

# 8

## Axiom Systems For Classical Sentential Logic

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## 1. Introduction

In the current chapter, we exhibit several deductive (axiom) systems for classical sentential logic (CSL). One of these systems – AS1 – will play a prominent role in subsequent chapters on soundness and completeness.

The choice of AS1 is largely arbitrary. Ultimately, what we want to show is that

*there is a deductive system that adequately characterizes CSL.*

One way to prove this is by demonstrating that a *particular* deductive system adequately characterizes CSL, from which we obtain the desired result simply by existential-introduction! We happen to choose AS1 for this task. If we were so inclined, we could go back and prove the same results for many other axiom systems, including those listed below [and the inquisitive reader is urged to do just that!]

Basically, in writing down a deductive (axiom) system for a logic, the usual goal is to write down a reasonably compact set rules that generate all and only the valid arguments of that logic. In this connection, it is customary to write the system in minimal terms. Specifically, rather than write down rules for all connectives, one writes down rules for a restricted subset of “primitive” connectives. This is acceptable insofar as, and only insofar as, the chosen subset is *expressively complete* (recall Chapter 4). The remaining connectives are then introduced by way of definitions.

## 2. Axiom System AS1

Axiom system AS1 is written exclusively in terms of just ‘ $\sim$ ’ and ‘ $\rightarrow$ ’. The remaining connectives are introduced by way of definitions, which are given as follows.

### AS1 Definitions:

(d1)	$\alpha \vee \beta$	$=_{df}$	$\sim \alpha \rightarrow \beta$
(d2)	$\alpha \& \beta$	$=_{df}$	$\sim (\alpha \rightarrow \sim \beta)$
(d3)	$\alpha \leftrightarrow \beta$	$=_{df}$	$\sim [(\alpha \rightarrow \beta) \rightarrow \sim (\beta \rightarrow \alpha)]$
(d4)	$\times$	$=_{df}$	$\sim (\alpha \rightarrow \alpha)$

Note that ‘ $\times$ ’ may be regarded as a sentential constant (zero-place connective). See Section 4, for examples of axiom systems that utilize it as a primitive.

In addition to definitions (d1)-(d4), axiom system AS1 prescribes four inference rules, given as follows.

**AS1 Rules:**

- (R1)  $\hookrightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$   
 (R2)  $\hookrightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$   
 (R3)  $\hookrightarrow (\sim \alpha \rightarrow \sim \beta) \rightarrow (\beta \rightarrow \alpha)$   
 (R4)  $\alpha, \alpha \rightarrow \beta \hookrightarrow \beta$

For the sake of brevity, we write the rules horizontally, using the curvy arrow symbol ( $\hookrightarrow$ ) to divide the input (which *can* be null) from the output (which *cannot* be null). As usual, the lower case Greek letters are metalinguistic variables ranging over formulas.

Notice that (R4) is simply *modus ponens* – from  $\alpha$  and  $\alpha \rightarrow \beta$ , one is entitled to infer  $\beta$ . Also, notice that (R1)–(R3) have no input; they are zero-place rules. [In elementary logic, the only zero-place rule is the rule of reflexivity of ‘=’]. In a zero-place rule, given *nothing*, one can write down the output. An alternative, more traditional-sounding, description is to say that AS1 involves three *axiom schemata* – (R1)–(R3) – plus the rule of *modus ponens*.

### 3. Examples of Derivations in System AS1

Recall the definitions of ‘derivation’ and ‘proof’, which may be stated as follows.

Def

Let  $\Sigma$  be a deductive system with formulas  $S$  and rules  $\mathcal{R}$ .

A *derivation of  $\alpha$  from  $\Gamma$  in  $\Sigma$*  is, by definition, a finite sequence of formulas of  $\Sigma$ , the last one of which is  $\alpha$ , and every line of which is either an element of  $\Gamma$  or follows from previous lines by a rule of  $\Sigma$ .

A *proof of  $\alpha$  in  $\Sigma$*  is a finite sequence of formulas of  $\Sigma$ , the last one of which is  $\alpha$ , and every line of which follows from previous lines by a rule of  $\Sigma$ .

Notice, given these definitions, that every proof is automatically a derivation – specifically, a derivation from the empty set.

The following are examples of derivations and proofs in AS1. We begin with a proof of ‘ $P \rightarrow P$ ’. In elementary logic, the proof of ‘ $P \rightarrow P$ ’ is incredibly easy, since elementary logic employs the method of conditional derivation. No such method is included in system AS1, so the proof of ‘ $P \rightarrow P$ ’ in AS1 is not nearly so trivial.

#### Example 1:

- |     |   |        |
|-----|---|--------|
| (1) | $P \rightarrow (P \rightarrow P \rightarrow P)$   | R1     |
| (2) | $P \rightarrow (P \rightarrow P \rightarrow P) \rightarrow (P \rightarrow P \rightarrow P) \rightarrow (P \rightarrow P)$ | R2     |
| (3) | $(P \rightarrow P \rightarrow P) \rightarrow (P \rightarrow P)$   | 1,2,R4 |

(4)	$(P \rightarrow . P \rightarrow P)$	R1
(5)	$P \rightarrow P$	3,4,R4

Note that, in order to reduce visual clutter, some parentheses are replaced by dots, according to the usual custom.

Notice also in the above display that there are three columns.

- (1) the column of line numbers
- (2) the column of formulas
- (3) the column of annotations.

Strictly speaking, only the column of formulas constitute the *derivation proper*. [Recall the official definition of derivation.] On the other hand, the two flanking columns are not strictly speaking part of the derivation. Rather, they constitute the surrounding informal metalinguistic argument to the conclusion that the given sequence – i.e., the column of formulas – is in fact a derivation in the system.

The above proof appeals to all the rules except R3. The following example – which proves ‘ $\sim P \rightarrow (P \rightarrow Q)$ ’ – utilizes R3.

### Example 2:

(1)	$\sim Q \rightarrow \sim P . \rightarrow . P \rightarrow Q$	R3
(2)	$(\sim Q \rightarrow \sim P . \rightarrow . P \rightarrow Q) \rightarrow . \sim P \rightarrow (\sim Q \rightarrow \sim P . \rightarrow . P \rightarrow Q)$	R1
(3)	$\sim P \rightarrow (\sim Q \rightarrow \sim P . \rightarrow . P \rightarrow Q)$	1,2,R4
(4)	$\sim P \rightarrow (\sim Q \rightarrow \sim P . \rightarrow . P \rightarrow Q) . \rightarrow . (\sim P \rightarrow . \sim Q \rightarrow \sim P) \rightarrow (\sim P \rightarrow . P \rightarrow Q)$	R2
(5)	$(\sim P \rightarrow . \sim Q \rightarrow \sim P) \rightarrow (\sim P \rightarrow . P \rightarrow Q)$	3,4,R4
(6)	$(\sim P \rightarrow . \sim Q \rightarrow \sim P)$	R1
(7)	$\sim P \rightarrow . P \rightarrow Q$	5,6,R4

Examples 1 and 2 are both proofs [derivations with no premises]. The following is a derivation with a single premise; in particular, it is a derivation of ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’ from ‘ $P \rightarrow P$ ’.

### Example 3:

(1)	$P \rightarrow (P \rightarrow Q) . \rightarrow . (P \rightarrow P) \rightarrow (P \rightarrow Q)$	R2
(2)	$\{ (P \rightarrow (P \rightarrow Q) . \rightarrow . (P \rightarrow P) \rightarrow (P \rightarrow Q)) \} \rightarrow$ $\{ [P \rightarrow (P \rightarrow Q) . \rightarrow . P \rightarrow P] \rightarrow [P \rightarrow (P \rightarrow Q) . \rightarrow . P \rightarrow Q] \}$	R2
(3)	$[P \rightarrow (P \rightarrow Q) . \rightarrow . P \rightarrow P] \rightarrow [P \rightarrow (P \rightarrow Q) . \rightarrow . P \rightarrow Q]$	1,2,R4
(4)	$(P \rightarrow P) \rightarrow [P \rightarrow (P \rightarrow Q) . \rightarrow . (P \rightarrow P)]$	R1
(5)	$P \rightarrow P$	Pr
(6)	$[P \rightarrow (P \rightarrow Q) . \rightarrow . P \rightarrow P]$	4,5,R4
(7)	$P \rightarrow (P \rightarrow Q) . \rightarrow . (P \rightarrow Q)$	3,6,R4

Notice that Example 3 *ostensively* ( $\neq$ ostensibly!) demonstrates that ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’ is derivable from ‘ $P \rightarrow P$ ’. This result can be summarized by the following metalanguage statement.

$$(m1) \quad P \rightarrow P \vdash (P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$$

Similarly, Example 1 *ostensively* demonstrates that ‘ $P \rightarrow P$ ’ is provable, which can be summarized as follows.

$$(m2) \quad \vdash P \rightarrow P$$

Recall that, in Chapter 6, we proved the following *general* theorem about deduction.

$$(G6) \quad \Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \beta \ . \rightarrow \Gamma \vdash \beta$$

The following is an immediate corollary.

$$(c) \quad \vdash \alpha \ \& \ \alpha \vdash \beta \ . \rightarrow \vdash \beta$$

In other words, if  $\alpha$  is provable, and  $\beta$  can be derived from  $\alpha$ , then  $\beta$  is provable [where, as usual, it is tacitly understood that ‘ $\alpha$ ’ and ‘ $\beta$ ’ are universally quantified]. Combining (c) with (m1) and (m2), we obtain the following result.

$$(m3) \quad \vdash (P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$$

This, of course, says that ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’ is provable.

Notice carefully that, although we have shown that ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’ is provable in system AS1, we have not presented an *actual* proof of ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’ in system AS1. Rather, we have demonstrated only that a *proof exists*.

However, recall that in proving (G6), we provide a general procedure for taking two derivations — one of  $\alpha$  from  $\Gamma$ , and one of  $\beta$  from  $\Gamma \cup \{\alpha\}$  — and constructing a derivation of  $\beta$  from  $\Gamma$ , thereby proving that such a derivation exists. How does this general procedure work for derivations 1 and 3? Quite simply, one takes derivation-3 — which derives ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’ from ‘ $P \rightarrow P$ ’ — and one takes derivation-1 — which proves ‘ $P \rightarrow P$ ’ — and interleaves them in the prescribed manner. Specifically, everywhere ‘ $P \rightarrow P$ ’ occurs in derivation-3, one inserts derivation-1. The following is the result.

(1)	$P \rightarrow (P \rightarrow Q) \ . \rightarrow . (P \rightarrow P) \rightarrow (P \rightarrow Q)$	R2
(2)	$\{ (P \rightarrow (P \rightarrow Q) \ . \rightarrow . (P \rightarrow P) \rightarrow (P \rightarrow Q)) \} \rightarrow$ $\{ [P \rightarrow (P \rightarrow Q) \ . \rightarrow . P \rightarrow P] \rightarrow [P \rightarrow (P \rightarrow Q) \ . \rightarrow . P \rightarrow Q] \}$	R2
(3)	$[P \rightarrow (P \rightarrow Q) \ . \rightarrow . P \rightarrow P] \rightarrow [P \rightarrow (P \rightarrow Q) \ . \rightarrow . P \rightarrow Q]$	1,2,R4
(4)	$(P \rightarrow P) \rightarrow [P \rightarrow (P \rightarrow Q) \ . \rightarrow . (P \rightarrow P)]$	R1
(5.1)	$P \rightarrow (P \rightarrow P \rightarrow P)$	R1
(5.2)	$P \rightarrow (P \rightarrow P \rightarrow P) \ . \rightarrow . (P \rightarrow . P \rightarrow P) \rightarrow (P \rightarrow P)$	R2
(5.3)	$(P \rightarrow . P \rightarrow P) \rightarrow (P \rightarrow P)$	1,2,R4
(5.4)	$(P \rightarrow . P \rightarrow P)$	R1
(5.5)	$P \rightarrow P$	3,4,R4
(6)	$[P \rightarrow (P \rightarrow Q) \ . \rightarrow . P \rightarrow P]$	4,5.5,R4
(7)	$P \rightarrow (P \rightarrow Q) \ . \rightarrow . (P \rightarrow Q)$	3,6,R4

As you can readily see, the sequence is (as promised!) a proof of ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’!

Although the above sequence is indeed a proof of ‘ $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ ’, it by no means the only proof of this formula. The following is a somewhat shorter proof.

(1)	$P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow P) \rightarrow (P \rightarrow Q)$	R2
(2)	$P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow P) \rightarrow (P \rightarrow Q) : \rightarrow :$ $P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow P) \rightarrow (P \rightarrow Q)$	R2
(3)	$P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow P) \rightarrow (P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow Q))$	1,2,R4
(4)	$P \rightarrow (P \rightarrow Q \rightarrow P) \rightarrow (P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow P))$	R2
(5)	$P \rightarrow (P \rightarrow Q \rightarrow P)$	R1
(6)	$P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow P)$	4,5,R4
(7)	$P \rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow Q)$	3,6,R4

#### 4. Other Axiom Systems for CSL

In the present section, we present five other axiom systems for CSL. The first two are based on the same connectives and connective definitions as AS1. The next two are based on ' $\times$ ' and ' $\rightarrow$ '. The final one is based on ' $\rightarrow$ ', ' $\sim$ ', '&', and ' $\vee$ '. Notice that the only non-axiomatic rule is modus ponens.

##### AS2:

- (R1)  $\hookrightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2)  $\hookrightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3)  $\hookrightarrow (\sim \alpha \rightarrow \sim \beta) \rightarrow ((\sim \alpha \rightarrow \beta) \rightarrow \beta)$
- (R4)  $\alpha, \alpha \rightarrow \beta \hookrightarrow \beta$

##### AS3:

- (R1)  $\hookrightarrow (\sim \alpha \rightarrow \alpha) \rightarrow \alpha$
- (R2)  $\hookrightarrow \alpha \rightarrow (\sim \alpha \rightarrow \beta)$
- (R3)  $\hookrightarrow (\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)]$
- (R4)  $\alpha, \alpha \rightarrow \beta \hookrightarrow \beta$

##### AS4:

- (R1)  $\hookrightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2)  $\hookrightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3)  $\hookrightarrow [(\alpha \rightarrow \times) \rightarrow \times] \rightarrow \alpha$
- (R4)  $\alpha, \alpha \rightarrow \beta \hookrightarrow \beta$
- (d)  $\sim \alpha =_{df} \alpha \rightarrow \times$

##### AS5:

- (R1)  $\hookrightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2)  $\hookrightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3)  $\hookrightarrow [(\alpha \rightarrow \beta) \rightarrow \alpha] \rightarrow \alpha$
- (R4)  $\hookrightarrow \times \rightarrow \alpha$
- (R5)  $\alpha, \alpha \rightarrow \beta \hookrightarrow \beta$
- (d)  $\sim \alpha =_{df} \alpha \rightarrow \times$

**AS6:**

- (R1)  $\hookrightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$
- (R2)  $\hookrightarrow [\alpha \rightarrow (\beta \rightarrow \gamma)] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)]$
- (R3)  $\hookrightarrow (\alpha \& \beta) \rightarrow \alpha$
- (R4)  $\hookrightarrow (\alpha \& \beta) \rightarrow \beta$
- (R5)  $\hookrightarrow \alpha \rightarrow [\beta \rightarrow (\alpha \& \beta)]$
- (R6)  $\hookrightarrow \alpha \rightarrow (\alpha \vee \beta)$
- (R7)  $\hookrightarrow \beta \rightarrow (\alpha \vee \beta)$
- (R8)  $\hookrightarrow (\alpha \rightarrow \gamma) \rightarrow [(\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma)]$
- (R9)  $\hookrightarrow (\alpha \rightarrow \beta) \rightarrow [(\alpha \rightarrow \sim \beta) \rightarrow \sim \alpha]$
- (R10)  $\hookrightarrow \sim \sim \alpha \rightarrow \alpha$
- (R11)  $\alpha, \alpha \rightarrow \beta \hookrightarrow \beta$

(d)  $\alpha \leftrightarrow \beta \text{ =}_{\text{df}} (\alpha \rightarrow \beta) \& (\beta \rightarrow \alpha)$