

Unit 6: Derivations in Description Logic

1. EXERCISES

Directions: For each of the following, construct a formal derivation of the conclusion (indicated by "/") from the premises (if any). In cases in which two formulas are separated by "//", construct a derivation of each formula from the other.

1. EXERCISE SET A

1. $\forall xFx ; \exists x[x=t] / F[t]$
2. $\forall xFx ; \exists x[x=x] / \exists xFx$
3. $F[t] ; \exists x[x=t] / \exists xFx$
4. $/ \forall x(Fx \rightarrow \exists yFy)$
5. $/ \forall x(\forall yFy \rightarrow Fx)$
6. $/ \forall x\exists y[x=y]$
7. $/ \forall x\forall y\{x=y \rightarrow (Fx \leftrightarrow Fy)\}$
8. $\sim\exists x[x=x] // \forall x(Fx \& \sim Fx)$
9. $R[s,t] ; \exists x[x=s] / \exists xR[x,t]$
10. $\sim\exists x\exists y[x=y] // \forall x[x \neq x]$

2. EXERCISE SET B

11. $/ \forall x\exists y[x=y]$
12. $/ \forall x(x = \imath yFy \leftrightarrow \forall y(Fy \leftrightarrow y=x))$
13. $\exists x[x = \imath xFx] / \forall x(Fx \leftrightarrow x = \imath xFx)$
14. $\exists x[x = \imath xFx] / F[\imath xFx]$
15. $\exists x[x = \imath xFx] // \exists x\forall y(Fy \leftrightarrow y=x)$
16. $/ \forall x[x = \imath y[y=x]]$
17. $/ \forall x\forall y\{x=y \leftrightarrow \imath z[z=x] = \imath z[z=y]\}$
18. $/ \forall x\{Fx \leftrightarrow F[\imath y[y=x]]\}$
19. $/ \forall x\forall y\{x=y \rightarrow [\imath zRxz = \imath zRyz]\}$
20. $\exists x\forall y(Fy \leftrightarrow y=x) / \forall x(Fx \leftrightarrow x = \imath xFx)$
21. $G[\imath xFx] ; \exists x[x = \imath xFx] / \exists x(Fx \& Gx)$
22. $G[\imath xFx] ; \exists x[x = \imath xFx] / \exists x(\forall y(Fy \leftrightarrow y=x) \& Gx)$
23. $G[\imath xFx] ; \sim\exists xGx / \sim\exists x[x = \imath xFx]$
24. $\exists x\forall y\{Fy \leftrightarrow y=x\} / \exists x(Fx \& x = \imath xFx)$

3. EXERCISE SET C

25. $\exists x\{\forall y(Fy \leftrightarrow y=x) \& Gx\} / G[1xFx]$
26. $\forall x\{Fx \rightarrow x=1xFx\} / \exists xFx \rightarrow F[1xFx]$
27. $t = 1xFx ; \exists x[x=t] / \exists x\{Fx \& \forall y[Fy \rightarrow y=x]\}$
28. $Fa ; \forall x\forall y\{(Fx \& Fy) \rightarrow x=y\} / a = 1xFx$
29. $t = 1xFx ; \exists x\exists y\{x \neq y \& Fx \& Fy\} / \sim \exists x[x=t]$
30. $\exists x\forall y([y \neq t] \leftrightarrow y=x) / \forall x([x=t] \vee [x=1x[x \neq t]])$
31. $\forall x\exists y\forall z(Rxz \leftrightarrow z=y) ; \forall x\forall y(Rxy \rightarrow Ryx) / \forall x[1yRxy = 1yRyx]$
32. $\exists x\forall y(Fy \rightarrow y=x) / \forall x(Fx \rightarrow x=1xFx)$
33. $\sim F[1xFx] ; \exists xFx / \exists x\exists y\{x \neq y \& Fx \& Fy\}$
34. $\forall x(Fx \rightarrow Gx) ; \exists x\forall y(Gy \leftrightarrow y=x) / \forall x(Fx \rightarrow [x = 1xGx])$
35. $\exists x\forall y\{Fy \rightarrow y=x\} / \forall x(Fx \rightarrow F[1xFx])$

2. ANSWERS TO UNIT 6 EXERCISES

#1:

(1)	$\forall xFx$	Pr
(2)	$\exists x[x = t]$	Pr
(3)	SHOW: $F[t]$	DD
(4)	$a = t$	2, \exists O
(5)	Fa	1, \forall O
(6)	$F[t]$	4,5,LL

#2:

(1)	$\forall xFx$	Pr
(2)	$\exists x[x=x]$	Pr
(3)	SHOW: $\exists xFx$	5, \exists I
(4)	$a=a$	2, \exists O
(5)	Fa	1, \forall O

#3:

(1)	$F[t]$	Pr
(2)	$\exists x[x = t]$	Pr
(3)	SHOW: $\exists xFx$	5, \exists I
(4)	$a = t$	2, \exists O
(5)	Fa	1,4,LL

#4:

(1)	SHOW: $\forall x(Fx \rightarrow \exists yFy)$	UCD
(2)	Fa	As
(3)	SHOW: $\exists yFy$	2, \exists I

#5:

(1)	SHOW: $\forall x(\forall yFy \rightarrow Fx)$	UCD
(2)	$\forall yFy$	As
(3)	SHOW: Fa	2, \forall O

#6:

(1)	SHOW: $\forall x\exists y[x=y]$	UD
(2)	SHOW: $\exists y[a=y]$	DD
(3)	$a=a$	REF=
(4)	$\exists y[a=y]$	3, \exists I

#7:

(0)	SHOW: $\forall x\forall y\{x=y \rightarrow (Fx \leftrightarrow Fy)\}$	U2CD
(1)	$a=b$	As
(2)	SHOW: $Fa \leftrightarrow Fb$	DD
(3)	$Fa \leftrightarrow Fa$	SL
(4)	$Fa \leftrightarrow Fb$	1,3,LL

#8a:

(1)	$\sim\exists x[x=x]$	Pr
(2)	SHOW: $\forall x(Fx \& \sim Fx)$	UD
(3)	SHOW: $Fa \& \sim Fa$	ID
(4)	$\sim(Fa \& \sim Fa)$	As
(5)	SHOW: \times	6,7,SL
(6)	$\sim[a=a]$	1, $\sim\exists$ O
(7)	$a=a$	REF

#8b:

(1)	$\forall x(Fx \ \& \ \sim Fx)$	Pr
(2)	SHOW: $\sim \exists x[x=x]$	ID
(3)	$\exists x[x=x]$	As
(4)	SHOW: \ast	6,SL
(5)	$a=a$	Ref=
(6)	$Fa \ \& \ \sim Fa$	1, \forall O

#9:

(1)	$R[s,t]$	Pr
(2)	$\exists x[x = s]$	Pr
(3)	SHOW: $\exists xR[x,t]$	\exists D
(4)	$a = s$	2, \exists O
(5)	$R[a,t]$	1,4,LL

#10a:

(1)	$\sim \exists x \exists y[x=y]$	Pr
(2)	SHOW: $\forall x[x \neq x]$	UD
(3)	SHOW: $a \neq a$	ID
(4)	$a=a$	As
(5)	SHOW: \ast	3,6,SL
(6)	$\sim[a=a]$	1, $\sim \exists$ O2

#10b:

(1)	$\forall x[x \neq x]$	Pr
(2)	SHOW: $\sim \exists x \exists y[x=y]$	ID
(3)	$\exists x \exists y[x=y]$	As
(4)	SHOW: \ast	6,7,SL
(5)	$\exists y[a=y]$	3, \exists O
(6)	$a \neq a$	1, \forall O
(7)	$a=a$	REF=

#11:

(1)	SHOW: $\forall x \exists y[x=y]$	UD
(2)	SHOW: $\exists y[a=y]$	3, \exists I
(3)	$a=a$	Ref=

#12:

(1)	SHOW: $\forall x\{x = \iota y Fy \leftrightarrow \forall y(Fy \leftrightarrow y=x)\}$	UD
(2)	SHOW: $a = \iota y Fy \leftrightarrow \forall y(Fy \leftrightarrow y=a)$	3,6 \leftrightarrow I
(3)	SHOW: $a = \iota y Fy \rightarrow \forall y(Fy \leftrightarrow y=a)$	CD
(4)	$a = \iota y Fy$	As
(5)	SHOW: $\forall y(Fy \leftrightarrow y=a)$	4, ι O
(6)	SHOW: $\forall y(Fy \leftrightarrow y=a) \rightarrow a = \iota y Fy$	CD
(7)	$\forall y(Fy \leftrightarrow y=a)$	As
(8)	SHOW: $a = \iota y Fy$	7, ι I

#13:

(1)	$\exists x[x = \iota x Fx]$	Pr
(2)	SHOW: $\forall x\{Fx \leftrightarrow x = \iota x Fx\}$	DD
(3)	$a = \iota x Fx$	1, \exists O
(4)	$\forall x(Fx \leftrightarrow x=a)$	3, ι O
(5)	$\forall x\{Fx \leftrightarrow x = \iota x Fx\}$	3,4,LL

#14:

(1)	$\exists x[x = 1xFx]$	Pr
(2)	SHOW: $F[1xFx]$	DD
(3)	$a = 1xFx$	1, \exists O
(4)	$\forall x(Fx \leftrightarrow x=a)$	3, \forall O
(5)	Fa	4,OO
(6)	$F[1xFx]$	3,5,LL

#15a:

(1)	$\exists x[x = 1xFx]$	Pr
(2)	SHOW: $\exists x\forall y(Fy \leftrightarrow y=x)$	\exists D
(3)	$a = 1xFx$	1, \exists O
(4)	$\forall x(Fx \leftrightarrow x=a)$	3, \forall O
(5)	$\forall y(Fy \leftrightarrow y=a)$	4,AV

#15b:

(1)	$\exists x\forall y(Fy \leftrightarrow y=x)$	Pr
(2)	SHOW: $\exists x[x = 1xFx]$	3, \exists I
(3)	$\forall y(Fy \leftrightarrow y=a)$	1, \exists O
(4)	$a = 1yFy$	3, \forall I
(5)	$a = 1xFx$	4,AV

#16:

(1)	SHOW: $\forall x[x = 1y[y=x]]$	UD
(2)	SHOW: $a = 1y[y=a]$	3, \forall I
(3)	SHOW: $\forall y\{y=a \leftrightarrow y=a\}$	UD
(4)	SHOW: $b=a \leftrightarrow b=a$	SL

#17:

(1)	SHOW: $\forall x\forall y\{x=y \leftrightarrow 1z[z=x] = 1z[z=y]\}$	UBD
(2)	$[\rightarrow] a=b$	As
(3)	SHOW: $1z[z=a] = 1z[z=b]$	2,3,LL
(10)	$[\leftarrow] 1z[z=a] = 1z[z=b]$	As
(11)	SHOW: $a=b$	DD
(12)	SHOW: $a = 1z[z=a]$	
(13)	see problem 16	
(14)	SHOW: $b = 1z[z=b]$	
(15)	see problem 16	
(17)	$a = b$	10,12,14,Equ(=)

#18:

(1)	SHOW: $\forall x\{Fx \leftrightarrow F[1y[y=x]]\}$	UD
(2)	SHOW: $Fa \leftrightarrow F[1y[y=a]]$	DD
(3)	$Fa \leftrightarrow Fa$	SL
(4)	SHOW: $a=1y[y=a]$	
(5)	see problem 16	
(6)	$Fa \leftrightarrow F[1y[y=a]]$	3,4,LL

#19:

(1)	SHOW: $\forall x\forall y\{x=y \rightarrow [1zRxz = 1zRyz]\}$	UD2
(2)	SHOW: $a=b \rightarrow [1zRaz = 1zRbz]$	CD
(3)	$a=b$	As
(4)	SHOW: $1zRaz = 1zRbz$	DD
(5)	$1zRaz = 1zRaz$	REF=
(6)	$1zRaz = 1zRbz$	3,5,LL

#20:

(1)	$\exists x \forall y (Fy \leftrightarrow y=x)$	Pr
(2)	SHOW: $\forall x (Fx \leftrightarrow x=1xFx)$	UD
(3)	SHOW: $Fa \leftrightarrow a=1xFx$	DD
(4)	$\forall y (Fy \leftrightarrow y=b)$	1, $\exists O$
(5)	$b = 1yFy$	4, 1I
(6)	$b = 1xFx$	5, AV
(7)	$Fa \leftrightarrow a=b$	4, $\forall O$
(8)	$Fa \leftrightarrow a=1xFx$	6, 7, LL

#21:

(1)	$G[1xFx]$	Pr
(2)	$\exists x [x = 1xFx]$	Pr
(3)	SHOW: $\exists x (Fx \& Gx)$	8, $\exists I$
(4)	$a = 1xFx$	2, $\exists O$
(5)	$\forall x (Fx \leftrightarrow x=a)$	4, 1O
(6)	Fa	5, OO
(7)	Ga	1, 4, LL
(8)	$Fa \& Ga$	8, 9, SL

#22:

(1)	$G[1xFx]$	Pr
(2)	$\exists x [x = 1xFx]$	Pr
(3)	SHOW: $\exists x (\forall y (Fy \leftrightarrow y=x) \& Gx)$	8, $\exists I$
(4)	$a = 1xFx$	2, $\exists O$
(5)	$\forall x (Fx \leftrightarrow x=a)$	4, 1O
(6)	$\forall y (Fy \leftrightarrow y=a)$	5, AV
(7)	Ga	1, 4, LL
(8)	$\forall y (Fy \leftrightarrow y=a) \& Ga$	6, 7, SL

#23:

(1)	$G[1xFx]$	Pr
(2)	$\sim \exists x Gx$	Pr
(3)	SHOW: $\sim \exists x [x = 1xFx]$	ID
(4)	$\exists x [x = 1xFx]$	As
(5)	SHOW: \times	7, 8, SL
(6)	$a = 1xFx$	4, $\exists O$
(7)	Ga	1, 6, LL
(8)	$\sim Ga$	2, $\sim \exists O$

#24:

(1)	$\exists x \forall y \{Fy \leftrightarrow y=x\}$	Pr
(2)	SHOW: $\exists x (Fx \& x=1xFx)$	3, $\exists I$
(3)	$\forall y (Fy \leftrightarrow y=a)$	1, $\exists O$
(4)	Fa	3, OO
(5)	$a = 1yFy$	3, 1I
(6)	$a = 1xFx$	5, AV
(7)	$Fa \& a=1xFx$	4, 6, SL

#25:

(1)	$\exists x \{ \forall y (Fy \leftrightarrow y=x) \& Gx \}$	Pr
(2)	SHOW: $G[1xFx]$	DD
(3)	$\forall y (Fy \leftrightarrow y=a) \& Ga$	1, $\exists O$
(4)	$a = 1yFy$	3a, 1I
(5)	$a = 1xFx$	4, AV
(6)	$G[1xFx]$	3b, 5, LL

#26:

(1)	$\forall x\{Fx \rightarrow x=1xFx\}$	Pr
(2)	SHOW: $\exists xFx \rightarrow F[1xFx]$	CD
(3)	$\exists xFx$	As
(4)	SHOW: $F[1xFx]$	DD
(5)	Fa	3, \exists O
(6)	$a = 1xFx$	1,5, $\forall \rightarrow$ O
(7)	$F[1xFx]$	5,6,LL

#27:

(1)	$t = 1xFx$	Pr
(2)	$\exists x[x = t]$	Pr
(3)	SHOW: $\exists x\{Fx \& \forall y[Fy \rightarrow y=x]\}$	(7,8)&, \exists I
(4)	$a = t$	2, \exists O
(5)	$a = 1xFx$	1,4,LL
(6)	$\forall x(Fx \leftrightarrow x=a)$	5,1O
(7)	Fa	6,OO
(8)	SHOW: $\forall y[Fy \rightarrow y=a]$	UCD
(9)	Fb	As
(10)	SHOW: $b=a$	DD
(11)	$Fb \leftrightarrow b=a$	6, \forall O
(12)	$b=a$	9,11,SL

#28:

(1)	Fa	Pr
(2)	$\forall x\forall y\{(Fx \& Fy) \rightarrow x=y\}$	Pr
(3)	SHOW: $a = 1xFx$	4,1I
(4)	SHOW: $\forall x(Fx \leftrightarrow x=a)$	UD
(5)	SHOW: $Fb \leftrightarrow b=a$	6,11, \leftrightarrow I
(6)	SHOW: $Fb \rightarrow b=a$	CD
(7)	Fb	As
(8)	SHOW: $b=a$	DD
(9)	$(Fb \& Fa) \rightarrow b=a$	2, \forall O2
(10)	$b=a$	1,7,9,SL
(11)	SHOW: $b=a \rightarrow Fb$	CD
(12)	$b=a$	As
(13)	SHOW: Fb	1,12,LL

#29:

(1)	$t = 1xFx$	Pr
(2)	$\exists x\exists y\{x \neq y \& Fx \& Fy\}$	Pr
(3)	SHOW: $\sim \exists x[x = t]$	ID
(4)	$\exists x[x = t]$	As
(5)	SHOW: \times	9a,13,SL
(6)	$a = t$	4, \exists O
(7)	$a = 1xFx$	1,6,LL
(8)	$\forall x(Fx \leftrightarrow x=a)$	7,1O
(9)	$b \neq c \& Fb \& Fc$	2, \exists O2
(10)	$Fb \leftrightarrow b=a$	8, \forall O
(11)	$Fc \leftrightarrow c=a$	8, \forall O
(11)	$b=a$	9b,10,SL
(12)	$c=a$	9c,11,SL
(13)	$b=c$	11,12,LL

#30:

(1)	$\exists x \forall y ([y \neq t] \leftrightarrow y = x)$	Pr
(2)	SHOW: $\forall x ([x = t] \vee [x = \iota x [x \neq t]])$	UD
(3)	SHOW: $a = t] \vee [a = \iota x [x \neq t]$	\vee ID
(4)	$a \neq t \ \& \ a \neq \iota x [x \neq t]$	As
(5)	SHOW: \times	4b, 11, SL
(6)	$\forall y ([y \neq t] \leftrightarrow y = b)$	1, \exists O
(7)	$a \neq t \leftrightarrow a = b$	6, \forall O
(8)	$a = b$	4a, 7, SL
(9)	$b = \iota y [y \neq t]$	6, ι I
(10)	$a = \iota y [y \neq t]$	8, 9, LL
(11)	$a = \iota x [x \neq t]$	10, AV

#31:

(1)	$\forall x \exists y \forall z (Rxz \leftrightarrow z = y)$	Pr
(2)	$\forall x \forall y (Rxy \rightarrow Ryx)$	Pr
(3)	SHOW: $\forall x [\iota y Rxy = \iota y Ryx]$	UD
(4)	SHOW: $\iota y Ray = \iota y Rya$	DD
(5)	$\exists y \forall z (Raz \leftrightarrow z = y)$	1, \forall O
(6)	$\forall z (Raz \leftrightarrow z = b)$	5, \exists O
(7)	$b = \iota z Raz$	6, ι I
(8)	$b = \iota y Ray$	7, AV
(9)	SHOW: $b = \iota y Rya$	ι D
(10)	SHOW: $\forall y (Rya \leftrightarrow y = b)$	UD
(11)	SHOW: $Rca \leftrightarrow c = b$	DD
(12)	$Rac \leftrightarrow c = b$	6, \forall O
(13)	$Rca \rightarrow Rac$	2, \forall O2
(14)	$Rac \rightarrow Rca$	2, \forall O2
(15)	$Rca \leftrightarrow c = b$	12-14, SL
(16)	$\iota y Ray = \iota y Rya$	8, 9, LL

#32:

(1)	$\exists x \forall y (Fy \rightarrow y = x)$	Pr
(2)	SHOW: $\forall x (Fx \rightarrow x = \iota x Fx)$	UD
(3)	SHOW: $Fa \rightarrow a = \iota x Fx$	CD
(4)	Fa	As
(5)	SHOW: $a = \iota x Fx$	5, ι I
(6)	SHOW: $\forall x (Fx \leftrightarrow x = a)$	UD
(7)	SHOW: $Fb \leftrightarrow b = a$	8, 17 \leftrightarrow I
(8)	SHOW: $Fb \rightarrow b = a$	CD
(9)	Fb	As
(10)	SHOW: $b = a$	DD
(11)	$\forall y (Fy \rightarrow y = c)$	1, \exists O
(12)	$Fa \rightarrow a = c$	11, \forall O
(13)	$Fb \rightarrow b = c$	11, \forall O
(14)	$a = c$	4, 12, SL
(15)	$b = c$	9, 13, SL
(16)	$b = a$	14, 15, LL
(17)	SHOW: $b = a \rightarrow Fb$	CD
(18)	$b = a$	As
(19)	SHOW: Fb	DD
(20)	Fb	4, 18, LL

#33:

(1)	$\sim F[\iota xFx]$	Pr
(2)	$\exists xFx$	Pr
(3)	SHOW: $\exists x\exists y\{x\neq y \ \& \ Fx \ \& \ Fy\}$	ID
(4)	$\sim \exists x\exists y\{x\neq y \ \& \ Fx \ \& \ Fy\}$	As
(5)	SHOW: \ast	6,18,SL
(6)	Fa	2, \exists O
(7)	SHOW: $\forall x(Fx \leftrightarrow x=a)$	UD
(8)	SHOW: $Fb \leftrightarrow b=a$	9,14 \leftrightarrow I
(9)	SHOW: $Fb \rightarrow b=a$	CD
(10)	Fb	As
(11)	SHOW: $b=a$	DD
(12)	$\sim(b\neq a \ \& \ Fb \ \& \ Fa)$	4, $\sim\exists$ O2
(13)	$b=a$	6,10,12,SL
(14)	SHOW: $b=a \rightarrow Fb$	CD
(15)	$b=a$	As
(16)	Fb	6,15,LL
(17)	$a = \iota xFx$	7, ι I
(18)	$\sim Fa$	1,17,LL

#34:

(1)	$\forall x(Fx \rightarrow Gx)$	Pr
(2)	$\exists x\forall y(Gy \leftrightarrow y=x)$	Pr
(3)	SHOW: $\forall x(Fx \rightarrow [x = \iota xGx])$	UCD
(4)	Fa	As
(5)	SHOW: $a = \iota xGx$	11, ι I
(6)	$Fa \rightarrow Ga$	1, \forall O
(7)	$\forall y(Gy \leftrightarrow y=b)$	2, \exists O
(8)	$Ga \leftrightarrow a=b$	7, \forall O
(9)	$a=b$	4,6,8,SL
(10)	$\forall y(Gy \leftrightarrow y=a)$	7,9,LL
(11)	$\forall x(Gx \leftrightarrow x=a)$	10,AV

#35:

(1)	$\exists x\forall y\{Fy \rightarrow y=x\}$	Pr
(2)	SHOW: $\forall x(Fx \rightarrow F[\iota xFx])$	UD
(3)	SHOW: $Fa \rightarrow F[\iota xFx]$	CD
(4)	Fa	As
(5)	SHOW: $F[\iota xFx]$	DD
(6)	SHOW: $\forall x\{Fx \leftrightarrow x=a\}$	UD
(7)	SHOW: $Fb \leftrightarrow b=a$	8,17 \leftrightarrow I
(8)	SHOW: $Fb \rightarrow b=a$	CD
(9)	Fb	As
(10)	SHOW: $b=a$	DD
(11)	$\forall y(Fy \rightarrow y=c)$	1, \exists O
(12)	$Fa \rightarrow a=c$	11, \forall O
(13)	$Fb \rightarrow b=c$	11, \forall O
(14)	$a=c$	4,12,SL
(15)	$b=c$	9,13,SL
(16)	$b=a$	14,15,LL
(17)	SHOW: $b=a \rightarrow Fb$	CD
(18)	$b=a$	As
(19)	SHOW: Fb	DD
(20)	Fb	4,18,LL
(21)	$a = \iota xFx$	6, ι I
(22)	$F[\iota xFx]$	4,21,LL