USE OF ANALOGIES AND SPATIAL TRANSFORMATIONS BY EXPERTS IN SOLVING MATHEMATICS PROBLEMS

John Clement

Department of Physics and Astronomy
University of Massachusetts
Amherst, Massachusetts 01003

August 2, 1983

To appear in the proceedings of the fifth annual meeting of the International Group for the Psychology of Mathematics Education, North American Chapter, Montreal, 1983.

Preparation of this paper was supported in part by a grant from the National Science Foundation program for Research in Science Education SED80-16567.

USE OF ANALOGIES AND SPATIAL TRANSFORMATIONS BY EXPERTS IN SOLVING MATHEMATICS PROBLEMS

John Clement

University of Massachusetts/Amherst

ABSTRACT

Since Polya, Wertheimer, and Hadamard's descriptions of qualitative reasoning strategies used by scientists and mathematicians, very little data has been collected on whether these strategies are actually used by experts. This study used video-taped thinking-aloud interviews to examine the problem solving strategies of professors and advanced graduate students in technical fields. Evidence from these interviews documents the use of analogies, visual transformations, extreme cases, and other plausible reasoning strategies used by experts. In the case of analogies, it was found that there is more than one method for generating analogies, and that in addition, the process of critically evaluating the analogy is very important.

Considering helpful analogous cases and extreme cases, breaking problems into analyzable parts, and performing simplifying spatial transformations are key reasoning processes in solving non-trivial problems. These processes allow talented scientists to attack problems outside the domain of familiar problems for which they have established algorithmic procedures. They allow them to attack problems they have never seen before, giving them a degree of problem solving power and scope that is truly impressive. Previous reports [1, 2, 3, 4] have documented the fact that these qualitative reasoning processes are used by expert scientists in solving physics problems. This paper examines the possibility that empirical evidence for these processes can also be found in the case of experts' solutions to mathematics problems. I will first describe results from the earlier physics problem study, and then describe those from the current mathematical problem study.

EXPERT REASONING ON A PHYSICS PROBLEM

In the previous study, ten expert subjects were asked to solve the following problem.

Spring Coils Problem

A weight is hung on a spring. The original spring is replaced with a spring made of the same kind of wire, with the same number of coils, but with coils that are twice as wide in diameter. Will the spring stretch from its natural length, more, less, or the same amount under the same weight? (Assume the mass of the spring is negligible compared to the mass of the weight.) Why do you think so?

All subjects were advanced doctoral candidates or professors in technical fields. The study concentrated most on documenting and analyzing the use of analogies.

Some examples of analogies generated for this problem are as follows. One subject thought about a horizontal saw blade held fixed at one end and loaded with a weight at the other end. He felt that a long blade would bend more easily than a short one, and this indicated to him that the wider spring might stretch more. Other examples of proposed analogies were that a longer horizontal "hairpin" shaped wire would extend more than a shorter one (see Fig. 1) and that a larger single "square coil" would stretch more than a smaller one.

Another subject examined the relationships between coil diameter, coiling angle, and wire length by thinking about mountain roads winding up narrow and wide mountains. The correct answer to the problem is that the wide spring will stretch farther (the stretch in fact increases with the cube of the diameter). This seems to correspond to most people's initial intuition about the problem. However, explaining <u>why</u> the wide spring stretches more (and explaining exactly where the stretch of the spring comes from), is a much more difficult task when taken seriously.

Some of the findings from this study were as follows.

(1) Spontaneously generated analogies were observed to play a significant role in problem solutions of scientifically trained subjects. Seven of the ten subjects generated at least one salient analogy.

(2) The subjects generated a large variety of analogous cases. Not all of the analogies were to situations familiar to the subject. Some were novel cases in the form of Gedanken experiments that appeared to be invented by the subject.

(3) In addition to the initial process of <u>generating</u> an analogy, there is a second process that is just as important in expert problem solving, that of <u>critically evaluating</u> the validity of the analogy.

(4) Analysis of the transcripts indicated that there was more than one type of analogy generation method: two of these methods are the associative leap and the generative transformation.

The subject using an <u>associative leap</u> jumps to an analogous situation that differs in many ways from the original problem. For example, one subject compared the wide and narrow springs to two blocks of foam rubber, one made with large air bubbles and one made with small air bubbles in the foam. He had a strong intuition that the foam with large air bubbles would be easier to compress.

However, the associative leap was not the only analogy generation method observed. Another pattern was observed in which a subject

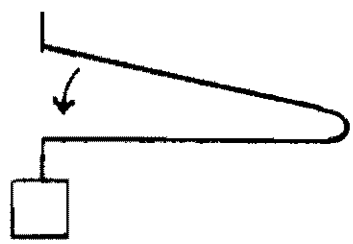

Figure 1

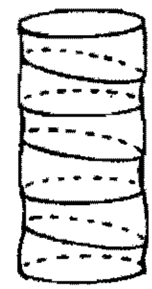

Figure 3

DOUGHNUT PROBLEM

Compute the volume of the torus (doughnut) below without taking an integral. Give an approximate answer if you cannot determine an exact one.

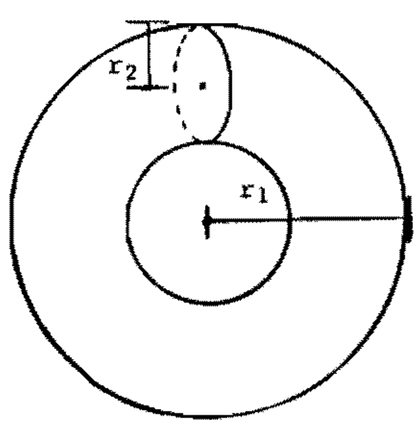

Figure 2

generates an analogy via a transformation which "warps" or changes the original situation A to produce the analogous situation B. These occur when a subject modifies an aspect of the original situation A that was previously assumed to be fixed. For example, some subjects "unrolled" the spring into a horizontal wire and thought about how much the wire would bend if the wire were held at one end and the weight were placed on the other end.

(5) Some subjects gave evidence of using spatial reasoning by referring spontaneously to imagining or picturing situations they were thinking about.

(6) Extreme cases such as considering a very narrow or very wide spring were observed as well. These were effective in adding confidence to a subject's prediction for the problem.

EXPERT REASONING ON A MATHEMATICS PROBLEM

A set of expert subjects were also asked to solve the mathematics problem shown in Figure 2.

All subjects were advanced graduate students or professors in technical fields. This paper reports on preliminary results from five of the solutions to this problem and looks in detail at one of the solutions. We are only beginning to analyze these protocols, but already some behaviors parallel to those in the solutions of the spring problem are emerging. A common analogy generated for this problem was to consider the case of the "straightened out" torus in the shape of a cylinder. Subjects conjectured that the volume of these two objects might be the same. The condensed transcript excerpt S5 below gives one example of this approach.

001 S: ...Ok. So here's a doughnut. Now the question is how to get its volume. Uhh, the first thing that comes to mind is that it's probably pretty close to a worm, er, I mean a
002 S: cylinder. Where you know, if you laid out the doughnut on the ground, uh, if you cut it open and laid it out, it would basically be the area of the base times the length around the middle. So let's see, I'll put down here number 1 is my first approximation which in fact may turn out to be the exact thing. Uh, I'll just turn it into a cylinder.

003 I: Mmm

004 S: In other words my hypothesis here is this volume of dough-
nut---and so that would--Pi r_2 squared would be the bottom
of the cylinder and then, uh, you know, I think the rele-
vant length of the cylinder would not be r_1 but the
distance uh, uh, to the middle there namely r_1 oops, r_1
minus r_2--uh, right that gets us to the middle, and then
times 2 Pi... [Writes $V = \pi r_2^2 (r_1 - r_2)(2\pi)$]

005 I: ...When you thought about the cylinder, do you know how
that arose? Did it just sort of flash in your head?

006 S: Well I mean I in fact, er, this little r_2 was drawn here
very nicely so I just imagine the knife cutting it open
and you know, laying it out...

008 S: ...I mean to say I thought of cutting it at one edge and
it sort of flopping down and then the uh, the doughnut
becomes a cylinder.

009 I: Ok

010 S: And so I guess uh, you know with er, I don't think I've
made any real algebraic mistakes here. I mean I think
this er, probably I mean I feel pretty confident about
that answer because what happens when you open the doughnut
is that the, the top of the doughnut kind of expands a
011 S: little. I mean, er, the part of the doughnut which was the
inside stretches out a bit but then on the other hand, the
part of the doughnut which was the outer perimeter gets
crunched a little bit. And so probably those two things
cancel. And you know, if the world is made correctly.
But now I guess-

012 I: Just put a confidence number on it at this point and then
you can go on.

013 S: Ok. I would say that on that I'm probably like 80% confi-
dent. Uhh, now what would happen if you did various
things to the doughnut? Certainly you could argue that
the answer gets closer and closer. I mean that this
answer [the formula for a cylinder] is closer and closer to
the correct one if uh, you know, if r_1 is much, much
greater than r_2, then in that limiting case, you've got
to get this. Because that's just gonna be like you know
it's gonna approach being a cylinder more and more. So
whatever the correct answer is, it's got to have that
[formula] as a limiting case if r_1 is much greater than
r_2...

026 S: ...I suppose the other way you could imagine doing if you
wanted to break it up would be to break it up into little
wedges of doughnuts. So that if you were looking at it
that way then you said ok, here's er, here's another
027 S: infinitesimal element which is a wedge like that, both
faces of which have an area of uh, Pi r_2 squared and
we'll call that now dZ and again you know, it would boil
down to that same equation again, if you added them all up

032 I: Yeah

033 S: into equation 2, right? And dZ essentially would be the midpoint, distance there. Ok. I mean I think I've er, I mean I think my confidence level at this point would be --like 95%.

The cylinder idea fits the definition of a spontaneous analogy as used here because it is a case which differs from the doughnut with respect to a feature (the shape of the doughnut) that is a fixed feature in the original problem. The observational definition of a spontaneous analogy used in both studies was the following:

(1) the subject, without provocation, considers another situation B where one or more features ordinarily assumed fixed in the original problem situation A are different; (2) the subject indicates that certain structural or functional relationships (as opposed to surface features alone) may be equivalent in A and B; and (3) the related case B is described at approximately the same level of abstraction as A.

The act of violating a feature previously assumed to be fixed is the creative aspect of producing an analogy. The difficulty of such acts is presumably the underlying source of Wertheimer's finding [5] that many students do not think to modify the shape of a parallelogram in order to compute its area.

RESULTS

Analogies. All 5 of the subjects spontaneously considered the analogy of the cylinder. Thus, spontaneous analogies were observed to occur in a number of expert solutions to a mathematics problem. Some subjects, such as S5, critically evaluated the analogy relation they had constructed by questioning whether the volume they had constructed was really the same as that of the torus. Earlier it was stated that the process of criticizing and evaluating an analogy is just as important as the process of generating it in solving science problems. This appears to be true in the case of mathematics problems as well. Subjects who think about an equivalent cylinder must choose a cylinder of the right length, and they often take pains to critically evaluate their choice of length.

For example, S5 above chooses the central or "average" circumference of the torus, $2\pi(r_1 - r_2)$, as the length of the cylinder. But he then evaluates this choice in lines 10 and 11 by giving a compensation argument about the inside stretching and the outer part getting "crunched". He also evaluates his prediction further by using an extreme case in line 13. If we call the circular curve passing through the centers of the circular cross sections of the torus its midline, then S5 seems to be implying that since the curvature of the midline gets less and less as r_1 gets very large, a local section of the torus will look more and more like a cylinder.

Analogy generation methods. A striking feature of this particular protocol is the explicit evidence for generating the analogy via a transformation rather than via an associative leap. The most explicit criterion used to code for a transformation is the subject referring to changing a fixed feature of the problem. Here the subject makes statements like: (line 2) "If you cut it open and laid it out...", and "...I'll just turn it into a cylinder...", referring explicitly to changing the shape. This method contrasts to an associative leap, where the subject is simply reminded of a familiar situation via a direct association (for example, if the subject were reminded of another problem he had seen about a torus). This protocol provides fairly explicit evidence for the act of generating an analogy via a transformation.

Spatial reasoning. The protocol also provides evidence for the role of spatial reasoning. First and most obviously there are references to spatial relations between objects that are primarily qualitative in nature, such as (line 2) "You know, if you laid out the doughnut on the ground"; and (line 11) "The part of the doughnut which was inside stretches out a little bit." Passages of this kind suggest that the subject is (1) imagining manipulating concrete or idealized objects, and (2) experiencing the anticipated outcomes of his manipulations via imagery or spatial reasoning.

Secondly, there are more explicit references to imagery. An imagery report is defined as occurring when the subject refers to imagining, picturing, "remembering a diagram for", hearing, or "feeling what it's like to manipulate" a situation. We refer to a dynamic imagery report if the reference is to imagining a situation which does not remain fixed but changes with time. In this study we are concerned here with spontaneous imagery reports where the interviewer does not ask the subject whether an image was used.

Examples of dynamic imagery reports in the protocol are: (line 6) "I just imagine the knife cutting it open.."; and (line 26) "You could imagine...if you wanted to...break it up into little wedges of doughnuts."

Thus it is possible to point to evidence in protocols which supports the hypothesis that spatial reasoning involving imagery of a qualitative and dynamic nature is involved in expert problem solving.

"Creative cutting". The reference to cutting the doughnut into wedge-shaped pieces and computing their volumes documents the strategy of breaking a problem into parts--in this case the subject partitions the problem symmetrically into a number of equivalent parts. One could treat this as the simple application of a heuristic, but the trouble with the heuristic "break the problem into simpler parts" is that it does not tell you which parts to form. Such an act can require considerable creativity and ingenuity. (Other subjects have been observed to attempt other partitions as well, which will be discussed in a future paper.)

By speaking of an infinitesimal slice, S5 is in danger of breaching the request in the instructions to refrain from taking an integral. Indeed, creative cutting and partitioning of just this kind is an essential skill for applying the integral calculus to non-trivial situations. As in the case of analogies, the breaking into parts process is in effect an attempt to find a conserving transformation which leaves one with one or more simpler problems. It is interesting to note that the wedges can be stacked alternately as shown in Fig. 3. In the limit, this can provide a connecting plausibility argument for the validity of the original analogy to a cylinder of length $2\pi(r_1 - r_2)$. The discovery of such conserving spatial transformations and equivalences can be a great source of satisfaction and appreciation for the interconnectedness of mathematical ideas. It was physical-spatial transformations of this type that apparently allowed Archimedes to develop (amazingly) the functional ideas underlying the integral calculus over two thousand years ago. (See description in [6].)

CONCLUSION

The ability to perform relevant spatial transformations, the ability to consider and evaluate analogous cases and extreme cases, and the ability to break problems into parts intelligently are crucial skills for solving non-trivial problems. The fact that experts use these processes can be documented in problem solving protocols, and the nature of the processes can be analyzed. In saying that these strategies played an important role in a number of the problem solutions we mean that they were involved in a serious attempt to understand or solve the problem and were not just proposed by the subjects as an ornamental side comment or as a check on a firm answer. We are currently investigating the abilities of students as well as experts to use these processes, and as we understand more about their nature, we should be able to design more effective instructional experiences which foster them.

REFERENCES

[1] Clement, J., "Analogy Generation in Scientific Problem Solving", proceedings of the Third Annual Conference of the Cognitive Science Society, Berkeley, California, 1981.

[2] Clement, J., "Analogical Reasoning Patterns in Expert Problem Solving", proceedings of the Fourth Annual Meeting of the Cognitive Science Society, Ann Arbor, Michigan, 1982.

[3] Clement, J., "Spontaneous Analogies in Problem Solving, Part I: The Progressive Construction of Mental Modes", presented at AERA annual meeting, New York City, March, 1982.

[4] Clement, J., "Observed Methods for Generating Analogies in Scientific Problem Solving", presented at AERA annual meeting, Montreal, Canada, March, 1983.

[5] Wertheimer, M., Productive Thinking. Harper & Row Publishers, New York: 1959.

[6] Polya, G., Mathematics and Plausible Reasoning, Vol. 1: Induction and Analogy in Mathematics. Princeton University Press, New Jersey: 1954, p. 155.

[7] Hadamard, J., The Psychology of Invention in the Mathematical Field. Princeton University Press, New Jersey: 1945.