two facts, and Lemma 2, we obtain $\vDash\alpha$.

4. The Strong Soundness Theorem

Having warmed up by proving the weak soundness result, we now turn to the strong soundness theorem, which is proven nearly the same way.

Strong Soundness Theorem

Let V be the usual truth-functional semantics for CSL, and let $\mathbb{A}$ be AS1, as in previous sections; define $\vDash$ and $\vdash$ as in Section 1. Then for any formula α, and any set Γ of formulas,

$$\Gamma\vdash\alpha \;\longrightarrow\; \Gamma\vDash\alpha$$

Proof:
In virtue of the definition of '$\vdash$', this amounts to the following,

$$\forall\Gamma\forall\alpha\{\exists d[d \text{ is a derivation of } \alpha \text{ from } \Gamma] \;\longrightarrow\; \Gamma\vDash\alpha\},$$

which is equivalent by QL to:

$$\forall\Gamma\forall\alpha\forall d\{d \text{ is a derivation of } \alpha \text{ from } \Gamma \;\longrightarrow\; \Gamma\vDash\alpha\}$$

Now, every derivation d has a length len(d), so it is sufficient to prove the following.

$$\forall n\forall\Gamma\forall\alpha\forall d\{d \text{ is a derivation of } \alpha \text{ from } \Gamma \text{ of length } n \;\longrightarrow\; \Gamma\vDash\alpha\}$$

The latter can be proven by strong induction, as follows.

Inductive Case: Where n is any number, suppose (IH) to show (IS), given as follows.

(IH) $\forall m\{m<n \;\longrightarrow\; \forall\Gamma\forall\alpha\forall d\{d \text{ is a derivation of } \alpha \text{ from } \Gamma \text{ of length } m \;\longrightarrow\; \Gamma\vDash\alpha\}\},$
(IS) $\forall\Gamma\forall\alpha\forall d\{d \text{ is a derivation of } \alpha \text{ from } \Gamma \text{ of length } n \longrightarrow \Gamma\vDash\alpha\}.$

Suppose D is a derivation of α from Γ of length n, to show $\Gamma\vDash\alpha$. Every line of D, including its last line (i.e., α) must (1) be an axiom, or (2) be a premise (an element of Γ), or (3) follow from previous lines by MP.
Case 1: α is an axiom; so by Lemma 3, $\Gamma\vDash\alpha$.
Case 2: α is a premise ($\alpha\in\Gamma$); so by Lemma 4, $\Gamma\vDash\alpha$.
Case 3: α follows from previous lines by MP. In virtue of the form of MP, one of the previous lines – call it i – is a conditional whose antecedent is the other line – call it j – and whose consequent is α. Thus, $D(i) = \gamma\rightarrow\alpha$, and $D(j) = \gamma$. Those lines up to and including P(i) constitute an i-long derivation of $\gamma\rightarrow\alpha$ from Γ. Similarly those lines up to and including P(j) constitute a j-long derivation of γ from Γ. But $i, j < n$, so we can apply the inductive hypothesis (which is universally quantified over Γ,α) to these two derivations to obtain, respectively, (i) $\Gamma\vDash\gamma\rightarrow\alpha$, and (j) $\vDash\gamma$. From these two facts, and Lemma 5, we obtain $\vDash\alpha$.

5. Appendix: Supporting Lemmas

The following are lemmas used in the proofs of the soundness theorem(s). The proofs of these lemmas are left as an exercise.

(L1) α is an axiom of AS1 $\longrightarrow\ \vDash\alpha$

(L2) $\vDash\alpha\rightarrow\beta\ \&\ \vDash\alpha\ .\!\longrightarrow\ \vDash\beta$

(L3) α is an axiom of AS1 $\longrightarrow\ \Gamma\vDash\alpha$

(L4) $\alpha\in\Gamma\longrightarrow\Gamma\vDash\alpha$

(L5) $\Gamma\vDash\alpha\rightarrow\beta\ \&\ \Gamma\vDash\alpha\ .\!\longrightarrow\ \Gamma\vDash\beta$