7 Abstract Logics

1. Single-Conclusion Logics

Logics can be characterized semantically (model theoretically) and deductively (axiomatically, proof theoretically).

In the former case, one defines validity in terms of truth; an argument is *semantically valid* iff it is impossible for the conclusion to be false while the premises are true. Possibility is explicated in terms of admissible valuations; specifically, an argument Γ/α is invalid if there is at least one admissible valuation v such that $v(\Gamma)=T$ and $v(\alpha)=F$.

In the latter case, one defines validity in terms of deducibility; an argument is *deductively valid* iff it is possible to deduce the conclusion from the premises using the rules of the formal system in question.

Each method characterizes a logic. The object characterized may also be characterized abstractly, simply as an ordered pair $(S,\Vdash)$, where S is a non-empty set and $\Vdash$ is a relation from $\mathcal{P}(S)$ to S; the former are the sentences; the latter is the entailment relation. Of course, not just any pair $(S,\Vdash)$ of this sort counts as a logic; logical entailment has various properties, which every abstract logic should satisfy.

The abstractly characterizable properties of all logics are made specific in the following definition of an SC-logic. 'SC' stands for 'single conclusion'; besides abstract logics of this category (multiple premises, single conclusion), there are other categories as well, which we examine later.

D1

A *single-conclusion logic* (SC-logic) is, by definition, a pair $(S,\Vdash)$, where S is a non-empty set, and $\Vdash$ is a relation from $\wp(S)$ to S, satisfying the following restrictions.

(r1) $\alpha\in\Gamma \;\rightarrow\; \Gamma\Vdash\alpha$

(r2) $\Gamma\subseteq\Delta \;\rightarrow.\; \Gamma\vdash\alpha \;\rightarrow\; \Delta\vdash\alpha$

(r3) $\Gamma\Vdash\alpha \;\&\; \Gamma\cup\{\alpha\}\Vdash\beta \;.\!\rightarrow\; \Gamma\Vdash\beta$

Basically, condition (r1) says that a set entails each of its elements. (r2) says that if a set entails a formula, then every superset of that set does as well; in other words, adding extra premises to a valid argument doesn't make it invalid; this is known as *monotonicity*. Finally, condition (r3) is a generalization of transitivity. One can prove (exercise) that the associated binary entailment relation is transitive, which is to say that every SC-logic satisfies the following.

(t) $\alpha\Vdash\beta \;\&\; \beta\Vdash\gamma \;.\!\rightarrow\; \alpha\Vdash\gamma$

Here, '$\alpha\Vdash\beta$' is short for the official '$\{\alpha\}\Vdash\beta$'.

2. Consequence Operators

Every entailment relation gives rise to a corresponding function, called a *consequence operator*. First of all, every relation R from A to B yields a corresponding function f_R from A to $\wp(B)$, implicitly defined as follows.

(d1) $f_R(x) = \{y: xRy\}$

Similarly, every function f from A to $\wp$(B) yields a corresponding relation from A to B, implicitly defined as follows.

(d2) $x\ R_f\ y$ iff: $y \in f(x)$

Now, the entailment relation $\vdash$ described in Part 1 is a relation from $\wp$(S) to S, so there is an associated function from $\wp$(S) to $\wp$(S), denoted $\mathcal{C}$ (short for 'consequence'), defined as follows.

(d3) $\mathcal{C}(\Gamma) = \{\alpha: \Gamma \Vdash \alpha\}$

In other words, $\mathcal{C}(\Gamma)$ (read "the consequences of Γ") is the set of formulas entailed by the set Γ.

Consequence operators can also be abstractly defined, as follows.

D2

Let S be any non-empty set, and let $\mathcal{C}$ be a function from $\wp$(S) into $\wp$(S). Then $\mathcal{C}$ is said to be a consequence operator on S iff it satisfies the following conditions.

(c1) $\Gamma \subseteq \mathcal{C}(\Gamma)$
(c2) if $\Gamma \subseteq \Delta$, then $\mathcal{C}(\Gamma) \subseteq \mathcal{C}(\Delta)$
(c3) $\mathcal{C}(\mathcal{C}(\Gamma)) \subseteq \mathcal{C}(\Gamma)$

Now, one can prove (exercise) that every SC-logic (S,$\vdash$) gives rise to an associated consequence operator $\mathcal{C}$ on S, defined by (d3). Similarly, one can prove (exercise) that every consequence operator $\mathcal{C}$ on S gives rise to an SC-logic (S,$\Vdash$), where $\Vdash$ is implicitly defined as follows.

[[[This has to be rewritten, since finitary-cut does not imply infinitary-cut, which is required for $\Vdash$ to yield a consequence operator.]]

(d4) $\Gamma \Vdash \alpha \ \leftrightarrow\ \alpha \in \mathcal{C}(\Gamma)$

In other words, Γ entails α iff α is a consequence of Γ.

3. Deductive Entailment is a Species of Logical Entailment

Recall from Chapter 6 that we can define deductive entailment in terms of the notion of a deductive system. The following are our official definitions.

Def

A *deductive system* is, by definition, a pair (S,$\mathcal{R}$), where S is the set of sentences of a formal language, and $\mathcal{R}$ is a collection of inference rules on S.

Def

A *derivation of* α *from* Γ *in* (S,$\mathcal{R}$) is, by definition, a finite sequence of formulas of S, the last one of which is α, and everyone of which is either an element of Γ, or follows from previous formulas by a rule in $\mathcal{R}$.

Given the notion of derivation in a deductive system, one can define the following notion of deductive entailment (relative to (S,$\mathcal{R}$).

(d) $\Gamma \vdash \alpha \quad =_{\text{df}}$ there is a derivation of α from Γ [in system (S,$\mathcal{R}$)]

In this connection one can prove (exercise) the following theorem.

Th

Let (S,$\mathcal{R}$) be a deductive system, and let ⊢ be defined as in (d). Then the following are true, for any α, β, Γ, Δ.

(1) $\alpha \in \Gamma \rightarrow \Gamma \vdash \alpha$
(2) $\Gamma \subseteq \Delta \rightarrow . \ \Gamma \vdash \alpha \rightarrow \Delta \vdash \alpha$
(3) $\Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \beta \ . \!\rightarrow \ \Gamma \vdash \beta$

Comparing clauses (1)-(3) with the restrictions (r1)-(r3) defining SC-logics, we can informally restate this theorem as follows.

Deductive entailment is a species of abstract logical entailment.

4. Semantic Entailment is a Species of Logical Entailment

Recall from Chapter 5 that, at the minimum, a semantics for a language L, specifies a set V of admissible valuations on the set of sentences of L. The following are the relevant definitions.

Def

Let S be a non-empty set of sentences. Then a *valuation* on S is, by definition, any function from S into {T,F}.

Def

Let S be a non-empty set of sentences. Then V is a *truth-value semantics* on S iff every element of V is a valuation on S.

Here, V is the set of admissible valuations on S. In this context, we will omit the modifier 'truth-value' and simply write 'semantics'.

Every (truth-value) semantics gives rise to an associated semantic entailment relation, defined as follows.

(d) $\Gamma \vDash \alpha \quad =_{\text{df}} \quad \sim\exists v \in V : \forall\gamma(\gamma \in \Gamma \rightarrow v(\gamma)=\varepsilon) \ \& \ v(\alpha)=F$

In this connection one can prove (exercise) the following theorem.

Th

Let (S,V) be a truth-value semantic system, and let $\vDash$ be defined as in (d). Then the following are true, for any α, β, Γ, Δ.

(1) $\alpha \in \Gamma \rightarrow \Gamma \vDash \alpha$
(2) $\Gamma \subseteq \Delta \rightarrow. \Gamma \vDash \alpha \rightarrow \Delta \vDash \alpha$
(3) $\Gamma \vDash \alpha \ \& \ \Gamma \cup \{\alpha\} \vDash \beta \ . \rightarrow \ \Gamma \vDash \beta$

Comparing clauses (1)-(3) with the restrictions (r1)-(r3) defining SC-logics, we can informally restate this theorem as follows.

Semantic entailment is a species of abstract logical entailment.

5. Every SC-Logic can be Semantically Specified.

An interesting further question is whether logical entailment is a species of semantic entailment. In other words, can every SC-logic be semantically specified? In this connection, we introduce further definitions.

Def

Let $\mathcal{L}$ [=(S,$\Vdash$)] be an SC-logic, let V be a semantics for S, and let $\vDash$ be the associated semantic entailment relation. Then

V is *sound* for $\mathcal{L}$ iff: for every $\Gamma \subseteq S$, $\alpha \in S$, if $\Gamma \vDash \alpha$ then $\Gamma \Vdash \alpha$.

V is *complete* for $\mathcal{L}$ iff: for every $\Gamma \subseteq S$, $\alpha \in S$, if $\Gamma \Vdash \alpha$ then $\Gamma \vDash \alpha$

In other words, soundness amounts to the claim that every semantically valid argument is logically valid, and completeness amounts to the claim that every logically valid argument is semantically valid.

V *specifies* $\mathcal{L}$ iff: V is both sound and complete for L,

$\mathcal{L}$ [=(S,⊩)]is *semantically specifiable* iff: there is a semantics V that is both sound and complete for $\mathcal{L}$.

[[Notice that the definitions could be reversed; we could define '$\mathcal{L}$ is sound/complete for V' in a similar manner. In that case we would obtain the following result: $\mathcal{L}$ is sound/complete for V iff V is complete/sound for $\mathcal{L}$.]]

The earlier question, whether every SC-logic can be semantically specified, can now be answered in the affirmative.

(T) Let $\mathcal{L}$ be an SC-logic. Then there exists a semantics V that is both sound and complete for $\mathcal{L}$.

Proof: Let $\mathcal{L}$ = (S,⊩) be an SCL. Define C($\mathcal{L}$) = {Cn(Γ): Γ⊆S}. For each A in C($\mathcal{L}$), define v_A so that $v_A(\alpha)$=T iff $\alpha \in$ A. Define V to be the set $\{v_A$: A∈C(L)}. Claim: V is sound and complete for L.

Soundness: Suppose not(Γ⊩α), to show for some v that v(Γ)= T, v(α)=F. Consider v_A, where A=Cn(Γ). Then $v_A(\Gamma)$=T, since $\Gamma \subseteq Cn(\Gamma)$, and $v_A(\alpha)$=F, since $\alpha \notin Cn(\Gamma)$, by hypothesis.

Completeness: Suppose Γ⊩α, to show that there is no v such that v(Γ)=T and v(α)=F. Suppose otherwise. Then there is an A in C(L) such that $v_A(\Gamma)$=T and $v_A(\alpha)$=F, where A=Cn(Δ) for some Δ. This amounts to: $\Gamma \subseteq Cn(\Delta)$. But then $Cn(\Gamma) \subseteq Cn(Cn(\Delta))$. Also, $Cn(Cn(\Delta)) \subseteq Cn(\Delta)$. so $Cn(\Gamma) \subseteq Cn(\Delta)$. By hypothesis, $\Gamma \vdash \alpha$, so $\alpha \in Cn(\Gamma)$, so $\alpha \in Cn(\Delta)$, so $v_A(\alpha)$=T, which contradicts an earlier assumption.

6. Not Every SC-Logic is Categorical.

Every SC-logic has a semantic specification. A remaining question is whether every SC-logic has a *unique* semantic specification. This gives rise to the following definition.

D

Let $\mathcal{L}$ be an SC-logic. Then $\mathcal{L}$ is *categorical* iff there is *exactly one* semantics V that is sound and complete for $\mathcal{L}$.

Given the earlier theorem that every logic is specified by at least one semantics V, to prove that a logic $\mathcal{L}$ is categorical, one must prove that V_1 and V_2 both specify $\mathcal{L}$ only if V_1=V_2.

The question, whether every SC-logic is categorical, is answered in the negative.

(T) Not every SC-logic is categorical.

Proof: it is sufficient to produce an SCL that is not categorical. Consider the following SCL.

S = {a,b} a≠b
{a}⊩b, {a,b}⊩a, {a,b}⊩b, {a}⊩a, {b}⊩b

Now, consider the following two sets of valuations.

$V_1 = \{v_1, v_2\}$;

$V_2 = \{v_1\}$

$v_1(a) = F$, $v_1(b) = T$
$v_2(a) = T$, $v_2(b) = T$

Routine calculation shows that the semantic entailments relative to V_1 are the very same as the semantic entailments relative to V_2, being the entailments of $\mathcal{L}$, above. Both V_1 and V_2 specify $\mathcal{L}$; yet $V_1 \neq V_2$.

This is a simple example, but a counter-example to a claim doesn't have to be complex. If one doesn't feel comfortable with "unreal" counterexamples, then one need merely consider any fragment of classical first order logic, ordinarily construed, to obtain a "real" counterexample. Proving that classical logic is not categorical requires more work, which we postpone until we have more formal resources.

7. Other Categories of Logics

As mentioned in Section 1, SC-logics are not the only category of logic. They are perhaps the most natural, to the extent that they closely resemble the natural deduction characterization of logic. Still, there are other categories; we list the three categories that interest us.

(c1)	single conclusion logics	category: (m,1)
(c2)	assertional logics	category: (0,1)
(c3)	multiple conclusion logics	category: (m,m)

The category (c_1,c_2) refers to how many premises and conclusions the logic countenances. For example, a single conclusion logic (category (m,1)) countenances multiple premises but a single conclusion; an assertional logic (category (0,1)) countenances no premises and a single conclusion. Basically, an assertional logic says what formulas are logically true, but it does not say explicitly what arguments are valid. Historically, a number of logics have been presented in this manner. Finally, a multiple conclusion logic (category (m,m) countenances both multiple premises and multiple conclusions; logics of this sort were pioneered by Gentsen. Semantically speaking, a multiple conclusion argument (Γ/Δ) is *in*valid if it is possible for all the premises to be true while all the conclusions are false; otherwise, it is valid.

Notice, of course, that one can define other categories of logics; e.g., (m,0) : multiple premise, no conclusion; or (1,m): single premise, multiple conclusion; however, these other categories are not historically instantiated, so we will ignore them.

We have already formally defined SC-logics; we now define assertional logics and MC-logics.

> D3
>
> An assertional logic is, by definition, a pair $(S,\Vdash)$, where S is a non-empty set, and $\Vdash$ is a subset of S.

The elements of $\vdash$ are the theses (assertions) of the logic; it is customary to write '$\Vdash\alpha$' in place of '$\alpha \in \Vdash$'; intuitively, '$\Vdash$' is treated as a one-place predicate '...is a thesis'. Notice that there are no abstract restrictions on what counts as a thesis.

D4

A *multiple-conclusion logic* (MC-logic) is, by definition, a pair $(S,\Vdash)$, where S is a non-empty set, and $\Vdash$ is a relation from $\wp(S)$ to $\wp(S)$, satisfying the following restrictions.

(r1) if $\alpha\in\Gamma$ and $\alpha\in\Delta$, then $\Gamma\Vdash\Delta$
(r2) if $\Gamma\Vdash\Delta$, and $\Gamma\subseteq\Gamma'$, and $\Delta\subseteq\Delta'$, then $\Gamma'\Vdash\Delta'$
(r3) if $\Gamma'\Vdash\Delta'$, for every $\Gamma'\supseteq\Gamma$, $\Delta'\supseteq\Delta$ s.t. $\Gamma'\cup\Delta'=S$, then $\Gamma\Vdash\Delta$

MC-logics are the most general category. Given any MC logic, one can define an affiliated SC logic, simply by restricting the MC entailment relation to single-conclusion arguments. Similarly, given any SC logic (and hence, given any MC logic), one can define an affiliated assertional logic, simply by considering those arguments with no premises. This is formulated as follows, where '$\Vdash$' is used ambiguously; on the right, it refers to MC-entailment relation; on the left, it refers to SC-entailment (resp., assertion "predicate").

(d5) $\Gamma\Vdash\alpha \quad =_{df} \quad \Gamma\Vdash\{\alpha\}$
(d6) $\Vdash\alpha \quad =_{df} \quad \varnothing\Vdash\{\alpha\}$

8. Generalized Semantic Entailment is a Species of MC-Entailment.

We now turn to MC-logics. First we define generalized semantic entailment in a manner similar to ordinary semantic entailment.

(D) Let S be a non-empty set of sentences, and let V be a semantics for S. Then *generalized semantic entailment* w.r.t. V is the relation $\vDash$ from $\wp(S)$ to $\wp(S)$ defined as follows.

(d) $\Gamma\vDash\Delta$ iff: no v in V is such that $v(\Gamma)=T$ and $v(\Delta)=F$

In other words, Δ follows from Γ (relative to V) iff it is impossible (relative to V) for every sentence in Γ to be true while every sentence in Δ is false.

One can prove that the generalized semantic entailment relation $\vDash$, as defined above, satisfies the restrictions on an MC-logic (exercise). In other words, GS-entailment is a species of MC-entailment.

9. Every MC-Logic can be Semantically Specified.

One is naturally led to ask whether MC-entailment is a species of GS-entailment, alternatively stated, whether every MC-logic can be can be semantically specified. This is answered in the affirmative in the following theorem. First, we introduce a definition and supporting lemma.

(D) Let S be any non-empty set, and let S_1 and S_2 be (possibly empty) subsets of S. Then (S_1,S_2) is a *quasi-partition* of S iff $S_1\cap S_2=\varnothing$ and $S_1\cup S_2=S$

(L) Let $L = (S,\Vdash)$ be an MCL. Suppose not$(\Gamma\Vdash\Delta)$. Then there exists a quasi-partition (S_1,S_2) of S such that $\Gamma\subseteq S_1$, $\Delta\subseteq S_2$, and not$(S_1\Vdash S_2)$.

Proof: exercise.

(T) Let $(S,\Vdash)$ be an MCL. Then there is a semantics V that is sound and complete for $(S,\Vdash)$.

Proof: Let $\mathcal{L} = (S,\Vdash)$ be an MCL. Consider the subset $P(\mathcal{L})$ of quasi-partitions of S defined as follows: $P(\mathcal{L}) = \{(S_1,S_2): \text{not}(S_1\Vdash S_2)\}$. For each P in $P(\mathcal{L})$, define v_P so that $v_P(\alpha)=T$ if $\alpha\in S_1$, $v_P(\alpha)=F$ if $\alpha\in S_2$. Evidently, given the definition of a quasi-partition, every such function is a valuation on S. Define V to be the set $\{v_P: P\in P(\mathcal{L})\}$. Claim: V is sound and complete for $\mathcal{L}$.

Completeness: Suppose $\text{not}(\Gamma\Vdash\Delta)$, to show for some v in V, $v(\Gamma)=T$, $v(\Delta)=F$. By Lemma (L), there exists a quasi partition (S_1,S_2) such that $\Gamma\subseteq S_1$, $\Delta\subseteq S_2$, and $\text{not}(S_1\Vdash S_2)$; call it P. Consider v_P. Evidently, $v_P(\Gamma)=T$, and $v_P(\Delta)=F$.

Soundness: Suppose $\Gamma\Vdash\Delta$, to show that no v in V is such that $v(\Gamma)=T$ and $v(\Delta)=F$. Suppose otherwise. Then there is a P in $P(\mathcal{L})$ such that $v_P(\Gamma)=T$ and $v_P(\Delta)=F$, where $P = (S_1,S_2)$, and $\text{not}(S_1\Vdash S_2)$. This amounts to: $\Gamma\subseteq S_1$, $\Delta\subseteq S_2$; so by (r2), $S_1\Vdash S_2$, which yields a contradiction.

10. Every MC-Logic is Categorical.

In the previous section, we saw that every MCL is specified by at least one semantics. Next, we show that, unlike SCL's, every MCL is categorical; every MCL is specified by exactly one semantics.

(T) Every MCL is categorical.

Proof: Let $(S,\Vdash)$ be an MCL; suppose that V_1 and V_2 both specify $(S,\Vdash)$, to show $V_1=V_2$. Suppose otherwise. Then (without loss of generality) there is a valuation on S that is in V_1 but not in V_2; call it v. Consider the argument (T_v/F_v), where $T_v = \{\alpha:v(\alpha)=T\}$, and $F_v = \{\alpha:v(\alpha)=F\}$. Clearly, this argument is refuted by v, and hence by V_1; T_v does not entail F_v, relative to V_1. Since V_1 specifies $(S,\Vdash)$, by hypothesis, we have $\text{not}(T_v\Vdash F_v)$. But V_2 also specifies $(S,\Vdash)$, so V_2 must also contain a valuation, call it w, which refutes the argument (T_v/F_v). Insofar as w refutes (T_v/F_v), $w(T_v)=T$, and $w(F_v)=F$. So by extensionality, v=w, which means $v \in V_2$, which contradicts an earlier assumption.