# 9

# Initial Theorems about Axiom System AS1

## 1.      Theorems <u>in</u> Axiom Systems versus Theorems <u>about</u> Axiom Systems

We begin by reviewing some of the relevant definitions.

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Def

Let $\mathbb{A}$ be an axiom system with sentences S and rules $\mathcal{R}$; i.e., $\mathbb{A} = (S, S)$.
Let $\alpha$ be a formula in $S$.  Let $\sigma$ be a finite sequence of formulas of $S$.
Then:

| | | |
|---|---|---|
| $\sigma$ is a *proof in* $\mathbb{A}$ | $=_{df}$ | every item in $\sigma$ follows from previous items by a rule in $\mathcal{R}$. |
| $\sigma$ is a *proof of* $\alpha$ *in* $\mathbb{A}$ | $=_{df}$ | $\sigma$ is a proof in $\mathbb{A}$ the last item of which is $\alpha$. |
| $\alpha$ is *provable in* $\mathbb{A}$ | $=_{df}$ | there is a proof of $\alpha$ in $\mathbb{A}$. |
| $\alpha$ is a *theorem/thesis* of $\mathbb{A}$ | $=_{df}$ | $\alpha$ is provable in $\mathbb{A}$. |
| $\vdash \alpha$ | $=_{df}$ | $\alpha$ is a theorem [of $\mathbb{A}$]. |

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Observe that, just as with the double turnstile, the single turnstile is used both as a two-place predicate and as a one-place predicate.  The ambiguity is harmless, even useful, since it is easy to prove the following theorem (Chapter 6).

(G3)    $\vdash \alpha \leftrightarrow \varnothing \vdash \alpha$

In other words, $\alpha$ is provable in $\mathbb{A}$ if and only if $\alpha$ is derivable in $\mathbb{A}$ from the empty set.  Thus, we can understand '$\vdash \alpha$' as an additional primitive expression in the metalanguage, or as a lazy way of writing '$\varnothing \vdash \alpha$'.

Based on results from the previous chapter, we already have the following list of theorems of axiom system AS1.

(o1)    $P \rightarrow P$
(o2)    $\sim P \rightarrow (P \rightarrow Q)$
(o3)    $[P \rightarrow (P \rightarrow Q)] \rightarrow (P \rightarrow Q)$

We regard this to be a list, written in the metalanguage, of formulas (of the object language).  What occurs above are not the formulas of the object language, but rather names of those formulas.  Similarly, a list of senators does not consist of senators; rather, it consists of names of senators.  Each expression in this list is a complex noun phrase in the metalanguage referring in the obvious way to its counterpart in the object language.

If we want to write the corresponding *assertions* that the formulas cited in (o1)-(o3) are theorems, we take these noun phrases and attach the turnstile-predicate, which results in the following metalanguage sentences.

(m1)    ⊢ P→P
(m2)    ⊢ ~P→(P→Q)
(m3)    ⊢ [P→(P→Q)]→(P→Q)

The resulting expressions are sentences in the metalanguage, not sentences in the object language, so they are not theorems *in* AS1.  Rather they are theorems *about* AS1.  To be more explicit,

P→P

is a theorem <u>in</u> AS1, but

'P→P' is a theorem in AS1

is not a theorem <u>in</u> AS1, but rather a theorem <u>about</u> AS1.


## 2.    **Proofs <u>about</u> Axiom Systems**

How do we show the following?

⊢ P→P

[Note: for the time being, we understand '⊢' as relativized to AS1.]  The direct way goes as follows.  First, we display an object language proof in AS1 of P→P.  The following will do.

(1)    P→(P→P.→P)                                                          R1
(2)    P→(P→P.→P) .→. (P→.P→P)→(P→P)                              R2
(3)    (P→.P→P)→(P→P)                                                  1,2,R4
(4)    (P→.P→P)                                                            R1
(5)    P→P                                                                  3,4,R4

The embedded sequence of formulas, call it Σ, is a proof of

P→P.

Σ is not, however, a proof of

⊢ P→P.

The latter is a metalinguistic sentence, whose proof must be written in the metalanguage.   The metalinguistic proof is no big deal, so it is usually taken for granted.  It goes as follows.

> Inspection of the above annotated list reveals that the sequence Σ is a proof of 'P→P' in AS1.  Therefore, there is a proof of 'P→P' in AS1.  Therefore, 'P→P' is provable in AS1.  Therefore 'P→P' is a theorem of AS1 – i.e., ⊢P→P.

This is the *direct* way to prove '⊢P→P'.  Generally, the *direct* way to prove '⊢α' is to display a proof π(α) of α, then reason as follows.

> π(α) is a proof of α in AS1.  Therefore, there is a proof of α in AS1.  Therefore, α is provable in AS1.  Therefore α is a theorem of AS1 – i.e., ⊢α.

Usually, however, we are less direct.  For example, we can prove

$$\vdash Q \rightarrow Q$$

not by displaying a proof of $Q \rightarrow Q$, but by noting that the proof of $P \rightarrow P$ can be converted into a proof of $Q \rightarrow Q$ by substituting Q for P throughout. More generally, we can prove $\alpha \rightarrow \alpha$, for arbitrary $\alpha$, by taking the proof of $P \rightarrow P$ and substituting $\alpha$ for P. Thus, we have

$$\vdash \alpha \rightarrow \alpha.$$

By similar reasoning we obtain the following from our earlier proofs.

$$\vdash \sim\alpha \rightarrow (\alpha \rightarrow \beta)$$
$$\vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$$

The last three formulas of the metalanguage are understood to be universally quantified over the variables '$\alpha$' and '$\beta$'. If we wish to be more explicit, we can write the following.

(u1)    $\forall\alpha \vdash \alpha \rightarrow \alpha$
(u2)    $\forall\alpha\forall\beta \vdash \sim\alpha \rightarrow (\alpha \rightarrow \beta)$
(u3)    $\forall\alpha\forall\beta \vdash [\alpha \rightarrow (\alpha \rightarrow \beta)] \rightarrow (\alpha \rightarrow \beta)$

It is customary, however, to drop the universal quantifiers.

Whereas the topic of discussion is proofs in the object language, most proofs actually presented to us are proofs in the metalanguage about the object language. In this connection, recall the proofs about semantics. Proofs in the metalanguage employ the full resources of logic, as elaborated in an appendix. By contrast, proofs in the object language use only the resources afforded by the actual axiom system under consideration. For example, a proof in the metalanguage can use conditional derivation; on the other hand, a proof in the object language cannot, at least not in any axiom system presented so far.

The object language proofs presented in the previous chapter are pretty much the only such proofs actually displayed. In what follows, we do not actually display object language proofs; rather, we provide metalanguage proofs that object language proofs exist.

## 3.    Initial Examples of Proofs in the Metalanguage about AS1

Let us begin with some obvious theorems <u>about</u> AS1.

(T1)    $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$
(T2)    $\vdash [\alpha \rightarrow (\beta \rightarrow \gamma) \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
(T3)    $\vdash (\sim\alpha \rightarrow \sim\beta) \rightarrow (\beta \rightarrow \alpha)$

Note that all formulas are implicitly universally quantified.

These are all corollaries to a general theorem, proven earlier.

(G5)    $\text{Axiom}[\alpha] \rightarrow \vdash\alpha$

Recall that an axiom is, by definition, any formula that is output by a zero-place rule. For example, R1 outputs formulas of the form $\alpha \rightarrow (\beta \rightarrow \alpha)$, so every instance of this form is an axiom of AS1, so every instance is provable in AS1. Similarly, with R2 and R3. Notice carefully that, in (T1)-(T3), there is something to prove; it's just that the proof is rather simple.

At this point, since we have now fully ascended to the metalanguage, we are faced with the problem of how to use/mention the various logical symbols – '→', '~', etc. Recall the material in Chapter 1 on use/mention in formal languages. Also recall that, in Chapter 5, we employed the usual logical symbols in three distinct ways.

(1)   as (abbreviations of) connectives in the metalanguage;
(2)   as names in the metalanguage of the usual logical symbols in the object language;
(3)   as names of the truth-functions associated with the usual logical symbols.

In this chapter, and subsequent chapters, we will continue to use the logical symbols ambiguously. So context will be exceedingly important in deciphering the various formulas. To help visually, we will often write the metalinguistic if-then somewhat bigger.

The next theorem illustrates our metalinguistic convention.

(T4)   ⊢α  &  ⊢α→β  .$\longrightarrow$  ⊢β

Ignoring the implicit universal quantifiers, the main functor of this formula is '$\longrightarrow$', which is the metalinguistic 'if…then' connective. Similarly, '&' is the metalinguistic 'and' connective. Translating this into English, we have:

if ⊢α, and ⊢α→β, then ⊢β.

The remaining occurrence of '→' is under the predicate '⊢'; it is accordingly a proper noun referring to the conditional connective of the object language. When we translate '⊢' as 'is a theorem', we obtain:

if α is a theorem, and α→β is a theorem, then β is a theorem.

The informal proof of (T4) goes as follows.

> Proof: Suppose ⊢α and ⊢α→β, to show ⊢β. Then, α and α→β are provable; so there is a proof of α, call it $P_1$, and there is a proof of α→β, call it $P_2$. Consider the sequence $P_1+P_2+\langle β\rangle$, obtained by appending $P_2$ to $P_1$, and then appending β to that. Claim: $P_1+P_2+\langle β\rangle$ is a proof of β, from which it follows that ⊢β. To prove this claim, we must prove that $P_1+P_2+\langle β\rangle$ is a proof the last line of which is β. Clearly, the last line of $P_1+P_2+\langle β\rangle$ is β, so the question is whether $P_1+P_2+\langle β\rangle$ is a proof. This amounts to the following: every line of $P_1+P_2+\langle β\rangle$ follows from previous lines by a rule. So, suppose δ is a line of $P_1+P_2+\langle β\rangle$; there are 3 cases. c1: δ is in $P_1$; c2: δ is in $P_2$; c3: δ=β. Case 1 is ok, because $P_1$ is a proof (by hyp). Similarly, case 2 is ok, since $P_2$ is a proof. That leaves case 3; now, $P_1$ is a proof of α, so its last line is α; similarly, $P_2$ is a proof of α→β, so its last line is α→β. Accordingly, both α and α→β are lines prior to δ (=β); but by R4 (mp), β follows from α and α→β, so case 3 is ok.

The next theorem is an immediate consequence of T1 and T4.

(T5)   ⊢α  $\longrightarrow$  ⊢β→α

> Proof: Suppose ⊢α, to show ⊢β→α. By T1, ⊢α→(β→α); so by the hypothesis and T4, we have ⊢β→α.

With T5 in hand, we can prove a simple form of the transitive law for the conditional.

(T6)    ⊢α→β  &  ⊢β→γ .→ ⊢α→γ

Proof:  Suppose ⊢α→β and ⊢β→γ, to show ⊢α→γ. By hyp2, T5, and QL, we obtain
⊢α→(β→γ).  By T2, we have ⊢[α→(β→γ)]→[(α→β)→(α→γ)], so by T4, we have
⊢(α→β)→(α→γ).  So with hyp1, T4, we have ⊢α→γ.

At this point, we re-introduce our earlier theorems into the new numbering scheme.

(T7)    ⊢α→α
(T8)    ⊢~α→(α→β)
(T9)    ⊢[α→(α→β)]→(α→β)

Again, note that the universal quantifiers are omitted.   The following is an informal proof of (T7) in the
metalanguage.  It appeals to earlier theorems, plus first-order logic.

By T1 and QL, ⊢α→[(α→α)→α].  By T2,
⊢{α→[(α→α)→α]}→{[α→(α→α)]→[α→α]}.  So, by T4 and SL,
⊢[α→(α→α)]→[α→α].  But, by T1, ⊢α→(α→α), so by T4, ⊢α→α.

The proofs of (T8) and (T9) can be carried out in a similar way.

We conclude this section with several theorems that we employ in proving the Deduction
Theorem, which is the topic of the next few sections.

(T10)  ⊢α  →  Γ⊢α                                                                      [a.k.a. G2]

Proof:  Suppose ⊢α.  Then there is a proof of α. Every proof of α is automatically a
derivation of α from any set Γ; hence Γ⊢α.

(T11)  Γ⊢α→α

Proof:  By T7, ⊢α→α, so by T10 (G2), Γ⊢α→α.

(T12)  α∈Γ  →  Γ⊢α                                                                     [a.k.a. G0]

Proof:  If α∈Γ, then the singleton sequence of α is a derivation of α from Γ.

(T13)  Γ⊢α→(β→α)

Proof: By T1, ⊢α→(β→α), so by T10 (G2), Γ⊢α→(β→α).

(T14)  Γ⊢α  &  Γ⊢α→β .→ Γ⊢β

Proof: Completely analogous to the proof of T4.

(T15)  Γ⊢α→(β→γ)  &  Γ⊢α→β .→ Γ⊢α→γ

Proof:  Suppose Γ⊢α→(β→γ) & Γ⊢α→β.  Then, by T2,
⊢[α→(β→γ)]→[(α→β)→(α→γ)], so by T10 (G2),
Γ⊢[α→(β→γ)]→[(α→β)→(α→γ)], so by T14, Γ⊢(α→β)→(α→γ), so by T14,
Γ⊢α→γ.

### 4. The Deduction Theorem

In the next few sections, we prove the single most important purely axiomatic theorem, called the Deduction Theorem. [In later chapters, we prove more important theorems, but they are not purely axiomatic, since they concern the relation between deductive entailment and semantic entailment.]

In proving the deduction theorem, we employ *mathematical induction*, which is discussed in Chapter 3, and in Section 5.

The Deduction Theorem may be stated as follows. As usual, the variables are sortal ($\Gamma$ is a set of formulas, and $\alpha$ and $\beta$ are formulas), and they are understood to be universally quantified.

> DT
>
> $$\Gamma \cup \{\alpha\} \vdash \beta \;\rightarrow\; \Gamma \vdash \alpha \rightarrow \beta$$

In other words, if one can derive $\beta$ from $\Gamma$ together with $\alpha$, then one can derive the conditional $\alpha \rightarrow \beta$ from $\Gamma$ alone.

The following is a special case, obtained by substituting $\varnothing$ for $\Gamma$.

(DT1) $\alpha \vdash \beta \;\rightarrow\; \vdash \alpha \rightarrow \beta$

The latter says that if one can derive $\beta$ from $\alpha$, then one can prove $\alpha \rightarrow \beta$ (i.e., derive it from $\varnothing$).

Note carefully that the seemingly anthropocentric 'one can derive' is simply a shorthand way of saying that a derivation of a certain sort exists; recall the definition of derivation. Most derivations cannot be accomplished by a human (or any mortal being, for that matter!). Abstract objects can nevertheless be proven to exist, even when they cannot be directly displayed.

One might notice that (DT) is reminiscent of the method of conditional derivation. The similarity is important, but potentially misleading. It is therefore crucial at this point to keep the following in mind.

> CD
>
> Conditional derivation states, <u>as a rule</u>, that
> a derivation of $\beta$ from $\alpha$ (given $\Gamma$) ***is tantamount to***
> a derivation of $\alpha \rightarrow \beta$ (given $\Gamma$).

The Deduction Theorem makes a completely different claim.

> DT
>
> The Deduction Theorem states that,
> if there exists a derivation of $\beta$ from $\alpha$ plus $\Gamma$,
> then there also exists a derivation of $\alpha \rightarrow \beta$ from $\Gamma$.

In particular, unlike CD, the Deduction Theorem does not say anything about what the derivation of α→β from Γ looks like, but only that such a derivation exists.

To see that the Deduction Theorem is not trivial, consider the simplest case imaginable. It is very easy to derive P from P (indeed, in any system!). The singleton sequence ⟨P⟩ is such a derivation; every line is either a premise or follows by a rule, and the last line is P. Accordingly we have P⊢P, so by the Deduction Theorem (special case), we have ⊢P→P.

But the Deduction Theorem does not declare that the derivation of P from P *is* a proof of P→P. It merely says that a proof of P→P exists, without giving so much as a clue about what that proof looks like. Now, as it turns out, we happen to know what the proof of P→P looks like, since we proved it in the previous chapter; recall the proof of P→P involves five non-trivial lines. Thus, although the derivation of P from P is completely trivial, the proof of P→P is considerably less trivial.

Before proceeding with the Deduction Theorem, it might be helpful to consider the corresponding semantic theorem, which was proven in an earlier chapter.

(S)      $\Gamma\cup\{\alpha\}\vDash\beta \longrightarrow \Gamma\vDash\alpha\rightarrow\beta$

Given that (S) is a theorem about CSL, and given that we ultimately intend the semantic relation ⊨ and the deductive relation ⊢ to be coextensive, we must prove the Deduction Theorem at some point, either directly or indirectly. In principle, we could prove the Deduction Theorem indirectly by proving ⊢ and ⊨ are coextensive. However, in order to prove the latter, it is easiest first to prove the Deduction Theorem directly.

What we must prove is that, if there is a derivation of β from Γ∪{α}, then there is also a derivation (somewhere!) of α→β from Γ. In order to prove this, we need mathematical induction, which is discussed in the next section.

## 5.      Using Mathematical Induction to do Proofs <u>about</u> Derivations

The proof of the Deduction Theorem involves mathematical induction. Recall from an earlier chapter that the basic idea about mathematical induction is that, if one wants to prove that every natural number has a given property $\mathbb{P}$, one proves that 0 has $\mathbb{P}$, and one proves that if a number n has $\mathbb{P}$, then so does its successor $n^+$. This is known as *weak induction*.

Recall that there is also the method of *strong induction*. According to this method, if one wants to prove that every number has $\mathbb{P}$, one proves that a number n has $\mathbb{P}$ *provided* every predecessor of n has P. Notice that since 0 has no predecessors, this automatically includes proving the usual base case. Sometimes the 0-case has to be done separately; other times it comes along for free.

Mathematical induction is great for arithmetic. But how do we employ an arithmetic proof technique in a metalogical context? The trick centers around what counts as a property. In particular, observe that property $\mathbb{P}$ doesn't have to be an *intrinsic* property of numbers; it can be any formula that refers to numbers. For example, one such "property" of the number 1 is that there is a derivation of 'P' from 'P' that has length 1; similarly, one such "property" of the number 5 is that there is a proof of 'P→P' that has length 5.

Inductive proofs about derivations are based on the following argument form.

(P0)    every derivation has some length (1,2,3, etc.)
(P1)    every derivation of length 1 has property $\mathbb{P}$
(P2)    every derivation of length 2 has property $\mathbb{P}$
(P3)    every derivation of length 3 has property $\mathbb{P}$
        etc.
        therefore,
(C)     every derivation has property $\mathbb{P}$

An informal proof of this argument form might go as follows. Consider an arbitrary derivation D; by P0, every derivation has some length, so D has some length, call it k; but according to premise Pk, every derivation of length k has property $\mathbb{P}$; so D has property $\mathbb{P}$.

The above argument has infinitely many premises, so in order to formalize the reasoning involved, we must reduce the number of premises to a finite number. One way is to substitute a universal formula for the infinite sequence P1, P2, …. This yields the following formalized argument scheme, where 'd' ranges over derivations, and 'n' ranges over natural numbers, and 'len(d)' refers to the length of d. Note that this argument is valid by quantifier logic (exercise).

(P0)    $\forall d \exists n[len(d)=n]$
(P*)    $\forall n \, \forall d\{len(d)=n \rightarrow \mathbb{P}\}$
(C)     $\forall d \mathbb{P}$

Now, (P0) follows from the fact that derivations are finite sequences. So, in order to prove (C), one need merely prove (P*). But (P*) is a statement of the form

$$\forall n \mathbb{F}$$

where $\mathbb{F}$ is

$$\forall d\{len(d)=n \rightarrow \mathbb{P}\}.$$

In order to prove that every number has "property" $\mathbb{F}$, we proceed by strong mathematical induction. In other words, we prove the following. [Recall there is no official base case in strong induction.]

(IC)    $\forall n\{\forall k\{k<n \rightarrow \forall d\{len(d)=k \rightarrow \mathbb{P}\}\} \rightarrow \forall d\{len(d)=n \rightarrow \mathbb{P}\}\}$

The latter is proven by UCD, which amounts to assuming the inductive hypothesis,

(IH)    $\forall k\{k<n \rightarrow \forall d\{len(d)=k \rightarrow \mathbb{P}\}\}$,

and proving the inductive step,

(IS)    $\forall d\{len(d)=n \rightarrow \mathbb{P}\}.$

(IH) says that every derivation of length less than n has $\mathbb{P}$; (IS) says that every derivation of length n has $\mathbb{P}$.

## 6.    Setting up the Proof of the Deduction Theorem

In this section, we set up the proof of the Deduction Theorem.

(DT)    $\Gamma \cup \{\alpha\} \vdash \beta \rightarrow \Gamma \vdash \alpha \rightarrow \beta$

First, we apply the definition of '⊢' to the antecedent (only), thus obtaining the following.

$$\exists d[d \text{ is a derivation of } \beta \text{ from } \Gamma\cup\{\alpha\}] \longrightarrow \Gamma\vdash\alpha\rightarrow\beta$$

Restoring the implicit universal quantifiers, the overall form of the latter is:

$$\forall\Gamma\forall\alpha\forall\beta\{\exists d\mathbb{F} \rightarrow \mathbb{G}\}$$

Since 'd' does not occur free in $\mathbb{G}$, this is equivalent by QL to:

$$\forall d \ \forall\Gamma\forall\alpha\forall\beta\{\mathbb{F} \rightarrow \mathbb{G}\}$$

The latter formula has the form,

$$\forall d\mathbb{P}$$

where $\mathbb{P}$ is the formula

$$\forall\Gamma\forall\alpha\forall\beta(\mathbb{F} \rightarrow \mathbb{G})$$

As mentioned in the previous section, formulas of the form $\forall d\mathbb{P}$ can be proven by strong induction, which amounts to the following.

(IH)    assume:    $\forall k(k<n \rightarrow \forall d\{len(d)=k \rightarrow \mathbb{P}\})$

(IS)    show:    $\forall d\{len(d)=n \rightarrow \mathbb{P}\}$

Recall the abbreviations.

| | | |
|---|---|---|
| $\mathbb{P}$ | $=_{df}$ | $\forall\Gamma\forall\alpha\forall\beta(\mathbb{F} \rightarrow \mathbb{G})$ |
| $\mathbb{F}$ | $=_{df}$ | d is a derivation of $\beta$ from $\Gamma\cup\{\alpha\}$ |
| $\mathbb{G}$ | $=_{df}$ | $\Gamma\vdash\alpha\rightarrow\beta$ |
| so | | |
| $\mathbb{P}$ | $=_{df}$ | $\forall\Gamma\alpha\beta(d \text{ is a derivation of } \beta \text{ from } \Gamma\cup\{\alpha\} \rightarrow \Gamma\vdash\alpha\rightarrow\beta)$ |

Upon substituting this into (IH) and (IS), above, we obtain the following.

(IH)    assume:
$\forall k\{k<n \longrightarrow \forall d\{len(d)=k \rightarrow.$
$\forall\Gamma\alpha\beta(d \text{ is a derivation of } \beta \text{ from } \Gamma\cup\{\alpha\} \rightarrow \Gamma\vdash\alpha\rightarrow\beta)\}\}$

(IS)    show:
$\forall d\{len(d)=n \longrightarrow.$
$\forall\Gamma\alpha\beta(d \text{ is a derivation of } \beta \text{ from } \Gamma\cup\{\alpha\} \rightarrow \Gamma\vdash\alpha\rightarrow\beta)\}$

## 7.    Informal Proof of the Deduction Theorem

Having set up the proof, we now proceed to complete it.  First, we informally read the inductive hypothesis (IH) and inductive step (IS) as follows.

(IH)    for any derivation – <u>of length less than n</u> – of $\beta$ from $\Gamma\cup\{\alpha\}$, there is a derivation of $\alpha\rightarrow\beta$ from $\Gamma$ (for any $\Gamma,\alpha,\beta$).

(IS)     for any derivation – <u>of length n</u> – of β from Γ∪{α}, there is a derivation of α→β from Γ
         (for any Γ,α,β).

## The Proof:

[Note: the proof appeals to four supporting lemmas — dt1-dt4 — which are proven in Section 8.]

Where n is any number, suppose (IH) to show (IS).
Suppose D is a derivation of length n of β from Γ∪{α}, to show: Γ⊢α→β. Since D is a
derivation of β, β is the last line of D. Every line of D must be (1) an axiom, (2) a
premise, or (3) follow from previous lines by MP. This applies, of course, to β, so there
are three cases to consider. Case 1: β is an axiom. In this case, by Lemma dt1, Γ⊢α→β.
Case 2: β is a premise; i.e., β∈Γ∪{α}. By set theory, this divides into two cases. Case
1: β∈Γ. In this case, by Lemma dt2, Γ⊢α→β. Case 2: β=α. In this case, the question
whether Γ⊢α→β amounts to the question whether Γ⊢α→α. This is the content of
Lemma dt3. Case 3: β follows from previous lines by MP. In virtue of the form of MP, if
β follows from lines by MP, then one of the previous lines – call it i – is a conditional
whose antecedent is the other line – call it j – and whose consequent is β. Thus, D(i) =
γ→β, and D(j) = γ. Those lines up to and including D(i) constitute an i-long derivation of
γ→β from Γ∪{α}. Similarly those lines up to and including D(j) constitute a j-long
derivation of γ from Γ∪{α}. But i, j < n, so we can apply the inductive hypothesis (which
is universally quantified over Γ,α,β) to these two derivations to obtain, respectively, (i)
Γ⊢α→(γ→β), and (j) Γ⊢α→γ. From these two facts, and Lemma dt4, we obtain
Γ⊢α→β.

## 8.    The Lemmas Supporting the Deduction Theorem

In the previous section, we prove the Deduction Theorem appealing to four lemmas – dt1 – dt4.

(dt1)    β is an axiom  →  Γ⊢α→β

(dt2)    β∈Γ  →  Γ⊢α→β

(dt3)    Γ⊢α→α

(dt4)    Γ⊢α→(γ→β)  &  Γ⊢α→γ  .→  Γ⊢α→β

We can in fact go back and routinely rewrite our proof so that it proves something more general.

GT

        Suppose axiom system 𝔸 is based on a language with '→';
        suppose that 𝔸 has exactly one multi-place rule – modus ponens.
        Further suppose that 𝔸 satisfies (dt1)-(dt4).
        Then 𝔸 satisfies the Deduction Theorem.

Not every logical system satisfies the antecedent conditions of GT. For example, the original systems of
C.I. Lewis (S1-S5) do not satisfy the deduction theorem for Lewis' proposed conditional connective.
Neither does Relevance Logic nor Quantum Logic. On the other hand, both Classical Logic and
Intuitionistic Logic satisfy the antecedent conditions of GT, and hence satisfy the Deduction Theorem.

We are interested primarily in Classical Sentential Logic, in particular AS1, so the question is whether AS1 delivers dt1-dt4. We have already explicitly proven dt3 [=T11] and dt4 [=T15]. We have not explicitly proven dt1 or dt2.

(dt1)   β is an axiom  →  $\Gamma \vdash \alpha \rightarrow \beta$

Proof: Suppose β is an axiom. Then the singleton sequence ⟨β⟩ is a proof of β, so $\vdash \beta$, so by T5, $\vdash \alpha \rightarrow \beta$, so by T10 (G2), $\Gamma \vdash \alpha \rightarrow \beta$.

(dt2)   $\beta \in \Gamma$ → $\Gamma \vdash \alpha \rightarrow \beta$

Proof: Suppose $\beta \in \Gamma$. Then by G0, $\Gamma \vdash \beta$. But by T1, $\vdash \beta \rightarrow (\alpha \rightarrow \beta)$, so by T10 (G2), $\Gamma \vdash \beta \rightarrow (\alpha \rightarrow \beta)$, so by T14, $\Gamma \vdash \alpha \rightarrow \beta$.

## 9.    Rules R1 and R2 are Required for any DT-MP-Logic

The reader will notice that, in the proof of the Deduction Theorem, no use has been made of Rule R3. We have proven the Deduction Theorem appealing to R1, R2, and R4 (a.k.a. MP). This can be described as follows.

> Th
>
>         R1, R2, and R4 are sufficient to produce the Deduction Theorem.

A natural remaining question is whether they are necessary. First of all, they are <u>not</u> necessary as *primitive* rules. The more precise question is: are they necessary as *derivative* rules? This, of course, requires an elucidation of the concept of *derivative rule*. First, an example; the following rule is a derivative rule of AS1.

(R5)   $\alpha \rightarrow \beta, \beta \rightarrow \gamma \;\rightarrowtail\; \alpha \rightarrow \gamma$

By this, we mean the following. Consider adding R5 to system AS1; call the resulting system AS1+R5. The resulting system obviously admits many more derivations than AS1. BUT (and this is the key point) these additional derivations do not produce any additional deductive entailments. In other words we have the following theorem.

> Th
>
>         $\Gamma \vdash \alpha$ (AS1+R5)  →  $\Gamma \vdash \alpha$ (AS1)

In other words, if Γ entails α according to AS1+R5, then Γ entails α according to AS1. Another way to describe this situation is to say that Rule R5 is *redundant* in system AS1+R5.

More generally, we can define *rule-admissibility* as follows.

> **Def**
>
> Let Σ be an axiom system underwritten by language L.  Let ℝ be a rule of inference over L.  Let Σ+ℝ be the system obtained by adding rule ℝ to Σ.
> Then ℝ is said to be *admissible* by Σ iff the following obtains, for any Γ and α.
>
> $$\Gamma \vdash \alpha \ (\Sigma+\mathbb{R}) \ \rightarrow \ \Gamma \vdash \alpha \ (\Sigma)$$
>
> Alternatively, we say that ℝ is a *derivative rule* of Σ.

Thus, the question whether R1, R2, and R4 are necessary for the Deduction Theorem reduces the question whether the following is true.

> **???**
>
> Suppose axiom system 𝔸 is based on a language with '→';
> further suppose that 𝔸 satisfies the Deduction Theorem.
> Then does 𝔸 admit rules R1, R2, and R4?

The answer is negative, but that is because we are asking too much.  We need to restate the problem.  First, we presume '→' is some sort of conditional (if-then) connective.  There is still plenty of leeway here, but one thing seems clear to the logical community — if a formal connective '→' does not satisfy modus ponens, then it is not a conditional connective, no matter how we choose to write it.  In other words modus ponens is a *sine qua non* for any connective that purports to be an if-then connective.

So the more useful question concerns whether a logical system that satisfies the Deduction Theorem *and* admits *modus ponens* also admits rules R1 and R2.  In this case, the answer is unequivocal.

> **Th**
>
> Suppose axiom system 𝔸 is based on a language with '→';
> suppose that 𝔸 satisfies the Deduction Theorem,
> and further suppose that 𝔸 admits *modus ponens*.
> Then 𝔸 admits rules R1, R2?

Before proving this theorem, we state a simple but useful set of lemmas, whose proofs are left as an exercise.

(s1)    𝔸 admits R1 $\leftrightarrow$ $\vdash \alpha \rightarrow (\beta \rightarrow \alpha)$ (𝔸)
(s2)    𝔸 admits R2 $\leftrightarrow$ $\vdash [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$ (𝔸)
(s3)    𝔸 admits R3 $\leftrightarrow$ $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$ (𝔸)

Proof:
Suppose $\mathbb{A}$ admits modus ponens and satisfies the Deduction Theorem.  Given Lemma (s3),
$\mathbb{A}$ satisfies the following.

(MP)   $\{\alpha, \alpha{\rightarrow}\beta\} \vdash \beta$
(DT)   $\Gamma{\cup}\{\alpha\}\vdash\beta \rightarrow \Gamma\vdash\alpha{\rightarrow}\beta$

We wish to show that $\mathbb{A}$ admits rules R1 and R2.  Given Lemmas (s2) and (s3), it is
sufficient to show that $\mathbb{A}$ satisfies the following.

(r1)    $\vdash\alpha{\rightarrow}(\beta{\rightarrow}\alpha)$
(r2)    $\vdash[\alpha{\rightarrow}(\beta{\rightarrow}\gamma)]{\rightarrow}[(\alpha{\rightarrow}\beta){\rightarrow}(\alpha{\rightarrow}\gamma)]$

r1:  The singleton sequence $\langle\alpha\rangle$ is a derivation of $\alpha$ from $\{\alpha,\beta\}$, so $\{\alpha,\beta\}\vdash\alpha$.  Now,
$\{\alpha,\beta\} = \{a\}\cup\{\beta\}$, so $\{\alpha\}\cup\{\beta\}\vdash\alpha$, so by DT, $\{\alpha\}\vdash\beta{\rightarrow}\alpha$.  Also $\{\alpha\}=\varnothing\cup\{\alpha\}$, so by
DT, $\varnothing\vdash\alpha{\rightarrow}(\beta{\rightarrow}\alpha)$, so by G3, $\vdash\alpha{\rightarrow}(\beta{\rightarrow}\alpha)$.

r2:  Consider arbitrary $\alpha,\beta,\gamma$.  Consider the following *quasi*-derivation.

| | | |
|---|---|---|
| (1) | $\alpha{\rightarrow}(\beta{\rightarrow}\gamma)$ | Pr |
| (2) | $\alpha{\rightarrow}\beta$ | Pr |
| (3) | $\alpha$ | Pr |
| (4) | $\beta{\rightarrow}\gamma$ | 1,3,MP |
| (5) | $\beta$ | 2,3,MP |
| (6) | $\gamma$ | 4,5,MP |

If MP is a primitive rule of $\mathbb{A}$, then the above sequence is already a genuine derivation in
$\mathbb{A}$.  If MP is not a primitive rule of $\mathbb{A}$, since $\mathbb{A}$ admits MP as a rule, there is a genuine
derivation that can be substituted for the above quasi-derivation.  In either case, there is a
derivation in $\mathbb{A}$ of $\gamma$ from $\{\alpha{\rightarrow}(\beta{\rightarrow}\gamma), \alpha{\rightarrow}\beta, \alpha\}$, so $\{\alpha{\rightarrow}(\beta{\rightarrow}\gamma), \alpha{\rightarrow}\beta, \alpha\}\vdash\gamma$.  Taking
this result, and applying DT three times, plus G3, we get
$\vdash[\alpha{\rightarrow}(\beta{\rightarrow}\gamma)]{\rightarrow}[(\alpha{\rightarrow}\beta){\rightarrow}(\alpha{\rightarrow}\gamma)]$.

## 10.    The Converse of the Deduction Theorem and Modus Ponens

        In the remainder of the present chapter, we prove several more theorems about AS1.  A complete
list of theorems proven up to this point is included at the end.  Many of these theorems are later used in the
proof of the Soundness and Completeness Theorems for AS1.

        Some of the proofs appeal to the Deduction Theorem, which is abbreviated by 'DT'.  The
expression '(c)' means 'corollary'; a corollary to a theorem T is generally a fairly immediate consequence
of T, and usually a special case of T.  As with lemmas, the distinction between theorems and corollaries is
purely pragmatic.

        We start with a proof of the converse of DT, followed by an immediate corollary.

(T17)  $\Gamma\vdash\alpha{\rightarrow}\beta \rightarrow \Gamma{\cup}\{\alpha\}\vdash\beta$

Proof:  Suppose $\Gamma\vdash\alpha{\rightarrow}\beta$, to show $\Gamma{\cup}\{\alpha\}\vdash\beta$.  Given the hyp, there is a derivation of
$\alpha{\rightarrow}\beta$ from $\Gamma$.  This is automatically also a derivation of $\alpha{\rightarrow}\beta$ from $\Gamma{\cup}\{\alpha\}$, so we have
$\Gamma{\cup}\{\alpha\}\vdash\alpha{\rightarrow}\beta$.  Also, $\alpha{\in}\Gamma{\cup}\{\alpha\}$, so by G0, $\Gamma{\cup}\{\alpha\}\vdash\alpha$.  So by T14, $\Gamma{\cup}\{\alpha\}\vdash\beta$.

(c)      ⊢α→β  →  α⊢β

Proof: Suppose ⊢α→β, to show {α}⊢β. By hyp, and T10 (G2), ∅⊢α→β, so by T17, ∅∪{α}⊢β, but ∅∪{α}={α}, so we have {α}⊢β, and hence α⊢β.

The following theorem is a general theorem about deductive systems.

> Th
>
> Let 𝔸 be an axiom system with a two-place connective →. Then 𝔸 satisfies the Converse Deduction Theorem if and only if 𝔸 admits modus ponens as a valid rule.

Proof:
[→] Suppose 𝔸 satisfies CDT.  Now, {α→β}⊢α→β, so by CDT {α→β}∪{α}⊢β, so by set theory, {α→β, α}⊢β, so MP is valid in 𝔸, so by Lemma (s3) 𝔸 admits MP.
[←] Suppose 𝔸 admits MP.  Suppose Γ⊢α→β, to show Γ∪{α}⊢β.  Then there is a derivation of α→β from Γ, call it D,  Consider the sequence D+⟨α,β⟩; claim: this sequence is a derivation of β from Γ∪{α}.  Its last line is β, so it is a derivation of β; the question is whether it is a derivation from Γ∪{α}; this amounts to the question whether every line is an axiom, a premise, or follows by MP.  Consider an arbitrary line, L; there are three cases: c1: L is in D; D is a derivation from Γ, so it is automatically a derivation from Γ∪{α}; c2: L=α; α is a premise, since α∈Γ∪{α}; c3: L=β; by hypothesis, D is a derivation of α→β, so the last line of D is α→β; thus, L (a.k.a. β) follows from previous lines by MP.  If MP is a primitive rule, then D+⟨α,β⟩ is a derivation of β from Γ∪{α}.  If MP is a derivative rule, then D+⟨α,β⟩ can be converted into a derivation of β from Γ∪{α}.

## 11.    Some General Theorems About ⊢

Next, we prove some theorems that hold of the deducibility relation ⊢, irrespective of the particular axiom system.  These have already been formally proven in Chapter 6.  We offer informal proofs here.

(T18)  Γ⊆Δ  →. Γ⊢α  →  Δ⊢α                                                    [a.k.a. G1]

Proof: Suppose Γ⊆Δ and Γ⊢α, to show Δ⊢α.  Then there is a derivation of α from Γ, call it D.  Since D is a derivation of α from Γ, and since Γ⊆Δ (by hyp), D is also a derivation of α from Δ.

(T19)  Γ⊢α  &  Γ∪{α}⊢β .→  Γ⊢β                                                [a.k.a. G6]

Proof: Suppose Γ⊢α  &  Γ∪{α}⊢β, to show: Γ⊢β.  Given As1, there is a derivation of α from Γ, call it D1.  Given As2, there is a derivation of β from Γ∪{α}, call it D2.  Take D2, and everywhere α occurs, replace it by D1; call the resulting sequence D3.  Claim: D3 is a derivation of β from Γ.  Clearly, D3 is a derivation of β, so the question is whether every line derives from Γ, which is to say that every line is an element of Γ or follows by a rule.  Consider line L, there are two cases: c1: L occurs in a copy of D1; in this case, L derives from Γ, since D1 is a derivation from Γ; c2: L does not occur in a copy of D1.

Then, L cannot be α, since every occurrence of α in D2 was replaced by D1. So it must be a line from the original derivation D2. By hypothesis, D2 is a derivation from Γ∪{α}, so every line of D2 is an element of Γ∪{α} or follows by a rule. That L=α has already been ruled out, so L must either be an element of Γ or follows by a rule, which is to say that L derives from Γ.

(c1)     Γ⊢α  &  α⊢β .→ Γ⊢β

Proof: Suppose Γ⊢α  and  α⊢β [=$_{df}$ {α}⊢β], to show Γ⊢β. By As2, since {α}⊆Γ∪{α}, we have Γ∪{α}⊢β; the latter, together with As1 and T19 (G6), yields Γ⊢β.

(c2)     {α,β}⊢γ  &  {α,γ}⊢δ .→ {α,β}⊢δ

Proof: Suppose {α,β}⊢γ and {α,γ}⊢δ, to show: {α,β}⊢δ. Since {α,γ} ⊆ {α,β,γ}, by As2 and T18 (G1), {α,β,γ}⊢δ. So {α,β}∪{γ}⊢δ, since {α,β,γ}={α,β}∪{γ}. So, combining this with As1 and T19 (G6), we obtain {α,β}⊢δ.

## 12.    **Further Theorems About AS1**

In the present section, we prove a rather large number of theorems about AS1. We start by proving that the laws of double negation are valid in AS1 (which, of course, is mandatory, if AS1 is adequate for the job it is intended).

(T20)  ⊢~~α→α

Proof:  By T8, ⊢~~α→(~α→~~~α), and by T3, ⊢(~α→~~~α)→(~~α→α), so by T6, ⊢~~α→(~~α→α), but by T9, ⊢[~~α→(~~α→α)]→(~~α→α), so by T14, ⊢~~α→α.

(c)        ~~α⊢α

Proof: Immediate from T20 and T17(c).

(T21)  ⊢α→~~α

Proof:  By T20, ⊢~~~α→~α, and by T3, ⊢(~~~α→~α)→(α→~~α), so by T6, ⊢α→~~α.

(c)       α⊢~~α

Proof: Immediate from T21 and T17(c).

(T22)  ⊢(α→β)→(~~α→~~β)

Proof: {α,α→β}⊢β (since MP is a rule), and {~~α} ⊢ α, by T20, so by T19 (G6), {~~α, α→β} ⊢ β; but {β}⊢~~β, so by T19(c1), {~~α, α→β} ⊢ ~~β, so by DT (twice), ⊢(α→β)→(~~α→~~β).

(T23)  ⊢(α→β)→(~β→~α)

Proof: By T22, ⊢(α→β)→(~~α→~~β), and by T3, ⊢(~~α→~~β)→(~β→~α), so by T6, ⊢(α→β)→(~β→~α).

(T24)  ⊢α→((α→β)→β)

Proof: {α, α→β}⊢β (since MP is a rule), so by DT (twice), we have ⊢α→((α→β)→β).

(T25)  ⊢α→(~β→~(α→β))

Proof: By T24, ⊢α→((α→β)→β), and by T23, ⊢((α→β)→β)→(~β→~(α→β)), so by T6, ⊢α→(~β→~(α→β)).

(T26)  ⊢(~α→α)→(β→α)

Proof:  By T8, ⊢~α→(α→~β); by T2, ⊢[~α→(α→~β)]→[(~α→α)→(~α→~β)], so by T6, ⊢[(~α→α)→(~α→~β)]. By T3, ⊢(~α→~β)→(β→α), so by T6, ⊢[(~α→α)→(β→α).

(T27)  ⊢(~α→α)→α

Proof: By T26, ⊢(~α→α)→((~α→α)→α).  By T2, ⊢[(~α→α)→((~α→α)→α)]→{[(~α→α)→(~α→α)]→[(~α→α)→α]}, so by T6, ⊢[(~α→α)→(~α→α)]→[(~α→α)→α].  But by T7, ⊢(~α→α)→(~α→α), so by T6, ⊢(~α→α)→α.

(T28)  ⊢(~β→~α)→[(~α→β)→(~β→β)]

Proof: The sequence (~β→~α, ~α→β, ~β, ~α, β( is a derivation of β from {~β→~α, ~α→β, ~β}, so {~β→~α, ~α→β, ~β} ⊢ β.  Therefore, applying DT three times, we have ⊢(~β→~α)→[(~α→β)→(~β→β)]

(T29)  Γ⊢α→(β→γ)  &  Γ⊢γ→δ .→  Γ⊢α→(β→δ)

Proof: Suppose Γ⊢α→(β→γ) and Γ⊢γ→δ, to show: Γ⊢α→(β→δ).  By As1 and T17 (twice) Γ∪{α}∪{β}⊢γ; by As2 and T17, Γ∪{γ}⊢δ, so by T18 (G1), Γ∪{α}∪{β}∪{γ}⊢δ, so by T19 (G6), Γ∪{α}∪{β}⊢δ, so by DT (twice), Γ⊢α→(β→δ).

(c)     ⊢α→(β→γ)  &  ⊢γ→δ .→  ⊢α→(β→δ)

Proof: this is obtained from T29 by substituting '∅' for 'Γ', then noting that ⊢α iff ∅⊢α, for any α.

(T30)  ⊢(α→β)→[(~α→β)→β]

Proof: By T23, ⊢(α→β)→(~β→~α); by T28, ⊢(~β→~α)→[(~α→β)→(~β→β)], so by T6, ⊢(α→β)→[(~α→β)→(~β→β)].  Also, by T8, ⊢(~β→β)→β, so by T29(c), ⊢(α→β)→[(~α→β)→β].

(T31)  {α→β, β→γ, α} ⊢ γ

Proof:  For any formulas α,β,γ, the sequence ⟨α→β, β→γ, α, β, γ⟩ is a derivation of γ from {α→β, β→γ, α}.  Thus, {α→β, β→γ, α}⊢γ

(c1)    {α→β, β→γ} ⊢ α→γ.
(c2)    {α→β} ⊢ (β→γ)→(α→γ).
(c3)    ⊢(α→β)→[(β→γ)→(α→γ)].

Proof: these are obtained by application of DT, once, twice, thrice, respectively.

(T32)  ⊢(α→∼β)→(β→∼α)

Proof: By T31(c3), ⊢(∼∼α→α)→[(α→∼β)→(∼∼α→∼β)].  By T20, ⊢∼∼α→α, so by T6, ⊢(α→∼β)→(∼∼α→∼β).  By T3, ⊢(∼∼α→∼β)→(β→∼α), so by T6, ⊢(α→∼β)→(β→∼α).

(T33)  ⊢(α→∼α)→∼α

Proof: By T23, ⊢(α→∼α)→(∼∼α→∼α); by T27, ⊢(∼∼α→∼α)→∼α, so by T6, ⊢(α→∼α)→∼α.

(T34)  {α→β, β→∼α} ⊢ ∼α

Proof: By T33, ⊢(α→∼α)→∼α, so by T17(c), {α→∼α}⊢∼α. Also, by T31(c1), {α→β, β→∼α} ⊢ α→∼α, so by T19(c1), {α→β, β→∼α} ⊢ ∼α.

(T35)  {α→β, α→∼β} ⊢ ∼α

Proof: By T32, ⊢(α→∼β)→(β→∼α), so by T17(c), {α→∼β} ⊢ β→∼α, so by T18 (G1), {α→β, α→∼β} ⊢ β→∼α.  Also, by T34, {α→β, β→∼α} ⊢ ∼α, so by T19(c2), {α→β, α→∼β} ⊢ ∼α.

(c)      ⊢[α→β]→[(α→∼β)→∼α]

Proof: Apply DT twice to T35.

(T36)  {∼α→β, ∼α→∼β} ⊢ α

Proof: By T35, {∼α→β, ∼α→∼β} ⊢ ∼∼α, and by T20(c), ∼∼α ⊢ α, so by T19(c1), {∼α→β, ∼α→∼β} ⊢ α.

(c)      ⊢[∼α→β]→[(∼α→∼β)→α]

Proof: Apply DT twice to T36.

## 13.    Appendix: Summary of Theorems about AS1

As usual, each theorem is universally quantified over the relevant variables, sorted in the usual way.

(T1)      ⊢ α→(β→α)

(T2)      ⊢ [α→(β→γ)]→[(α→β)→(α→γ)]

(T3)      ⊢ (∼α→∼β)→(β→α)

(T4)      ⊢α & ⊢α→β .→ ⊢β

(T5)      ⊢α  →  ⊢β→α

(T6)      ⊢α→β & ⊢β→γ .→ ⊢α→γ

(T7)      ⊢ α→α

(T8)      ⊢ ~α→(α→β)

(T9)      ⊢ [α→(α→β)]→(α→β)

(T10)     ⊢α → Γ⊢α                         [a.k.a. G2]

(T11)     Γ⊢α→α

(T12)     α∈Γ → Γ⊢α                        [a.k.a. G0]

(T13)     Γ⊢α→(β→α)

(T14)     Γ⊢α & Γ⊢α→β .→ Γ⊢β

(T15)     Γ⊢α→(β→γ) & Γ⊢α→β .→ Γ⊢α→γ

(T16)     Γ∪{α}⊢β → Γ⊢α→β                  [a.k.a. DT]

(c)       {α}⊢β → ⊢α→β

(T17)     Γ⊢α→β → Γ∪{α}⊢β                  [converse deduction theorem]

(c)       ⊢α→β → {α}⊢β

(T18)     Γ⊆Δ →. Γ⊢α → Δ⊢α                 [a.k.a. G1]

(T19)     Γ⊢α & Γ∪{α}⊢β .→ Γ⊢β            [a.k.a. G6]

(c1)      Γ⊢α & {α}⊢β .→ Γ⊢β

(c2)      {α,β}⊢γ & {α,γ}⊢δ .→ {α,β}⊢δ

(T20)     ⊢~~α→α

(c)       {~~α} ⊢ α

(T21)     ⊢α→~~α

(c)       {α} ⊢ ~~α

(T22)     ⊢(α→β)→(~~α→~~β)

(T23)     ⊢(α→β)→(~β→~α)

(T24)     ⊢α→((α→β)→β)

(T25)     ⊢α→(~β→~(α→β))

(T26)     ⊢(~α→α)→(β→α)

(T27)     ⊢(~α→α)→α

(T28)     ⊢(~β→~α)→[(~α→β)→(~β→β)]

(T29)     Γ⊢α→(β→γ) & Γ⊢γ→δ .→ Γ⊢α→(β→δ)

(c)       ⊢α→(β→γ) & ⊢γ→δ .→ ⊢α→(β→δ)

(T30)     ⊢(α→β)→[(~α→β)→β]

(T31)     {α→β, β→γ, α} ⊢ γ

(c1)      {α→β, β→γ} ⊢ α→γ

(c2)      {α→β} ⊢ (β→γ)→(α→γ)

(c3)        $\vdash(\alpha\rightarrow\beta)\rightarrow[(\beta\rightarrow\gamma)\rightarrow(\alpha\rightarrow\gamma)]$

(T32)      $\vdash(\alpha\rightarrow\sim\beta)\rightarrow(\beta\rightarrow\sim\alpha)$

(T33)      $\vdash(\alpha\rightarrow\sim\alpha)\rightarrow\sim\alpha$

(T34)      $\{\alpha\rightarrow\beta,\ \beta\rightarrow\sim\alpha\}\vdash\ \sim\alpha$

(T35)      $\{\alpha\rightarrow\beta,\ \alpha\rightarrow\sim\beta\}\vdash\ \sim\alpha$

(c)         $\vdash(\alpha\rightarrow\beta)\rightarrow[(\alpha\rightarrow\sim\beta)\rightarrow\sim\alpha]$

(T36)      $\{\sim\alpha\rightarrow\beta,\ \sim\alpha\rightarrow\sim\beta\}\vdash\alpha$

(c)         $\vdash(\sim\alpha\rightarrow\beta)\rightarrow[(\sim\alpha\rightarrow\sim\beta)\rightarrow\alpha]$