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The Deductive Characterization of Logic

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1. Derivations

Having provided a formal semantic (model-theoretic) characterization of logic, we now turn to the deductive (proof-theoretic) characterization of logic. The analogy with introductory logic is straightforward; in intro logic, the semantic characterization of SL is given in terms of truth tables, whereas the deductive characterization of SL is given in terms of derivations in a natural deduction system.

Whereas truth tables are easy to describe in a logically and mathematically rigorous manner, in terms of valuations and truth-functions, derivations are more difficult to characterize mathematically. The full derivation system of intro logic is quite complicated, involving as it does provisional assumptions, show-lines, boxing, cancelling, etc.

On the other hand, *simple derivations*, which form the backbone of all derivations, are not so complicated. A simple derivation does not involve show-lines or provisional assumptions. You start with the premises and apply inference rules (repeatedly, as necessary) until you reach the conclusion, at which point you have a simple derivation of the conclusion from the premises. The following is the official (but informal) definition.

Def

A *simple derivation* of conclusion C from premises P_1, P_2, \dots, P_n is, by definition, a list of formulas, the last line of which is the conclusion C , and every line of which is either a premise, or follows from previous lines by an inference rule.

Simple derivations serve as the archetype in the metalogical description of deductive systems. Accordingly, we offer the following formal account. Notice that we drop the modifier ‘simple’; from the point of view of metalogic, all derivations are “simple”.

Def

A *derivation* of α from Γ is, by definition, a finite sequence of formulas, the last one of which is α , and every one of which is either an element of Γ , or follows from previous formulas by an inference rule.

By way of formally rendering the notions of *last* and *previous*, we introduce explicit sequence notation, as follows.

Def

A *derivation* of α from Γ is, by definition, a finite sequence, $\langle \sigma_1, \dots, \sigma_n \rangle$, of formulas satisfying the following restrictions.

- (1) $\sigma_n = \alpha$
- (2) for any $k \leq n$, $\sigma_k \in \Gamma$, or σ_k follows from $\{\sigma_i : i < k\}$ by an inference rule

Formally, a finite sequence is just like a finite string, the difference being purely pragmatic. Generally, a sequence σ has a first element σ_1 , a second element σ_2 , etc. If σ is n -long, then σ_n is the last element of σ . Also, to say that σ is a sequence of so-and-so's is to say that each σ_i is a so-and-so.

But what is an inference rule? What does it to say that something follows by one? We will largely leave this open. Basically, an inference rule is a computable relation R from sets of formulas to formulas. To say that α follows from Γ by R is to say that the pair $\langle \Gamma, \alpha \rangle$ stand in relation R . To say that a relation R is computable means that a computer can, in principle, decide whether a given pair $\langle \Gamma, \alpha \rangle$ stand in the relation. We will ignore this formal issue at the moment; suffice it to say that the rules one is accustomed to using in logic are all computable.

For example, modus ponens (MP) is an inference rule. Formally speaking, the pair $\langle \Gamma, \alpha \rangle$ bear the MP-relation if and only if Γ consists of exactly two elements, one of which is a conditional χ , the other of which is the antecedent of χ , and such that α is the consequent of χ .

With this in mind, we can now define what it means for something to follow by an inference rule.

Def

Let α be a formula, let Γ be a set of formulas, and let R be an inference rule. Then:

$$\alpha \text{ follows from } \Gamma \text{ by } R \quad =_{df} \quad \exists \Delta \{ \Delta \subseteq \Gamma \ \& \ \langle \Delta, \alpha \rangle \in R \}.$$

Notice that, as an immediate consequence of the definition, if α follows from Γ by R , and $\Gamma \subseteq \Delta$, then α follows from Δ by R .

2. Deductive Systems

In an earlier chapter, we saw how one can use a class V of admissible valuations to define various important logical notions, including validity and entailment. In what follows, we do the same in the deductive context — we define corresponding notions, not in terms of admissible valuations, but rather in terms of derivations.

First, we define the notion of *deductive system*.

Def

A *deductive system* is, by definition, a pair (S, \mathcal{R}) , where S is the set of sentences of a formal language, and \mathcal{R} is a collection of inference rules on S .

Def

A *derivation of α from Γ in (S, \mathcal{R})* is, by definition, a finite sequence of formulas of S , the last one of which is α , and everyone of which is either an element of Γ , or follows from previous formulas by a rule in \mathcal{R} .

Def

A *proof of α in (S, \mathcal{R})* is, by definition, a derivation of α from the empty set \emptyset .

3. Axioms in Deductive Systems

Note carefully that we allow zero-place rules. A well-known example in elementary logic is the reflexivity rule for identity (given “nothing”, one is entitled to write down ‘ $\tau = \tau$ ’ for any singular term). The existence of zero-place rules is critical if we are to have a non-trivial notion of proof, as defined in the previous section. In particular, a proof must have a first line; since such a line is first, it has no previous lines. Yet, by the definition of proof, it is still required to follow from “previous” lines by a rule of inference. The only way this can happen is for there to be at least one zero-place rule.

We have special terminology for such formulas – they are called *axioms* (of the deductive system). This may be summarized as follows.

Def

An *axiom* of a deductive system is, by definition, any formula that is delivered/generated by a zero-place rule.

For example, in the deductive system we propose for classical sentential logic [later chapter], we have the following rule of inference. (Generally, we use ‘ \hookrightarrow ’ as an in-line rule marker; specifically, ‘ $\epsilon_1, \dots, \epsilon_k \hookrightarrow \epsilon_0$ ’ means that one is entitled to infer ϵ_0 from $\epsilon_1, \dots, \epsilon_k$.)

$$\hookrightarrow \alpha \rightarrow (\beta \rightarrow \alpha)$$

Notice that the input side of this rule is empty; it is a zero-place rule. As with all rules, we use schematic letters from the metalanguage; in the above, α and β can be any formulas. If we substitute particular formulas – say ‘P’ and ‘Q’ – for ‘ α ’ and ‘ β ’, then we obtain the following instance.

$$\hookrightarrow P \rightarrow (Q \rightarrow P)$$

Since it is generated by a zero-place rule, the formula ‘ $P \rightarrow (Q \rightarrow P)$ ’ is an axiom of this system.

Whereas the object language expression ‘ $P \rightarrow (Q \rightarrow P)$ ’ is an axiom, the corresponding meta-language expression ‘ $\alpha \rightarrow (\beta \rightarrow \alpha)$ ’ is not, strictly speaking, an axiom, since it is not a formula of the object language. Rather, it is what is called an *axiom schema*.

Def

A *formula schema* is, by definition, a polynomial expression of the metalanguage that ranges over formulas of the object language. In particular, when one substitutes formula-names for all the schematic letters (metalinguistic variables), the result is a noun phrase that denotes a formula of the object language.

An *axiom schema* is, by definition, a formula schema every substitution instance of which denotes an axiom.

Without going into laborious detail, a *polynomial* is a fairly simple compound noun phrase built up pretty much like polynomials in algebra (e.g., $a^2 + 2ab + b^2$).

Suffice it to say that, at least in sentential logic, a schema may be obtained by “reverse substitution” — take a formula of the object language, and *appropriately* replace each atomic letter by a schematic letter.

4. Axiomatic Systems

Many texts on metalogic do not speak of deductive systems, but rather *axiomatic systems*. The following is the customary definition of axiomatic system, which is followed by the affiliated definitions of derivation and proof in an axiomatic system.

Def

An *axiomatic system* is, by definition, a triple $(S, \mathcal{A}, \mathcal{R})$, where S is a set of formulas of a formal language, \mathcal{A} is a (computable) subset of S , and \mathcal{R} is a (computable) collection of (1-place, 2-place, etc.) relations on S .

Def

A *derivation of α from Γ in $(S, \mathcal{A}, \mathcal{R})$* is, by definition, a finite sequence of formulas of S , the last one of which is α , and such that every line is either an axiom (i.e., an element of \mathcal{A}), an element of Γ , or follows from previous lines by a rule in \mathcal{R} .

Def

A *proof of α in $(S, \mathcal{A}, \mathcal{R})$* is, by definition, a finite sequence of formulas of S , the last one of which is α , and such that every line is either an axiom (i.e., an element of \mathcal{A}), or follows from previous lines by a rule in \mathcal{R} .

The difference between axiomatic systems and deductive systems, as we have defined them, is fundamentally trivial. On the one hand, deductive systems do not *officially* have axioms; they have zero-

place rules instead. On the other hand, axiomatic systems do not *officially* have zero-place rules; they have axioms instead.

Axiomatic systems and deductive systems are not interestingly different. Accordingly, we will use the terms ‘deductive system’ and ‘axiomatic system’ interchangeably; similarly, we will use the terms ‘0-place rule’ and ‘axiom *schema*’ interchangeably.

5. Validity and Entailment in the Deductive Context

Now, we are in a position to define the deductive versions of validity and entailment.

In what follows, we presuppose a deductive system (S, \mathcal{R}) relative to which all definitions are defined.

Def

Γ ***deductively entails*** α
if and only if
there is a derivation of α

Notation:

$\Gamma \vdash \alpha$

alternative terminology:
 α ***follows deductively from*** Γ
 α ***is a deductive consequence of*** Γ

Def

α is ***deductively valid***
if and only if
there is a proof of α .

Notation:

$\vdash \alpha$

alternative terminology:
 α ***is provable***
 α ***is a theorem***
 α ***is a thesis***

Def

Γ is ***deductively inconsistent***
if and only if
every formula is deductively entailed by Γ .

Notation:

$\Gamma \vdash$

Γ is ***deductively consistent***
if and only if

Notation:

Γ is not deductively inconsistent.

$\Gamma \nvdash$

Deductive entailment and deductive validity are the proof-theoretic counterparts of semantic entailment and semantic validity. Deductive inconsistency does not correspond *exactly* to any particular semantic notion; the closest semantic notion is the notion of unverifiability. The difficulty in providing an exact match between deductive and semantic entailment concerns the second argument of the predicate. In the case of semantic entailment, the second argument can be a set, even the empty set. However, in the usual characterization of deduction, allowing the second argument to be a (possibly empty) set does not make sense. What does it mean to derive the empty set from Γ ?

[[Fortunately, however, the usual characterization of deduction is not the only one. In a later chapter, we examine a deductive scheme that is equal in power to the semantic scheme.]]

6. Maximal Consistent Sets

In later chapters, we will prove two completeness theorems. In each case, we will prove a critical subordinate lemma called Lindenbaum's Lemma. This lemma claims that every deductively-consistent subset can be extended to a *maximal* deductively-consistent set. In order to understand this lemma, we must understand the term 'maximal', which is a general set theoretic term, defined as follows.

Def

Let K a collection of sets. Then a K -maximal set is, by definition, a set M satisfying the following conditions.

- (1) $M \in K$
- (2) $\forall X \{X \in K \rightarrow \sim[M \subset X]\}$
i.e.,
- (1*) M is a K -set
- (2*) M is not properly included in any K -set.

The application of this concept to any particular situation involves identifying the relevant class K . In our particular case, K is the class of deductively-consistent sets of formulas, in which case we obtain the following instance.

Def

Let Γ be a set of formulas. Then Γ is a maximal consistent set if and only if:

- (1) Γ is consistent
- (2) Γ is not properly included in any consistent set.

Notation:

$MC[\Gamma] \quad =_{df} \quad \Gamma \text{ is a maximal consistent set}$

One way to think of maximal consistent sets is as follows; if Γ is maximal consistent, then "adding" any formula α to Γ results in an inconsistent set. This may be formally stated as follows.

$$MC[\Gamma] \ \& \ \alpha \notin \Gamma \ .\rightarrow \ \Gamma \cup \{\alpha\} \vdash$$

This is a general theorem about \vdash ; it does not depend upon the details of the axiom system. The following is another general theorem about \vdash .

$$MC[\Gamma] \ \& \ \Gamma \vdash \alpha \ .\rightarrow \ \alpha \in \Gamma$$

In other words, a maximal deductively-consistent set contains all its deductive consequences.

These two general theorems about \vdash are included in the next section, which gives a more complete list.

7. General Theorems about \vdash

$$G0: \quad \alpha \in \Gamma \rightarrow \Gamma \vdash \alpha$$

$$G0c1: \quad \{\alpha\} \vdash \alpha$$

$$G0c2: \quad \{\alpha, \beta\} \vdash \alpha$$

$$G0c3: \quad \{\alpha, \beta, \gamma\} \vdash \alpha$$

etc.

$$G1: \quad \Gamma \subseteq \Delta \rightarrow. \Gamma \vdash \alpha \rightarrow \Delta \vdash \alpha$$

$$G1c: \quad \Gamma \subseteq \Delta \rightarrow. \Gamma \vdash \rightarrow \Delta \vdash$$

$$G2: \quad \vdash \alpha \rightarrow \Gamma \vdash \alpha$$

$$G3: \quad \vdash \alpha \leftrightarrow \emptyset \vdash \alpha$$

$$G4: \quad \vdash \alpha \leftrightarrow \forall \Gamma [\Gamma \vdash \alpha]$$

$$G5: \quad \text{Axiom}[\alpha] \rightarrow \vdash \alpha$$

$$G6: \quad \Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \beta \ .\rightarrow \ \Gamma \vdash \beta$$

$$G7: \quad \Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \ .\rightarrow \ \Gamma \vdash$$

$$G8: \quad \forall \delta \{ \delta \in \Delta \rightarrow \Gamma \vdash \delta \} \ \& \ \Delta \vdash \beta \ .\rightarrow \ \Gamma \vdash \beta$$

$$G9: \quad \Gamma \vdash \alpha \ \& \ \alpha \vdash \beta \ .\rightarrow \ \Gamma \vdash \beta$$

$$G10: \quad MC[\Gamma] \ \& \ \alpha \notin \Gamma \ .\rightarrow \ \Gamma \cup \{\alpha\} \vdash$$

$$G11: \quad MC[\Gamma] \ \& \ \Gamma \vdash \alpha \ .\rightarrow \ \alpha \in \Gamma$$

$$G12: \quad \Gamma \vdash \alpha \rightarrow \exists \Gamma' \{ \text{finite}[\Gamma'] \ \& \ \Gamma' \subseteq \Gamma \ \& \ \Gamma' \vdash \alpha \}$$

$$G13: \quad \alpha \vdash \beta \ \& \ \beta \vdash \alpha \ .\rightarrow. \ \Gamma \cup \{\alpha\} \vdash \gamma \leftrightarrow \Gamma \cup \{\beta\} \vdash \gamma$$

$$G14: \quad \forall d \exists n [\text{length}(d)=n] \quad \quad \quad \text{[‘d’ ranges over derivations; ‘n’ ranges over numbers.]}$$

G15: $\forall \mathbb{P}: \forall n \forall d (\text{length}(d)=n \rightarrow \mathbb{P}[d]) \rightarrow \forall d \mathbb{P}[d]$

G16: $\forall \mathbb{P}: \forall n \forall d \forall v_1 \dots v_k (\mathbb{F}[d, v_1 \dots v_k] \ \& \ \text{len}(d)=n \rightarrow \mathbb{P}) \rightarrow \forall d \forall v_1 \dots v_k (\mathbb{F}[d, v_1 \dots v_k] \rightarrow \mathbb{P})$

[‘ \mathbb{P} ’ ranges over properties.]

8. Exercises

Prove every theorem in Section 7.

9. Answers to Selected Exercises

G0:

(1)	SHOW: $\alpha \in \Gamma \rightarrow \Gamma \vdash \alpha$	CD
(2)	$\alpha \in \Gamma$	As
(3)	SHOW: $\Gamma \vdash \alpha$	Def \vdash
(4)	SHOW: $\exists d [d \text{ derives } \alpha \text{ from } \Gamma]$	5, QL
(5)	SHOW: $\langle \alpha \rangle$ derives α from Γ	6, 7, Def derives
(6)	a: SHOW: $\text{last}(\langle \alpha \rangle) = \alpha$	ST
(7)	b: SHOW: $\forall \delta \in \langle \alpha \rangle: \delta \in \Gamma$ or δ follows by a rule ...	UCD
(8)	$\delta \in \langle \alpha \rangle$	As
(9)	SHOW: $\delta \in \Gamma$ or δ follows by a rule ...	10, 11, SL
(10)	$\delta = \alpha$	8, ST
(11)	$\delta \in \Gamma$	2, 11, IL

Note: $\langle \alpha \rangle$ is defined to be the singleton sequence of α ‘ $\delta \in \langle \alpha \rangle$ ’ means δ is “in” $\langle \alpha \rangle$. Notice that sequences do not satisfy ordinary extensionality; for example, supposing $a \neq b$, then $\langle a, b \rangle \neq \langle b, a \rangle$ even though $\langle a, b \rangle$ and $\langle b, a \rangle$ have the same “elements” – namely, a and b .

G1:

(1)	SHOW: $\Gamma \subseteq \Delta \rightarrow \Gamma \vdash \alpha \rightarrow \Delta \vdash \alpha$	CCD
(2)	$\Gamma \subseteq \Delta$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\Delta \vdash \alpha$	Def \vdash
(5)	SHOW: $\exists d [d \text{ derives } \alpha \text{ from } \Delta]$	8, QL
(6)	$\exists d [d \text{ derives } \alpha \text{ from } \Gamma]$	3, Def \vdash
(7)	D derives α from Γ	6, \exists O
(8)	SHOW: D derives α from Δ	9, 10, Def derives
(9)	a: SHOW: $\text{last}(D) = \alpha$	7, Def derives [a]
(10)	b: SHOW: $\forall \delta \in D: \delta \in \Delta$ or δ follows by a rule ...	11, 12, QL
(11)	$\forall \delta \in D: \delta \in \Gamma$ or δ follows by a rule ...	7, Def derives [b]
(12)	$\forall x \{x \in \Gamma \rightarrow x \in \Delta\}$	2, Def \subseteq

corollary:

(1)	SHOW: $\Gamma \subseteq \Delta \rightarrow \Gamma \vdash \rightarrow \Delta \vdash$	CCD
(2)	$\Gamma \subseteq \Delta$	As
(3)	$\Gamma \vdash$	As
(4)	SHOW: $\Delta \vdash$	Def $\Gamma \vdash$
(5)	SHOW: $\forall \alpha [\Delta \vdash \alpha]$	UD
(6)	SHOW: $\Delta \vdash \omega$	2,8,G1
(7)	$\forall \alpha [\Gamma \vdash \alpha]$	3,Def $\Gamma \vdash$
(8)	$\Gamma \vdash \omega$	7,QL

G2:

(1)	SHOW: $\vdash \alpha \rightarrow \Gamma \vdash \alpha$	CD
(2)	$\vdash \alpha$	As
(3)	SHOW: $\Gamma \vdash \alpha$	Def $\Gamma \vdash \alpha$
(4)	SHOW: $\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	9,QL
(5)	$\exists p[p \text{ proves } \alpha]$	2, Def $\vdash \alpha$
(6)	P proves α	5, \exists O
(7)	last(P)= α & $\forall \delta \in P$: δ follows by a rule ...	6, Def proves
(8)	last(P)= α & $\forall \delta \in P$: $\delta \in \Gamma$ or δ follows by a rule ...	7,QL
(9)	P derives α from Γ	8,Def derives

G3:

(1)	SHOW: $\vdash \alpha \leftrightarrow \emptyset \vdash \alpha$	\leftrightarrow D
(2)	SHOW: $\vdash \alpha \rightarrow \emptyset \vdash \alpha$	G2
(3)	SHOW: $\emptyset \vdash \alpha \rightarrow \vdash \alpha$	CD
(4)	$\emptyset \vdash \alpha$	As
(5)	SHOW: $\vdash \alpha$	Def $\vdash \alpha$
(6)	SHOW: $\exists p[p \text{ proves } \alpha]$	QL
(7)	$\exists d[d \text{ derives } \alpha \text{ from } \emptyset]$	4, Def $\emptyset \vdash \alpha$
(8)	D derives α from \emptyset	9, \exists O
(9)	SHOW: D proves α	10,11 Def proves
(10)	a: SHOW: last(D)= α	8, Def derives [a]
(11)	b: SHOW: $\forall \delta \in D$: δ follows by a rule ...	UD
(12)	SHOW: δ follows by a rule ...	13,14,QL
(13)	$\forall \delta \in D$: $\delta \in \emptyset$ or δ follows by a rule ...	8, Def derives [b]
(14)	$\sim \exists x[x \in \emptyset]$	ST

G4:

(1)	SHOW: $\vdash \alpha \leftrightarrow \forall \Gamma[\Gamma \vdash \alpha]$	\leftrightarrow D
(2)	SHOW: \rightarrow	CD
(3)	$\vdash \alpha$	As
(4)	SHOW: $\forall \Gamma[\Gamma \vdash \alpha]$	UD
(5)	SHOW: $\Gamma \vdash \alpha$	3,G2
(6)	SHOW: \leftarrow	CD
(7)	$\forall \Gamma[\Gamma \vdash \alpha]$	As
(8)	SHOW: $\vdash \alpha$	9,G3
(9)	$\emptyset \vdash \alpha$	7,QL

G5:

(1)	SHOW: $Ax[\alpha] \rightarrow \vdash \alpha$	CD
(2)	$Ax[\alpha]$	As
(3)	SHOW: $\vdash \alpha$	Def \vdash
(4)	SHOW: $\exists p[p \text{ proves } \alpha]$	5,QL
(5)	SHOW: $\langle \alpha \rangle \text{ proves } \alpha$	6,7, Def proves
(6)	a:SHOW: $\text{last}(\langle \alpha \rangle) = \alpha$	ST
(7)	b:SHOW: $\forall \delta \in \langle \alpha \rangle: \delta \text{ follows by a rule ...}$	UCD
(8)	$\delta \in \langle \alpha \rangle$	As
(9)	SHOW: $\delta \text{ follows by a rule ...}$	12,QL
(10)	$\alpha \text{ follows by a zero-place rule}$	2, Def Ax
(11)	$\delta = \alpha$	8, Def $\langle \alpha \rangle$
(12)	$\delta \text{ follows by a zero-place rule}$	10,11,IL

G6:

(1)	SHOW: $\Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \beta \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \cup \{\alpha\} \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def \vdash
(5)	SHOW: $\exists d[d \text{ derives } \beta \text{ from } \Gamma]$	12,QL
(6)	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	2, Def \vdash
(7)	$\exists d[d \text{ derives } \beta \text{ from } \Gamma \cup \{\alpha\}]$	3, Def \vdash
(8)	$D_1 \text{ derives } \alpha \text{ from } \Gamma$	6, $\exists O$
(9)	$D_2 \text{ derives } \beta \text{ from } \Gamma \cup \{\alpha\}$	7, $\exists O$
(10)	$\exists d\{d = D_2[D_1/\alpha]\}$	ST*
(11)	$D_3 = D_2[D_1/\alpha]$	10, $\exists O$
(12)	SHOW: $D_3 \text{ derives } \beta \text{ from } \Gamma$	13,15,Def derives
(13)	a:SHOW: $\text{last}(D_3) = \beta$	14,Def D_3 , ST
(14)	$\text{last}(D_2) = \beta$	9, Def derives [a]
(15)	b:SHOW: $\forall \delta \in D_3: \delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	UCD
(16)	$\delta \in D_3$	As
(17)	SHOW: $\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	19-30,SC/SC
(18)	$\delta \in D_2[D_1/\alpha]$	11,16,IL
(19)	$\{\delta \in D_1\} \text{ or } \{\delta \in D_2 \ \& \ \delta \neq \alpha\}$	18, Def $\sigma[\pi/\varepsilon]$
(20)	c1: $\delta \in D_1$	As
(21)	$\forall \delta \in D_1\{\delta \in \Gamma \text{ or } \delta \text{ follows by a rule}\}$	8,Def derives [b]
(22)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	20,21,QL
(23)	c2: $\delta \in D_2 \ \& \ \delta \neq \alpha$	As
(24)	$\forall \delta \in D_2\{\delta \in \Gamma \cup \{\alpha\} \text{ or } \delta \text{ follows by a rule}\}$	9,Def derives [b]
(25)	$\delta \in \Gamma \cup \{\alpha\} \text{ or } \delta \text{ follows by a rule}$	23,24,QL
(26)	c1: $\delta \in \Gamma \cup \{\alpha\}$	As
(27)	$\delta \in \Gamma$	23b,26,ST
(28)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	27,SL
(29)	c2: $\delta \text{ follows by a rule ...}$	As
(30)	$\delta \in \Gamma \text{ or } \delta \text{ follows by a rule ...}$	29,SL

*Note: $D_2[D_1/\alpha]$ is defined to be the sequence that results when every occurrence of α in D_2 is replaced by sequence D_1 .

G7:

(1)	SHOW: $\Gamma \vdash \alpha \ \& \ \Gamma \cup \{\alpha\} \vdash \cdot \rightarrow \Gamma \vdash$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\Gamma \cup \{\alpha\} \vdash$	As
(4)	SHOW: $\Gamma \vdash$	Def \vdash
(5)	SHOW: $\forall \beta [\Gamma \vdash \beta]$	UD
(6)	SHOW: $\Gamma \vdash b$	2,8,G6
(7)	$\forall \beta [\Gamma \cup \{\alpha\} \vdash \beta]$	3, Def \vdash
(8)	$\Gamma \cup \{\alpha\} \vdash b$	7,QL

G8:

(1)	SHOW: $\forall \delta \{ \delta \in \Delta \rightarrow \Gamma \vdash \delta \} \ \& \ \Delta \vdash \beta \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\forall \delta \{ \delta \in \Delta \rightarrow \Gamma \vdash \delta \}$	As
(3)	$\Delta \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	Def \vdash
(5)	SHOW: $\exists d [d \text{ derives } \beta \text{ from } \Gamma]$	10,QL
(6)	$\exists d [d \text{ derives } \beta \text{ from } \Delta]$	3, Def \vdash
(7)	$D_0 \text{ derives } \beta \text{ from } \Delta$	6, \exists O
(8)	$\exists d \{ d = D_0[D_k/\delta_k : \delta_k \in \Delta] \}$	ST
(9)	$D = D_0[D_k/\delta_k : \delta_k \in \Delta]$	8, \exists O
(10)	SHOW: $D \text{ derives } \beta \text{ from } \Gamma$	
(11)	unfinished	

G9:

(1)	SHOW: $\Gamma \vdash \alpha \ \& \ \alpha \vdash \beta \rightarrow \Gamma \vdash \beta$	&CD
(2)	$\Gamma \vdash \alpha$	As
(3)	$\alpha \vdash \beta$	As
(4)	SHOW: $\Gamma \vdash \beta$	2,7,G8
(5)	$\{\alpha\} \vdash \beta$	3, Def \vdash
(6)	$\{\alpha\} \subseteq \Gamma \cup \{\alpha\}$	ST
(7)	$\Gamma \cup \{\alpha\} \vdash \beta$	5,6,G1

G10:

(1)	SHOW: $MC[\Gamma] \ \& \ \alpha \notin \Gamma \rightarrow \Gamma \cup \{\alpha\} \vdash$	&CD
(2)	$MC[\Gamma]$	As
(3)	$\alpha \notin \Gamma$	As
(4)	SHOW: $\Gamma \cup \{\alpha\} \vdash$	5,6,QL
(5)	$\Gamma \subset \Gamma \cup \{\alpha\}$	3,ST
(6)	$\forall \Delta \{ \Gamma \subset \Delta \rightarrow \Delta \vdash \}$	2, Def MC [b]

(1)	SHOW: $MC[\Gamma] \ \& \ \Gamma \vdash \alpha \rightarrow \alpha \in \Gamma$	&CD
(2)	$MC[\Gamma]$	As
(3)	$\Gamma \vdash \alpha$	As
(4)	SHOW: $\alpha \in \Gamma$	ID
(5)	$a \notin \Gamma$	As
(6)	SHOW: \times	8,11,SL
(7)	$\Gamma \subset \Gamma \cup \{\alpha\}$	5,ST
(8)	$\Gamma \not\vdash$	2, Def MC [a]
(9)	$\forall \Delta \{ \Gamma \subset \Delta \rightarrow \Delta \vdash \}$	2, Def MC [b]
(10)	$\Gamma \cup \{\alpha\} \vdash$	7,9,QL
(11)	$\Gamma \vdash$	3,10,G7

(1)	SHOW: $\forall d \exists n[\text{len}(d)=n]$	UD
(2)	SHOW: $\exists n[\text{len}(\delta)=n]$	DD
(3)	δ is a derivation	sortal assumption
(4)	δ is a finite sequence	3, Def derivation
(5)	$\exists n[\text{len}(\delta)=n]$	4, Def finite sequence

(1)	SHOW: $\forall n \forall d (\text{len}(d)=n \rightarrow \mathbb{F}[d]) \rightarrow \forall d \mathbb{F}[d]$	CD
(2)	$\forall n \forall d (\text{len}(d)=n \rightarrow \mathbb{F}[d])$	As
(3)	SHOW: $\forall d \mathbb{F}[d]$	UD
(4)	SHOW: $\mathbb{F}[\delta]$	2,6,QL
(5)	$\exists n [\text{len}(\delta)=n]$	G12,QL
(6)	$\text{len}(\delta)=n$	5, \exists O

(1)	$\text{SHOW: } \forall n \forall d \forall v_1 \dots v_k (\mathbb{F}[d, v_1 \dots v_k] \ \& \ \text{len}(d)=n \rightarrow \mathbb{P})$ $\rightarrow \forall d \forall v_1 \dots v_k (\mathbb{F}[d, v_1 \dots v_k] \rightarrow \mathbb{P})$	CD
(2)	$\forall n \forall d \forall v_1 \dots v_k (\mathbb{F}[d, v_1 \dots v_k] \ \& \ \text{len}(d)=n \rightarrow \mathbb{P})$	As
(3)	SHOW: $\forall d \forall v_1 \dots v_k (\mathbb{F}[d, v_1 \dots v_k] \rightarrow \mathbb{P})$	UCD
(4)	$\mathbb{F}[\delta, c_1 \dots c_k]$	As
(5)	SHOW: \mathbb{P}	4,6,7,QL
(6)	$\exists n[\text{len}(\delta)=n]$	G12,QL
(7)	$\text{len}(\delta)=n$	6, \exists O

Here, \mathbb{F} and \mathbb{P} are formulas, v_1, \dots, v_k are variables, and c_1, \dots, c_k are constants appropriately substituted for v_1, \dots, v_k .

In the following, we presuppose a given axiom system \mathbb{A} with respect to which all deductive notions are defined.

σ is a proof $\quad =_{df} \quad \sigma$ is a finite sequence of formulas every item of which follows from previous lines by a rule

σ proves α	$=_{df}$	
σ is a proof of α	$=_{df}$	σ is a proof, and $\text{last}(\sigma)=\alpha$
σ derives from Γ	$=_{df}$	
σ is a derivation from Γ	$=_{df}$	σ is a finite sequence of formulas every item of which follows from previous lines by a rule, or is an element of Γ
σ derives α from Γ	$=_{df}$	
σ is a derivation of α from Γ	$=_{df}$	σ derives from Γ , and $\text{last}(\sigma)=\alpha$
α is an axiom	$=_{df}$	α results from applying a zero-place rule

2. Theoremhood, Deductive Entailment, and Deductive Consistency

$\vdash \alpha$	$=_{df}$	$\exists p[p \text{ is a proof of } \alpha]$	$[\alpha \text{ is a theorem/thesis}]$
$\Gamma \vdash \alpha$	$=_{df}$	$\exists d[d \text{ derives } \alpha \text{ from } \Gamma]$	$[\Gamma \text{ deductively entails } \alpha]$
$\Gamma \vdash$	$=_{df}$	$\forall \alpha[\Gamma \vdash \alpha]$	$[\Gamma \text{ is deductively inconsistent}]$
$\Gamma \nvdash$	$=_{df}$	$\sim[\Gamma \vdash]$	$[\Gamma \text{ is deductively consistent}]$

3. Maximal Consistent Sets

$\text{MC}[\Gamma]$	$=_{df}$	$\Gamma \nvdash \text{ \& } \forall \Delta\{\Gamma \subset \Delta \rightarrow \Delta \vdash\}$	$[\Gamma \text{ is maximal consistent}]$
$\Gamma \subset \Delta$	$=_{df}$	$\Gamma \subseteq \Delta \text{ \& } \sim[\Delta \subseteq \Gamma]$	$[\text{proper inclusion}]$