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THE CONCEPT OF VARIATION AND MISCONCEPTIONS IN CARTESIAN GRAPHING*

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Running Head: The Concept of Variation

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THE CONCEPT OF VARIATION AND MISCONCEPTIONS IN CARTESIAN GRAPHING

In this paper I propose some basic elements of a model of knowledge structures used in comprehending and generating graphs, with emphasis on the concept of covariation and on the analogical character of graphical representation. I then use this competetence model to attempt to organize and interpret some of the existing literature on misconceptions in graphing. Two types of common misconceptions, treating the graph as a picture, and slope-height confusions, will discussed, as will the earliest recorded use of graphs in the work of Oresme in 1361. One of the motives for studying concepts used in graphing is that it may help us understand the nature of the more general concepts of variable and function and the role that analogue spatial models play in representation.

Ponte (1984) found that many secondary students and pre-service teachers have difficulty in making connections between graphical and numerical data. The ability to interpret graphs is important for mathematical literacy and for understanding the concepts of function and variable, as well as for developing basic concepts in calculus. A minimal interpretation of the relationship between an <u>algebraic equation</u> and its <u>graph</u> is quite simple: the graph displays the coordinates of points corresponding to pairs of numbers belonging to the solution set for the equation. The problems considered here, however, ask the student about the relation between a <u>practical situation</u> and its <u>qualitative graph</u>. The nature of this cognitive relationship is less clear. In particular it raises two important problems: (1) How are mental linkages formed between non-numerical aspects of a practical situation and a graph? (2) How are the concepts of <u>variation</u> and <u>covariation</u> represented? An example of a problem in qualitative graphing (due to G. Steven Monk) is: "Draw the shape of the graph of speed vs time for a bike rider coasting over the edge of a hill." One challenge then is to describe the knowledge structures that a successful student might use to represent ideas of covariation like "the speed is increasing with time."

COMPETENCE MODEL

Static model. Figure 1a shows a model, proposed in Clement (1986), of four types of static knowledge structures which could be used to comprehend such a graph. The model depicts the knowledge needed to comprehend the meaning of only one point in the graph, but this could be repeated for other points. Conception (1) in Figure 1a is a naive practical representation incorporating everyday knowledge about the problem situation based on one's concrete experience with watching and riding bicycles, including the sensations of speeding up as one rides down a hill. Here I will simply refer to each knowledge structure as a conception, but they can also be thought of as occurring at different levels of representation. Conception (2) represents the idea that at a particular time, the bicycle is at a particular speed. Here, the subject must: (a) have adequately developed concepts for speed and time; (b) be able to isolate these variables in the problem situation; and (c) understand that the specified values occur together. Conception (3) shows the subject forming an analogue spatial model for each variable. The notation here is inspired by but not equivalent to that of Driver (1973). Time and speed are represented as lengths of line segments. A particular length representing the time is associated with a particular length representing the speed at that time. The segments in (3) can then be mapped onto distances in the graph in conception (4). Here the length representing time must be mapped onto the distance of a point from the y axis, and the length representing speed must be mapped onto distance from the x axis.

(Figures 1a and 1b about here)

Thus in this view a graph is essentially a model of a relationship between variables based on a metaphorical representation of variable values as lengths of line segments. The model is analogical in the sense that the relationships between line segments in the model are isomorphic to the relationships between variable values in the problem.

<u>The concepts of variation in a dynamic model</u>. One limitation of the model in Figure 1a is that it only shows correspondences between individual static values of the variables. The idea of variation seems to be missing. What is needed to go beyond a piecemeal understanding of what each individual point on the graph means is a representation of correspondences between <u>changes</u> in variables. An attempt to model some aspects of the idea of variation is illustrated in Figure 1b. Here conception (1) is the same as in Figure 1a, but in conception (2) one finds the variables of change in time and change in speed instead of simply time and speed. Conception (2) in Figure 1b embodies the idea that as the time elapsed increases the speed of the bicycle increases. (The subject may, for example, image the picture of a moving second hand on a stopwatch along with an increase in the perceived speed of the bicycle, including a rising pitch in the sound of the tires, etc.). That is, an increase in time is associated with the expectation of an increase in speed. This is the the idea that we have been seeking an explicit representation for--the idea that "the speed is increasing with time"--in which the variables are truly thought of as (co-)varying. It should be emphasized that this is essentially a direction-of-change idea that need not be quantitative or sophisticated in its most basic form. For example, one can know that "the more I work, the more tired I'll get," without thinking about ways to measure work or fatigue quantitatively.

In conception (3) the correspondence of direction of change in time with direction of change in speed is represented in terms of the spatial metaphor of line segments. An increasing value of speed is represented by an increasing length of line segment. Thus whereas the metaphor for the value of a variable was length, the metaphor for an increase in value is an increase in length, and the metaphor for continuous increase in value is a continuously growing line segment. In conception (4) the increases in the two variables are mapped onto changes in the x and y coordinates of the graph as the graph point moves. Thus motion of a point on the graph in two dimensions is used to represent changes in two variables at once. In this particular problem vertical motion on the graph represents a change in another kind of motion in the problem situation--an increase in speed. Notice these conceptions could be used to generate or interpret basic features of a qualitative graph without any numerical information on the axes. Students may use these more dynamic concepts in Figure 1b in conjunction with the more static concepts in Figure 1a.

In summary, this competence model proposes a hypothesis for some of the basic knowledge structures that are sufficient for comprehending qualitative graphs, and indicates that even at this low level, the knowledge required can be somewhat complex.

(Figure 2 about here)

GRAPHING MISCONCEPTIONS

Link to an Incorrect Graph Feature

In a study of science-oriented college students who anticipated difficulty in taking science courses, McDermott, Rosequist, Popp and van Zee (1983) asked the following question referring to the graph in Figure 2: "At the instant t = 2 seconds, is the speed of object A greater than, less than, or equal to the speed of object B?" As many as half the students answered "greater than" incorrectly. One interpretation of this error is that students are mistakenly using the graphical feature of height instead of slope to represent speed. The model proposed here for one cognitive source of this error is a misplaced link between a successfully isolated variable and an incorrect feature of the graph, as illustrated in Figure 2, solution 1. This figure shows the upper three levels of the static model for graph comprehension just discussed (Figure 1a). Here the link between levels 3 and 4 is faulty, as indicated by the (-) sign in the figure. For this problem, success based on understanding would seem to involve attending to covariation ideas. The problem can be solved qualitatively by associating speed with change in position of the object over time and using the dynamic conceptions in Figure 1b to decide that there is a greater change in position with time in graph B. However, as McDermott, et. al., point out, it is difficult to assign a single cause to this error since students have been observed to confound the physical concepts of speed and relative position at a conceptual level (level 2) in other tasks which do not involve graphs. In that case the two variables would not have been isolated successfully. This second interpretation is illustrated in Figure 2, solution 2.

(Figure 3 about here)

The height-for-slope error has also been reported by Janvier (1978). In one of the problems in his extensive thesis on graphing, he asked students to draw graphs of height vs. time for the water level in different jars being filled with water. In one task he showed them the graph (A) in Figure 3 for a wide jar and asked them to draw the graph for a narrower jar being filled from the same water source. Some of the students drew the parallel line of dots (B) in Figure 3 instead of a line (C) with an increased slope. Again, success based on understanding here would appear to require the types of dynamic concepts shown in Figure 1b, not just the static concepts in Figure 1a. Other errors not discussed here, such as height-for-difference, slope-for-height, and slope-for-curvature substitutions also fall into this general category of a misplaced link between a successfully isolated variable and an incorrect feature of the graph.

Treating the Graph as a Picture

In another type of error the student appears to treat the graph as a literal picture of the problem situation. This error can occur, for example, when students are asked to draw a graph of speed vs. time for a bicycle traveling over a hill. In classroom observations of a college science course we noted that many students would simply draw a picture of a hill. This can happen even when the student first demonstrates the ability to describe the changes in speed verbally.

This type of error has been discussed by Kerslake (1977), Janvier (1978), and McDermott, et. al. (1983). For example, Janvier interviewed students solving a problem about a graph of speed vs. distance travelled for a race car going around a track with a number of curves in it. (The car was assumed to slow down for each curve.) Many students erred when asked if they could tell how many curves were in the race track by looking at the oscillating graph and counting the number of curves in the graph itself.

(Figure 4 about here)

In making the bicycle problem error of drawing the shape of the hill, the student appears to be making a figurative correspondence between the shape of the graph and some visual characteristics of the problem scene, as shown in Figure 4. This simplistic approach contrasts with the relatively complicated process in the competence model shown in Figures 1 and 2. Notice that in the competance model the student must differentiate between and coordinate at least two separate images: (a) the problem situation and (b) the graph. Students making the error apparently have difficulty in maintaining this differentiation.

Two types of graph-as-picture errors. In classroom observations with college remedial mathematics students, we found that many students generating a graph such as the one shown in Figure 5 would say incorrectly that the two cars represented in the graph were passing each other at the point where the graphs cross. Here they seemed not to be drawing a whole picture, but to be mapping a local feature (same location for both cars) onto a similar feature of the two graphs (same location for both graphs at their point of intersection).

(Figure 5 about here)

We can then distinguish between two types of graph-as-picture errors. We refer to the picture of the hill error as a global correspondence error. Here the shape of an entire problem scene is matched to the shape of the entire graph in a global manner, as shown in Figure 4. This contrasts with the car problem error, which I call a local correspondence error. Here a local visual feature of the problem scene (same location) is matched to a specific feature of the graph (intersection). In both cases, however, the figurative matching process producing the error contrasts with the more complex process of metaphorical and functional symbolization shown in Figures 1 and 2.

CONNECTIONS TO HISTORY AND TEACHING

There is a connection between the above competence model, strategies for introducing graphs to students, and the earliest work on graphs in the history of mathematics. Oresme (1323-1382) is credited with producing the first written record of a graph in Paris in 1361. He used the graph shown in Figure 6 to solve the general form of the following problem which involves computing the average speed of a uniformly accelerating object: An object starts from rest and gains speed at a uniform rate for 4 seconds, after which it is going 20 feet per second. How far did it travel in this time?

(Figure 6 about here)

The vertical lines in Oresme's diagram represent the values of speed at equal time intervals, and the areas between vertical lines then represent the distance travelled in each interval. Realizing that the triangular area abc is equal to the rectangular area abde allowed him to conclude that the average speed v_a would be half of the final speed, or 20 m./sec. in this case. Although axes are not present, the basic idea of representing different variables with vertical and horizontal line segments is present.

One of the deep insights here is to realize, as Oresme put it, that "everything measurable can be represented by a line." This is the basic analogical representational leap represented as the link between levels two and three in Figure 1a. The length of a line can be used to represent the value of a variable, even when the variable itself is not a metric length. The idea that relationships between scalars such as speed, temperature, income, time, etc. can be modelled metaphorically by relationships between line segments is an immensely powerful one.

There is evidence that the origin of the idea for Oresme's diagram may also have been analogical in nature. Oresme refers to the horizontal and vertical positions in his diagram as "longitudes and latitudes", implying that the idea for the graph may have originated as an analogue to the existing coordinate system for maps of the world.

Oresme's diagram uses vertical lines in the same way as in level 4 in Figure 1a. The lines show explicitly the representation of speed as a vertical length but such vertical lines are usually omitted in modern notation. Instead, the curve on the graph represents the infinite set of uppermost points of all such vertical lines. Interestingly, Easley (personal communication) has found that 10 year old children who have had little or no training in graphing can represent quantities such as the speed of a car fairly readily using such vertical lines, when the horizontal dimension is used to represent distance. This suggests that such vertical line graphs are a fairly natural and intuitive symbolization device and that they would be a useful starting point in training students to draw graphs. More generally the links between the four levels shown in Figures 1a and 1b directly suggest specific subareas of focus for generating instructional activities. Thus the competance model appears to be useful in helping us to understand historical developments as well as new instructional strategies.

CONCLUSION

Here I have discussed only two of the many observable types of graphing errors. The partial taxonomy of misconceptions in graphing being discussed now looks like:

Type 1) Link to Incorrect Graph Feature

- a) Height for Slope
- b) Slope for Height
- c) Height for Difference in Height
- d) Slope for Curvature

(Only examples for 1a above were discussed for Type 1 misconceptions)

Type 2) Graph as Picture

- a) Global Correspondence
- b) Local Correspondence

More generally, one can refer to all of the above misconceptions as non-standard symbolization strategies. Non-standard symbolization strategies also occur in the area of algebra word problems (Clement, 1982). Such strategies are in some cases reasoned attempts to produce an intuitive notation, and are a source of non-random error patterns. When this is the case they act as "natural distractors" which can compete with the accepted symbol system conventions and make school learning more difficult. In addition it was noted that students can confound physical context variables within level (2) in the competence model, a difficulty that can occur in problems with or without graphs.

Several key features in the competence model were noted:

(A) In addition to corresponces between static value pairs in a function (Figure 1a), the model includes a separate mode of representation involving correspondences between <u>changes</u> in values (Figure 1b). In its most basic form, this simply tells the subject whether Y will increase or decrease with a change in Y. But this is the beginning of the powerful idea of covariation--of how Y varies with X.

(B) Four levels of knowledge structures are used within each mode: (1) practical representations; (2) isolated variable correspondences; (3) length models; and (4) graph models.

(C) The relation between level (2), isolated variable correspondences, and level (3), length models, is an analogical one. This analogical relationship stands at the core of what we mean when we say that the graph is a "model" of the relationship between variables in the problem situation for the subject.

(D) There is the potential to set up incorrect linkages between level (3) and the graph itself at level (4) leading to errors such as slope/height confusion.

(E) There is the potential to "short circuit" the processes in levels (2) and (3) by making a superficial figurative connection directly between levels (1) and (4), which leads to "graph as picture" errors.

The use of the concept of length as a model of continuous quantity in mathematics appears to be ubiquitous. The lower three levels of the competence model in Figures 1a and b may provide a way to conceptualize a foundation for the more general concept of function and covariation which underlies other representational systems besides graphs. The extension, refinement, and empirical evaluation of such competence models of basic knowledge structures are important tasks for future research.

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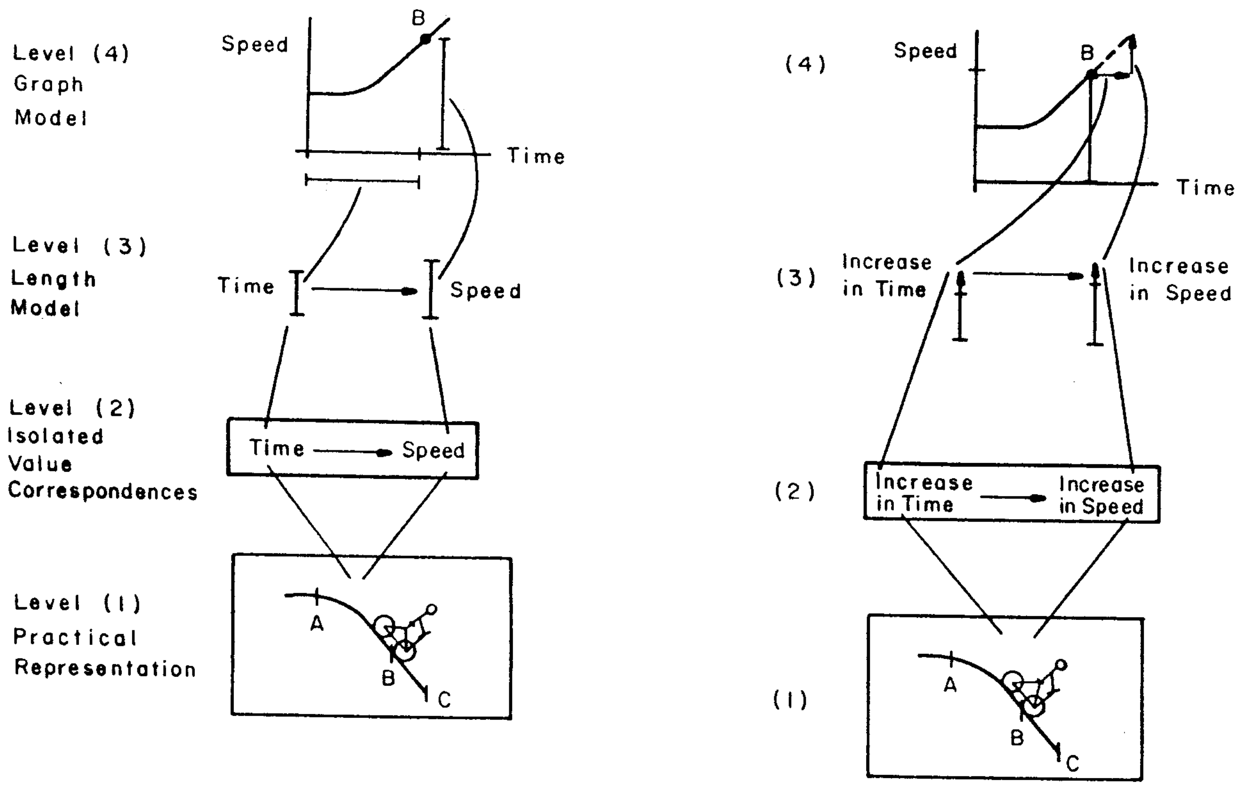

Figure 1a

Figure 1b

Figure 1.

Competence Model of: (A) Static Conceptions Used in Graphing; (B) Dynamic Conceptions Used in Graphing

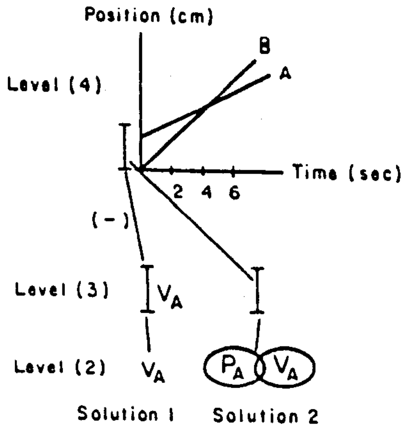

Figure 2.

Conceptions Involved in Slope-Height Confusion Error

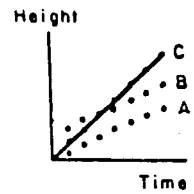

Figure 3.

Water Jar Problem: A) Presented Graph; B) Incorrect Answer; C) Correct Answer

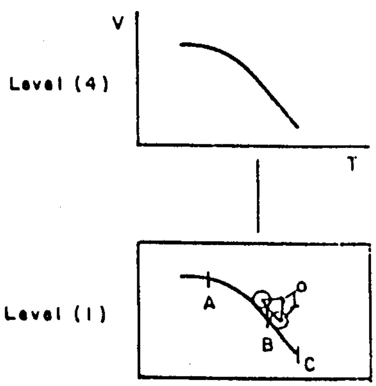

Figure 4.

Conceptions Involved in Graph as Picture Error

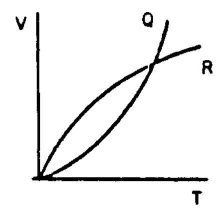

Figure 5.

Car Speed Problem. Student says one car is passing the other where graphs intersect

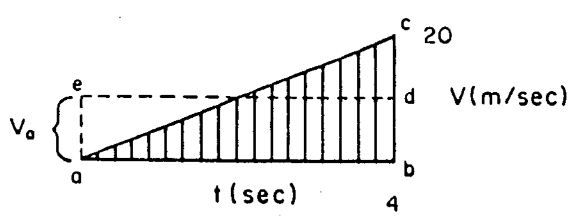

Figure 6.

Form of Earliest Recorded Graph due to Oresme in 1361. Object under constant acceleration has average speed v_a