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TRANSLATION DIFFICULTIES IN LEARNING MATHEMATICS

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Recent mathematics textbooks have increasingly emphasized applications. Mathematical modeling is a critical component of applications, as Rubin [8] points out. Unfortunately, results of written tests and of interviews suggest that the modeling process is far more complex than is generally imagined. In fact, rather than being an immediate aid to learning mathematics, the process of "translation" between a practical situation and mathematical notation presents the student with a fresh difficulty that must be overcome if the application (or the mathematics) is to make any sense to the student in the long run.

We became aware of this problem during a series of videotaped interviews [1] in which college science students were asked to talk aloud while working on simple problems. Using these interviews, we developed a set of written questions (shown in Table 1) that were put to freshman engineering majors from two universities, most of whom were taking calculus. The fact that fewer than 50 percent of these students could solve the problems indicates the difficulty of translating into and out of algebraic notation. The predominant error on the second two problems was reversing the variables in an equation, e.g., $4C = 5S$ instead of $5C = 4S$ in problem 2 in Table 1. The presence of such a consistent pattern suggests that the difficulty is not simply one of misunderstanding English. Furthermore, errors are also high for translations from pictures and data tables [2].

TABLE 1

	# of students tested	% correct
1. Write an equation of the form P_A = ______ for the price you should charge adults to ride your ferry in order to take in an average of D dollars on each trip. You have the following information: Your customers average 1 child for every 2 adults. Childrens' tickets are half-price. Your average load is L people (adults and children). Write your equation for P_A in terms of the variables D and L only.	497	12
2. Write an equation using the variables C and S to represent the following statement: At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel. Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered.	497 most common error: $4C = 5S$	39
3. Weights are hung on the end of a spring and the stretch of the spring is measured. The data are shown in the table below: Stretch S (cm) — Weight W (g) 3 — 100 6 — 200 9 — 300 12 — 400 Write an equation that will allow you to predict the stretch (S) given the weight (W).	381 most common error: $3S = 100W$	42

To make certain that our results were not the consequence merely of inattention, one of us included a problem similar to problem 2 as part of a final examination in calculus. More than 40 percent failed this problem.

Sources of Error—The Reversal. At first, the students' difficulty in translation greatly surprised us. We had not been looking for it and, in fact, had assumed that college students could translate between English and algebra, at least in simple situations. After discovering the difficulties, however, we recalled that students are rarely asked to *construct* a formula or to interpret one in a significant way. They are usually given a formula or asked to select the appropriate formula from a well-defined (and very short) list and then to manipulate it using techniques from algebra or calculus. The one place in the secondary school mathematics curriculum where translation plays a large role is in doing "story problems." But we suspect that teachers have tended to deemphasize such problems because students find them difficult.

To investigate the source of the errors we had observed, we collected data on the following simpler problem:

The Students-and-Professors Problem. Write an equation for the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

On a written test with 150 calculus-level students, 37 percent missed this problem, and two-thirds of the errors took the form of a reversal of variables such as $6S = P$. In a sample of 47 nonscience majors taking college algebra, the error rate was 57 percent.

We also interviewed 15 students who were asked to think out loud while solving problems like this one. The videotaped records provide a much more detailed account of the students' thought processes than is possible with written tests. They allow one to distinguish between insignificant careless errors and more serious conceptual problems. Several interviews on problems like this one lasted more than 5 minutes. In these interviews the students vacillated between correct and incorrect solutions and appeared to be thoroughly confused, not just guilty of making hasty mistakes.

By analyzing the transcripts we identified two distinct sources for the students' tendency to reverse variables. The first faulty approach, which we call "word order matching," is described by Paige and Simon as "syntactic translation" [**5**]. This is a literal, direct mapping of the words of English into the symbols of algebra. For example, one might make the translation:

There are 6 times as many students as professors

$$6 \cdot S = P$$

Here the direct mapping has led to a reversed solution. We note that some textbooks explicitly instruct students to translate by the syntactic method [**3**].

We call the second method of mistranslation the "static-comparison" method. The student who takes this approach understands that the sentence implies that the student population is much bigger than the faculty population; in some cases the student will draw a diagram to indicate that this is so. (See Fig. 1.) But the student still believes that this relationship should be

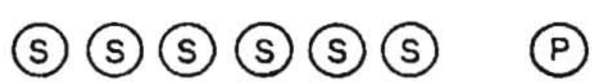

FIG. 1

represented by the equation $6S = P$. Apparently the expression "$6S$" is used to indicate the larger group and "P" to indicate the smaller group. The letter S is not understood as a variable that represents the *number* of students but rather is treated like a label or unit attached to the number 6. The equals sign expresses a comparison or association, not a precise equivalence. This interpretation of the equation is a literal attempt to symbolize the static comparison between two groups. This approach was used frequently by the students we interviewed. It may seem "more wrong" than the syntactic approach, but it does have the virtue of starting from a representation of the essential features of the problem. We regard this as an important first step in the translation process.

People who did a wide variety of problems correctly used a markedly different approach, which we call "operative translation." This approach requires the comprehension of the static-comparative approach, together with a much richer sense of what a mathematical equation is and says. In the students-and-professors problem, the number S is seen as bigger than P; therefore, the number P must be operated on by multiplication by 6 to produce a number that is the same as S. This is a lot to squeeze into the sentence $6 \cdot P = S$, but it is exactly what is required in order to understand the meaning of the simplest algebraic equations.

The reversal difficulty appears to be rather resilient and requires considerable attention and discussion before students can learn to overcome it. In a study of calculus students (primarily engineering and science majors) Rosnick and Clement [6] report on the effects of a 15–30 minute teaching unit involving worked examples and practice with several problems of this kind. Tapes of the individual teaching sessions showed that most of the students were not able to develop a reliable understanding of the issue after a moderate amount of tutoring.

Conclusion. The reversal error appears to be due not simply to carelessness but rather to a self-generated, stable, and persistent misconception concerning the meaning of variables and equations. The concepts of variable and equation are so fundamental that it is hard for practiced users of mathematics to imagine how such misconceptions can persist. However, it is not surprising that this misconception has not been affected by years of practice with manipulation of equations, since these techniques usually do not require one to understand the meaning of an equation.

Even after taking a semester or more of calculus, many students have difficulty expressing relationships algebraically. They cannot translate reliably between algebra and other symbol systems, such as English, data tables, and pictures. We do not believe that this is a trivial problem. Apparently it is rare for mathematicians to think solely in terms of algebraic symbols [4], [7], [9]. Rather, they often describe their thoughts as being like pictures, diagrams, or graphs. At some point, the mathematician is able to translate these ideas into algebraic notation; this translation is precisely what our students have not learned to do! The outlook is just as bleak for those who will never make an original mathematical contribution. They must learn to apply mathematics; that is, they must translate a problem usually expressed in words into algebraic notation and retranslate a solution back into words. Thus, translation skills are critically important in learning mathematics.

What makes teaching (and learning) of these translation skills so difficult is that behind them there are many unarticulated mental processes that guide one in constructing a new equation on paper. These processes are not identical with the symbols; in fact, the symbols themselves, as they appear on the blackboard or in a book, communicate to the student very little about the processes used to produce them. There seems to be no way to explain such translation processes to students quickly.

Our own experience suggests that one method for helping students to acquire these skills is to: (1) allocate time in courses for developing and practicing them as separate skills; (2) assign translation problems (such as those discussed above) that cannot be solved by trivial syntactic or other nonoperative approaches; (3) show by many examples the shortcomings of the latter methods; and (4) emphasize the operative nature of equations. These techniques have given us encouraging preliminary results. We are continuing our investigations into the best methods for teaching this important skill. We hope that some readers may be interested in trying experiments with their own students and that they will join us in investigating translation skills, a long-neglected component of mathematical literacy.

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