

LEARNING WITHOUT UNDERSTANDING: THE EFFECT OF TUTORING STRATEGIES ON ALGEBRA MISCONCEPTIONS¹

Peter Rosnick and John Clement

*Cognitive Development Project, Department of Physics and Astronomy,
University of Massachusetts, Amherst*

ABSTRACT

In this paper we discuss some research on students' abilities to translate English word problems into algebraic equations. In particular, we identify a common error pattern in very simple word problems, called the reversal error. Results are then described from a set of tutoring interviews in which an attempt was made to correct students' misconceptions of this kind. We conclude, based on these tutoring interviews, that for many students the reversal misconception is a resilient one which is not easily taught away. Although the surface behavior of the students changed, continued probing in the interviews revealed that many of the students' misconceptions remained unchanged. We believe these results underscore the importance of distinguishing between performance and understanding as outcomes of instruction.

INTRODUCTION

At universities across the country, more and more academic departments are requiring their students to take mathematics. The study of mathematics is no longer restricted to students in engineering and the physical sciences. From business to forestry, from hotel and restaurant management to nursing, students are

¹ Research reported in this paper was supported by NSF Award No. 78-22043 in the Joint National Institute of Education-National Science Foundation Program of Research on Cognitive Processes and the Structure of Knowledge in Science and Mathematics.

required to take at least precalculus math, and often calculus and statistics. In the spring of 1980, over 4,500 people were enrolled in precalculus and service calculus courses at the University of Massachusetts. (This is in addition to students in engineering or the physical sciences who take more rigorous calculus courses.) 4,500 represents approximately 25% of the undergraduate enrollment. Traditionally, this figure is higher in the fall.

What kind of learning is taking place for these 4,500 students? How well prepared are they to apply the large number of manipulative skills they have acquired to their fields of interest? Most of these students can solve quadratics, manipulate equations, find derivatives, and pass exams, but how do they fare at the interface between mathematical symbols and verbal descriptions of real world problems?

SOME BASIC MISCONCEPTIONS

To try to answer some of these questions we have been testing and interviewing students on some problems that require them to translate from one symbol system to another. In some tasks, we ask students to translate an English sentence to an algebraic equation, or vice versa. In others, we ask students to interpret information in tabular, graphic, or pictorial form into an algebraic equation. The results of these studies indicate that many students fare poorly at these translation activities. The following two problems are among those we have given on diagnostic tests:

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel." Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered.

In a group of 150 first-year engineering majors, only 63% were able to answer the *students and professors* problem correctly, and only 27% answered the *cheesecake* problem correctly (see Kaput and Clement, 1980, and Clement, Lochhead, and Monk, 1979). There was a very strong pattern in the errors on these problems: two thirds of the errors in both cases took the form of a *reversed equation*, such as $6S = P$ or $4C = 5S$. Further results indicate that students in the social sciences do considerably worse on these questions, as would be expected. (In preliminary tests, only 43% of these students answered the *students and professors* problem correctly.)

Initially, it was thought that these mistakes were simply careless misinterpretations due to specific wording of the problem. However, the reversal is also quite common in problems that call for translations from pictures to equations or data tables to equations. This suggests that the reversal error is not primarily due to the specific wording used in a word problem. In addition, lengthy videotaped interviews with students have indicated that difficulty is quite persistent in many cases.

Some students appeared to use a word order matching strategy by simply writing down the symbols $6S = P$ in the same order as the corresponding words in the text. Others, however, demonstrated a general semantic understanding of the problem (i.e., that there are, in fact, more students than professors), yet they persist in writing reversed equations. The interviews reveal disturbing difficulties in the students' conceptualization of the basic ideas of equation and variable. For example, some students have what we call a figurative concept of equation; i.e., that since there are more students than professors; the coefficient, 6, by virtue of the fact that it is bigger than 1, should be associated with the "bigger variable," S . This results in the reversed equation $6S = P$. (See Clement, 1980).

Other students have explicitly demonstrated erroneous and/or unstable conceptions of variable. Davis (1975), in analyzing the clinical interview of a 12-year-old solving an algebra problem, came to the conclusion that the student "... was not recognizing that x was, in fact, some *number*." Many of the college students

that we have interviewed and tested have demonstrated that they too do not recognize the use of letters as standing for numbers. They confuse the use of letter as a variable with the use of letter as a label or unit. These students also tend to write the reversed equation $6S = P$ as the answer to the *students and professors* problem. When questioned, they read the equation as "six students for every professor" and directly identify the letter S as a label standing for "students" rather than making the proper interpretation that S means the "number of students." Concomitant with this misconception of the use of letters in equations is a misconception of the use and meaning of the equal sign. Here, the equal sign apparently means "for every" or "is associated with" rather than "is numerically equal to."

EXAMPLES FROM PILOT TUTORING INTERVIEWS

After becoming convinced that the student's inability to do the *students and professors* problem and the *cheesecake* problem is a significant difficulty, we decided to address the question of the resiliency of these misconceptions. At first we thought it might simply be a matter of pointing out the mistake to the students. In taped pilot interviews, different teaching strategies were tried, including:

1. Simply telling the students that the reversal is incorrect.
2. Telling the student that the variable should be thought of as "number of students," not "students."
3. Pointing out (with pictures) that since "students" is a bigger group than "professors," one must multiply the professors by six to create an equality.
4. Asking the students to test the equations by "plugging in" numbers.
5. Asking the students to draw graphs and/or tables.
6. Specifically showing the students how to set up a proportion to solve the problem.
7. Demonstrating a correct solution to the students, using an analogous problem.

This was done with nine students, most of whom had taken one semester of calculus, who had all initially reversed the *students and professors* problem. These interviews were fairly informal, in that teaching strategies were picked at the discretion of the interviewer in response to a perceived misconception. With most students, several strategies were tried as the problem persisted. The interviews lasted between 45 minutes and one hour and covered several problems of the type above.

Our main conclusion, based on these pilot interviews, was that the reversal problem is a resilient one and that students' misconceptions pertaining to equation and variable are not quickly "taught" away. In fact, at least seven out of nine students demonstrated in one way or another that they maintained the reversal misconception, even after our attempts at remediation.

Dawn, for example, initially reversed the *students and professors* problem. During a session lasting more than 20 minutes, she was alternately "taught" and then interviewed. Several teaching strategies were used, including the use of tables, the focus on variable as number, the techniques of plugging in numbers to test an equation, and others. Throughout, she made comments like " $6P = S$, that's weird. I can't think of it that way." She claimed that the interviewer was "shaking all her foundations." Eventually, however, she agreed that $6P = S$ was correct and was able to translate her learning to the *oil and vinegar* problem, which follows. But then, she spontaneously redefined the problem to fit her preconceived notion, as seen in the following transcript segment:

The oil and vinegar problem asks for an equation which represents the fact that there is 3 times as much oil as vinegar in a salad dressing.

1. DAWN: [Draws a table showing 3, 6, and 9 under O, and 1, 2, and 3 under V.] My first impulse would be to write three times: three times O equals V.
2. INTERVIEWER: MmMm.
3. DAWN: So then, because that's wrong, I would change it to $3V = O$ because I know it's the other way around . . . So then I'm gonna plug that in. And that's right. [Writes $3V = O$.]

4. INTERVIEWER: Well, why don't you--; I-I'll.
5. DAWN: I know that's right [$3V = O$] because I make oil and vinegar dressing. And if you had a-if you had a cup-I'm rationalizing this-if you had a-
6. INTERVIEWER: Okay.
7. DAWN: a-if you had a cup of oil and vinegar [draws cups] you'd put this much oil in and that much--; I mean this much vinegar in and that much oil or else it would be really greasy . . .
8. [Dawn drew the following pictures. She pointed to the one on the left when indicating vinegar.]

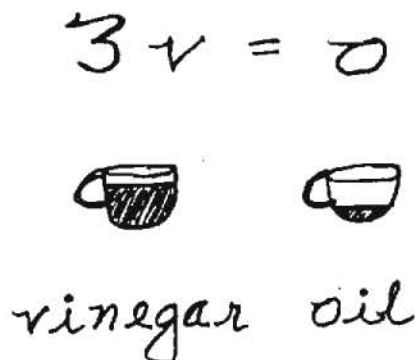


Figure 1. Reconstruction of Dawn's Diagram.

9. INTERVIEWER: So you're saying that what this equation is saying is you're putting in more vinegar than oil? [indicates equation $3V = O$.]
10. DAWN: Uh-huh.

What is striking about this example is that when Dawn first wrote $3V = O$ she knew that there was more oil than vinegar. However, on reexamining the equation, her reversal misconception apparently took over, causing her to lose sight of the original relationship.

Don, another precalculus student, had a similar experience. He, too, struggled through 30 minutes of work on the *students and professors* problem, where the interviewer tried several teaching strategies. One technique that he found helpful was graphing, and he applied this technique to the *goats and cows* problem, which reads as follows:

Write an equation using the variables G and C to express the fact that on a certain farm there are five times as many goats as cows. Let G stand for the number of goats and C for the number of cows.

The graph in Fig. 2a was appropriate, indicating that there were five times as many goats as cows. He then wrote the correct equation $1G = 5C$, while referring to his graph. However, he read it as follows: "1 goat equals 5 cows." In analyzing the equation further he said that "5 goats = 25 cows," having derived this from the equation by multiplying both sides by five. A complete shift had occurred. The interviewer then asked if his equation was consistent with the graph. The following ensued:

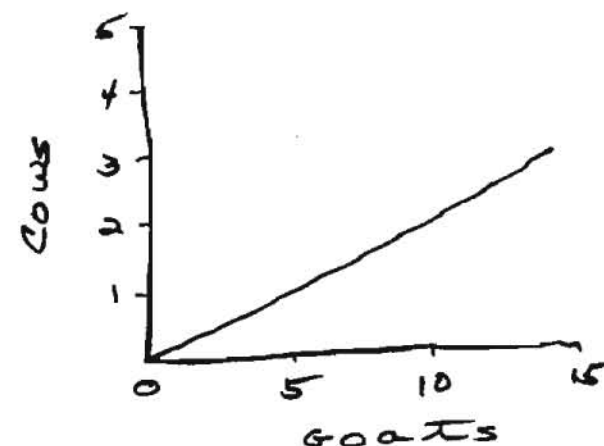


Figure 2a. After special instruction, Don made this correct graph to represent "five times as many goats as cows."

1. DON: If you had 1 goat, you'd have to have 5 cows. If you had 5 goats, you'd have to have 25 cows.
2. INTERVIEWER: And does this equation express that?
3. DON: Yup.
4. INTERVIEWER: ... does this graph express that?
5. DON: Mmm ... [6 sec] ... Nope. The goats and cows should be on the other side—so it should be the number of goats on the bottom, I mean number of cows ...
6. INTERVIEWER: Uh-huh.
7. [Don now crosses out the original labelling of the axes and relabels the vertical axis "goats" instead of "cows" and the horizontal axis "cows" instead of "goats." As a result of this change, his graph becomes incorrect. (See Fig 2b.).]

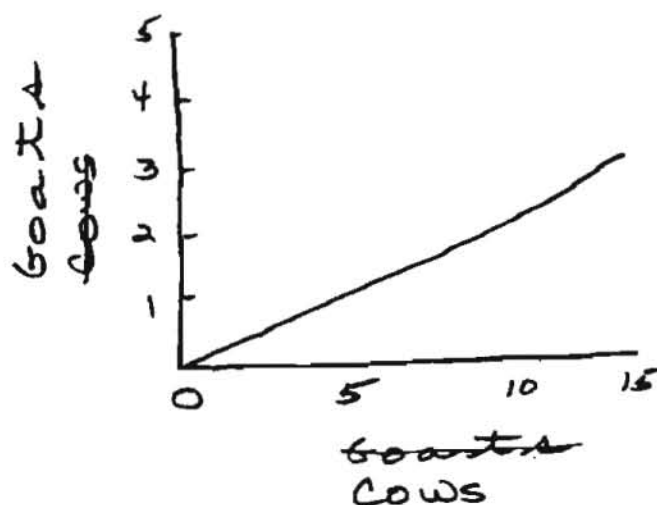


Figure 2b. After making the correct graph and writing the correct equation, Don reverted to his earlier method and modified his graph so as to produce this incorrect result.

8. DON: So that should be ... that should be the cows here, and that should be goats on this si— ... on the ... and this should be the cows here ... and then it'd be alright.

9. INTERVIEWER: Feel comfortable with that?
10. DON: Yup.

Don has totally lost sight of the original problem. Both he and Dawn learned to write the correct equation but then subsequently reversed the meaning of the original problem, enabling them to revert to their earlier way of writing variables.

Peter, a calculus and physics student, also learned to write the correct equation for the *China* problem, which states that there are 8 times as many people in China as there are in England. He, unlike Don and Dawn, was able to stay conscious of the original meaning of the problem. However, after 45 minutes, it became apparent that though his behavior had changed; i.e., he was able to write the correct equation, he still tenaciously retained his original misconceptions about variables and equations, as the following section of the transcript illustrates. Peter refers to the correct equation for the *China* problem, $8E = C$, to help him understand analogous problems.

1. PETER: I mean I feel confident here now.
2. INTERVIEWER: Okay. Um—
3. PETER: What's that—; I love that example where—; well the one with China, alright. (I'll) do that one again. So you have the ratio China ... to ... uh, England was equal to 8 over 1 ... so that's—
4. INTERVIEWER: Mmmm.
5. PETER: You know, that's how—; I should know that ... is that China is gonna equal $8E$. Just in a pure algebraically sense. But I don't think of it that way. I think there's 8 little—; 1 Chinese person for 8 little ... you know what I mean?
6. INTERVIEWER: So what does the C mean there? [Points to the equation $8E = C$.]
7. PETER: C means the English person ... uh, England itself.
8. INTERVIEWER: Uh-huh.

9. PETER: The number of people in England . . . naw, that's wrong . . . uh, yeah. The number of people in England, I'd say.
10. INTERVIEWER: And what does the E mean?
11. PETER: The number of people in China.

Peter's misconceptions are so resilient that he is willing to associate the letter E with China and the letter C with England rather than change his internal conceptualization.

Dennis is another student who, like Peter, tried hard to internalize the teaching strategies that he received. One of those strategies taught that since the group of professors is smaller than the group of students, it is the P that needs to be multiplied to achieve equality. He applied this strategy to get the correct answer on the *students and professors* problem and also on an analogous problem. Nevertheless, upon reading the *cheesecake* problem, he immediately reversed the equation. He wrote $4C = 5S$, which he read, "four cheesecakes equals five strudels." He validated this equation by showing that if you multiply both sides of the equation by five, it gives you the relationship that there are 20 cheesecakes for every 25 strudels. When questioned further, he attempted to apply the above teaching strategy to the problem, which led to the following discussion:

1. DENNIS: Well, there's the strudels . . . [9 sec] . . . and this will be the cheesecakes. And uh, overall they occupy the same area. The cheesecakes are just bigger . . . 1 cheesecake is bigger than 1 strudel.
2. [Dennis drew this picture where the figure on the left represents strudel and the one on the right, cheesecake.]



Figure 3. Reconstruction of Dennis's figure.

3. INTERVIEWER: I see. Is there any way that that can be expressed—the idea that each group of cheesecake is bigger than each group of strudel—by either this equation or some other equation?
4. DENNIS: Umm . . . well, you could say . . . one S equals a fractional amount of cheesecake 'cause it's not the complete thing. So if you say . . . umm . . . I don't know what that particular fraction would be . . . [16 sec] . . . well, say 8 . . . [5 sec] . . .
5. INTERVIEWER: And what did y—; you wrote down 80?
6. DENNIS: Point 8. [.8].
7. INTERVIEWER: Oh, point 8.
8. DENNIS: [unintelligible—three words] like 80%
9. INTERVIEWER: I see. Uh-huh.
10. DENNIS: 1 strudel is 80% the size of 1 cheesecake.
11. [Dennis then experiments with different equations and ratios, acknowledging the importance of the fraction 4/5ths.]
12. INTERVIEWER: Somehow you knew it had something to do with—you said the fraction 4/5ths—how did you know that? Where did . . . ?
13. DENNIS: Well, that was just by using the idea that . . . there are just larger than these; a larger value.
14. INTERVIEWER: You're pointing to the cheesecake?
15. DENNIS: Well, the cheesecakes are larger than the strudels.
16. INTERVIEWER: Larger than the strudels.
17. DENNIS: Yes . . . okay. So it says 4 cheesecakes and 5 strudels; that's a 4 to 5 ratio. That's where I came up with this point 8.
18. INTERVIEWER: Uh-huh.
19. DENNIS: For an individual strudel being 80% the size of an individual cheesecake.
20. INTERVIEWER: Now I—I'm getting confused again. When you're saying "individual strudel" what do you mean?
21. DENNIS: Well, if you were to take one . . . instead of messing with all these 5 and 4 of these things, you just pick 1 of each . . .

22. INTERVIEWER: I see.
23. DENNIS: . . . and look at 'em, you'll see that the strudel's only 80% as big as the cheesecake. . . from this ratio here . . .
24. INTERVIEWER: Uh-huh.
25. DENNIS: . . . this 4 to 5 ratio.
26. INTERVIEWER: As big in terms of . . . ?
27. DENNIS: Physical size.
28. INTERVIEWER: I see . . . okay.
29. DENNIS: Or you could use mass or whatever unit you want to define it by.

Dennis appears to have confused the notion of numerical or cardinal size with physical size. The problem explicitly states that C stands for the *number* of cheesecakes. The purpose of the instruction he received was to teach him to be aware of the relative size of the different groups in terms of their cardinality. Instead, he has focused on a system of relative sizes of individual pastries, a system which was not present in the original problem. This shift causes him to write an equation that is the reverse of the correct one.

Not all errors and misconceptions were as blatant as the preceding ones. However, if a student says, "I know how to get the right equation but it looks weird to me," or if a student reads the correct equation $6P = S$ as 6 professors for every student, we have concluded that the student does not truly understand the process of writing an equation from an English sentence. We now believe that writing a correct equation does not necessarily always imply understanding.

Criteria for Judging Conceptual Understanding

The following is a summary of the criteria we used in judging whether a student demonstrated a lack of conceptual understanding. These criteria helped us to distinguish the students who write the correct answer without understanding from those

students who truly understand the problem. We concluded that a subject had demonstrated a lack of conceptual understanding of the problem if:

1. S/he remained incapable of writing the correct answer throughout the interview.
2. After correcting the reversal mistake, s/he at a later time:
 - a) reverts back to the reversed equation,
 - b) accepts the correct equation but reverses the meaning of the original problem,
 - c) accepts the correct equation but switches the meaning of the original problem,
 - d) identifies the correct equation as being "weird" or "not making sense,"
 - e) acknowledges that the correct equation "works" but states that s/he doesn't know why it works.
3. The student reads the correct equation erroneously (e.g., $S = 6P$ is read "one student for every six professors").
4. After making a minor arithmetic mistake while checking the correct equation, s/he immediately doubts and discards the correct solution before rechecking the arithmetic (i.e., his/her belief in the correct equation is extremely tenuous).
5. The student demonstrates a clear misunderstanding of the use and meaning of letters in equations (e.g., by being unable to replace the letter with appropriate values).
6. After apparently learning how to solve the more elementary problems, the student:
 - a) makes no attempt to apply his/her learning to a more difficult problem,
 - b) does attempt to apply his/her learning but does so erroneously.

At least seven of the nine students in our pilot study demonstrated a lack of conceptual understanding in terms of the above criteria.

FOLLOW-UP STUDY USING STANDARDIZED TUTORING INTERVIEWS

On the basis of the initial set of nine interviews we became convinced that the reversal error and other related errors cannot be corrected by simply demonstrating the correct solution or by explaining to the student why his answer was wrong. They do not appear to be casual or careless mistakes that mere concentration can eliminate. Rather, they appear to be caused by deeply ingrained and resilient misconceptions.

To further test the hypothesis that the misconceptions are resilient we designed a more systematic teaching strategy. We wanted to give the students a written unit that contained a clear demonstration of how to do these problems and that had an explicit technique which the students could learn. This unit focused on the idea that letters in equations are variables that are meant to be replaced with appropriate numbers. This allows one to test whether an equation is an appropriate one. (This teaching unit appears in the Appendix.) The teaching unit is by no means our ideal approach to instruction. What we were interested in knowing was whether a fairly simple, traditional, algorithmic approach to teaching would be sufficient to help the students with the reversal error.

Whereas the nine students previously interviewed were drawn from various introductory math and physics courses, the six students to whom we gave the standardized unit were enrolled in the first year of a rigorous calculus course designed for engineers, scientists, and math majors. All six had reversed the equation for the *students and professors* problem on a written diagnostic test. These six students were interviewed and taped as they were working on the teaching unit. They were asked to "think out loud" as they worked their way through the various explanations and practice problems in the unit. Their performance on each problem was then graded in one of the following three ways: initially correct, initially incorrect but eventually correct (usually with prompting from the interviewer), and incorrect. Prompting took the form of reminding the students about the teaching strategy

and/or asking them to check their answers. The interviewer usually had the student work on each problem until it was correct. The results are shown in Table 1. These results might lead one to believe that some significant learning occurred. After all, by the time students reached the *sandwich* problem, four out of five initially got it right, and though four students initially erred on the *cheesecake* problem, all eventually worked to the correct answer.

TABLE 1

Problem	Initially Correct	Initially Incorrect but then Correct	Incorrect
<i>England</i> (analogous to <i>students and professors</i>)	4	2	
<i>Goats</i> (also analogous, with different wording)	4	2	
<i>Council</i> (analogous to <i>students and professors</i> , but additive)		5*	1
<i>Cheesecake</i>	2	4	
<i>Sandwiches</i> (analogous to <i>cheesecake</i>)	4	1	

*Data on the *Council* problem is somewhat tangential to our discussion of reversals because most of the errors there had to do with inappropriately assuming that the problem was multiplicative, an error that was not addressed by the teaching strategy.

Table 1. The results of using interviews and tutoring on 6 students majoring in mathematics, physical science, or engineering.

However, our conclusion is just the opposite. We have concluded that in at least five out of the six cases, significant learning did not occur. Though students' behavior for the most part was changed, we believe that their conceptual understanding of equation and variable remained, for the most part, unchanged.

EXAMPLES FROM STANDARDIZED TUTORING INTERVIEWS

Deirdre was able to write down the correct equations, but the following statement is evidence that her learning was merely procedural and not conceptual. She had just written a correct equation and said "this is probably right because it works. It works (by plugging in values) but I don't know why it works." Later, she said, "It works but I don't think it works." (!)

Carol similarly has acquired the procedure but makes the comment that the correct answer "is not what you would immediately write down but the opposite." Carol, late in the interview, goes back to the *students and professors* problem and looks at the correct equation, $S = 6P$, and tries to read what it says. "For every student there are . . . no, for every . . . see, it's not for every student there are 6 professors . . . I don't know. I'm confused now." An appropriate way to read $S = 6P$ is "the number of students equals six times the number of professors." *To know that the letters stand for numbers is an essential component to understanding the problem and was the main goal of the teaching unit.* However, that is still a very elusive idea for Carol.

Further evidence pointing to the fact that the learning that has occurred is not on a solid footing and is not backed by conceptual understanding was provided by two other students. Both, in checking their answers, made minor arithmetic mistakes. Rather than double check their arithmetic, their response was to scuttle their correct equations and try different equations that were just stabs in the dark. Mona, for example, had the correct equation $S/7 = H/9$ for one problem, but when it didn't check out she tried $S/7 = 9H$. She could provide no logical justification for this

last equation. That she had no qualms about abandoning the original equation suggests that she had little conceptual understanding of it. Mona also demonstrated confusion by concluding that $4C = 5S$ is incorrect for the *cheesecake* problem (which it is), by plugging in 2 for both the C and S. When asked what the 2 stood for, she said:

That would be the uh . . . the number of strudels . . . um, this equation doesn't really fit it. . . . For every 4 people who ordered cheesecake, 5 who ordered strudel. Say there's a group of . . . 2 times 4, which is eight. For every group of 4 . . . for every group of 4 equals . . . I guess C is like how many groups of 4 there are and S is, would be how many groups of 5.

Mona apparently does not understand the way letters are used as variables in an equation.

David demonstrates confusion in several different ways. He, on occasion, plugs numbers in incorrectly. He makes statements like, "It works, but it's wrong." He also becomes confused when he tries to read algebraic sentences, as shown in the following transcript excerpt.

1. INTERVIEWER: Is that right? [Writes $S = (5/4)C$]
2. DAVID: Yeah.
3. INTERVIEWER: Read this to me. I'm pointing to $S = (5/4)C$. What does that say?
4. DAVID: Alright, there's 5 strudels for every . . . 5 strudels is equal to $1\frac{1}{4}$ cheesecakes. I don't know . . . how did I get . . . ; I had the other one backwards . . .
5. INTERVIEWER: 5 strudels. Now where, where do you see 5?
6. DAVID: I mean 1 strudel.
7. INTERVIEWER: 1 strudel is equal to $1\frac{1}{4}$ cheesecakes?
8. DAVID: Yeah. I was looking up here [at the problem statement] just (jumbled); I was reading this over while I said it. I did it backwards the first time.

9. INTERVIEWER: Which, which is there more of, strudel or cheesecake?
10. DAVID: Strudel.
11. INTERVIEWER: Okay. So you said one strudel is equal to $1\frac{1}{4}$ cheesecake.
12. DAVID: Mm, yeah... [8 sec] ...
13. INTERVIEWER: Hmm. Confusing isn't it?
14. DAVID: Yeah.
15. INTERVIEWER: What are you thinking right now?
16. DAVID: I'm wondering why this one; this way here works, [points to $S = (5/4)C$.]
17. INTERVIEWER: You don't think it should work?
18. DAVID: Right.
19. [David proceeds to check the correct equation, $S = (5/4)C$, by appropriately plugging in numbers. Still confused, he checks the opposite equation $C = (5/4)S$ with the same numbers and finds that it is incorrect. But $S = (5/4)C$ doesn't "read" correctly for him so he continues to play with numbers for a long time. He finally decides that $S = (5/4)C$ is correct.]
20. DAVID: ... this one [$S = (5/4)C$]. I'd say, I'd say I'd stay with this one if I had to.
21. INTERVIEWER: Okay. Read this one to me again.
22. DAVID: ... No, maybe not.
23. INTERVIEWER: What ...
24. DAVID: If I read it to you, it seems wrong.
25. INTERVIEWER: Okay.
26. DAVID: If I say ...
27. INTERVIEWER: Read it to me then.
28. DAVID: 1 strudel ...
29. INTERVIEWER: Uh-huh.
30. DAVID: —is equal to $1\frac{1}{4}$ cheesecakes.

31. INTERVIEWER: Uh-huh. And that seems wrong?
32. DAVID: Yeah. Because it's ... ; 1 cheesecake is equal to $1\frac{1}{4}$ strudel like I had down here [$C = (5/4)S$], but this equation doesn't work.
33. [After this, David again plugs in numbers and after a good deal of time says:]
34. DAVID: So I'd say this one [$S = (5/4)C$] is right.
35. INTERVIEWER: ... could you read that for me?
36. DAVID: Um ... one strudel is equal to ... S is equal to $5/4$ ths strudels ... equal to $5/4$ ths the cheesecake ... it doesn't look like it works but it does.

David has finally learned to plug in numbers correctly and on the basis of that, decides on the correct equation. But he does so in a conceptual vacuum; more accurately, he does so with an incorrect conceptual framework that is resistant to change.

Criteria for Judging Conceptual Understanding

In analyzing the transcripts of the interviews with the six students, we were confronted with the difficult problem of judging whether the individual students conceptually understood each problem. We did this by categorizing the students' performance on each problem in one of the following three ways:

1. The student demonstrated a conceptual understanding of the problem; i.e., in the course of the execution and discussion of the problem, the student indicated that s/he understood that variables stand for numbers rather than individual objects, and that a larger coefficient is associated with the variable that represents the smaller group in order to equalize both sides of the equation.
2. The student demonstrated a lack of conceptual understanding of the problem. Criteria 1 through 5 of our previously discussed criteria for judging conceptual understanding were used to judge whether a student's solution to a particular problem should be categorized in this way.

3. Neither a conceptual understanding nor a lack of conceptual understanding of the problem were demonstrated, i.e., it is unknown whether or not the student understands the problem.

This third category, for the most part, was the modal one. However, it is our impression that the majority of students in this category would have been classified in category two if we had probed more deeply. Support for this view is provided by the data for the Cheesecake problem. Here, the students worked on and discussed the problem for the longest amount of time; time enough for the misconceptions to be revealed. Though all six students eventually wrote down the correct equation, five of them clearly demonstrated that they retained serious doubts and misconceptions about the problem. Their behavior was changed but their misconceptions remained. The results of this analysis appear in Table 2. (Compare Table 1.)

TABLE 2

Problem	Conceptual Understanding Demonstrated	Lack of Conceptual Understanding Demonstrated	Neither Conceptual Understanding Nor Lack of It Demonstrated
<i>England</i>	1	1	4
<i>Goats</i>	1	2	3
<i>Council</i>		2	4
<i>Cheesecake</i>		5	1
<i>Sandwiches</i>		2	3

Table 2. Although student performance seemed, on superficial measures, to improve as a result of tutoring, their *understanding* does not seem to improve. The *cheesecake* problem is probably the most revealing, because more extensive probing was employed on this problem to assess student understanding.

These results confirm the fact that the misconceptions students possess pertaining to variable and equation are deep seated and resistant to change. The results also underscore the fact that the ability to learn procedural techniques for solving problems does not entail an understanding of the essence of these problems. In this case, students' ability to write down the correct answer to a problem is a poor indicator of whether or not they understand what they are doing.

CONCLUSION

An important question remains: how can students learn to solve these problems with understanding? We believe that one answer to this question is that the fundamental concepts of variable and equation should not be treated lightly in high schools and colleges, nor should we assume that our students will develop the appropriate concepts by osmosis. We also believe that the answer lies in encouraging students to view equations in an operative way—that equations represent active operations on variables that create an equality. This contrasts with the view of an equation as a static statement, where the larger coefficient is associated simplistically and incorrectly with the larger variable. Furthermore, we believe that it is essential that students be able to view variables as standing for *number*. Simple as it may seem, this last conception is a fairly abstract one and, for that reason, a very difficult one to teach. The development of specific teaching strategies that would adequately address these issues is an important task in need of further investigation.

Several members of our research group are finding, in pilot studies, that students' misconceptions are not limited to the reversal of equations, but that there are a number of other deep seated misconceptions pertaining to semantic aspects of algebra. The implications of these and the present study are that more attention must be paid to conceptual development in mathematics education. The level of mathematical incompetence of these students is evidence for the shortcomings of an educational system that focuses primarily on manipulative skills. That many

students can succeed in a curriculum to the point of becoming engineering and science students, yet somehow have missed the mathematically essential notions of equation and/or variable is disturbing. That so many science-oriented students are confused at the interface between algebraic symbols and their meaning is also disturbing. It suggests that an even larger proportion of non-science students are not gaining the skills that would be helpful in their careers. It also suggests that large numbers of students may be slipping through their education with good grades and little learning.

REFERENCES

- Clement, J. "Algebra Word Problem Solutions: Analysis of a Common Misconception." Paper presented at American Educational Research Association meeting, Boston, April 1980.
- Clement, J., J. Lochhead, and G. Monk. "Translation Difficulties in Learning Mathematics." Technical report. Cognitive Development Project, Department of Physics and Astronomy, University of Massachusetts, Amherst (1979).
- Davis, R. "Cognitive Processes Involved in Solving Simple Algebraic Equations." *The Journal of Children's Mathematical Behavior*, vol. 1, no. 3 (1975).
- Kaput, J. and J. Clement. Letter to the editor. *The Journal of Children's Mathematical Behavior*, vol. 2, no. 2 (Spring 1979).

APPENDIX I: TEACHING UNIT GIVEN TO STUDENTS IN THE SECOND PART OF THE STUDY

We reproduce here the written material handed to the students for the remedial teaching part of the study.

Page 1:

Writing Algebraic Equations

Writing an algebraic equation from an English sentence is a deceptive task. We have found that a surprising number of people become confused when trying to write these equations. For that reason, we have developed a small unit for learning this skill. We ask that you follow the outline of the unit carefully.

In this unit, we ask you to go through three steps in writing an equation. THE THIRD STEP IS IMPORTANT!

Step 1. Understand the English sentence and describe what is asked for in your own words. Find numbers that would fit the relationship.

Step 2. Attempt to write an equation.

Step 3. CHECK YOUR ANSWER in the following way: REPLACE the letters in your equation with the numbers you found in Step 1 and see if both sides of the equation really are equal. If not, repeat Step 2.

On the next page is an example using these three steps.

Page 2:

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

Step 1. What this means to me is that there are more students than professors, specifically, 6 times more. So if there were 2 professors, there would be 12 students. If there were 10 professors, there would be 60 students.

Step 2. Attempt an equation. I'll try $6S = P$.

Step 3. Check by replacing the letters with numbers from Step 1.

I said 2 professors and 12 students. I replace S with 12 and P with 2 to get: $6(12) = 2$. THIS IS NOT TRUE. So I will attempt another equation.

Step 2. $6P = S$.

Step 3. Replacing S with 12 and P with 2, I get: $6(2) = 12$, which is true. So $6P = S$ is the correct equation.

Now you try one.

Page 3:

Write an equation to represent the following statement:

"There are 8 times as many people in China as there are in England." Let C be the population of China. Let E be the population of England.

Please go through all three steps.

Page 4:

Write an equation using the variables G and C to represent the following statement: "On a nearby farm, the number of goats is five times the number of cows." Use G for the number of goats and C for the number of cows.

Don't forget to check!!

Page 5:

Write an equation to represent the following statement:

"A certain council has 9 more men than women in it."

Use M for the number of men and W for the number of women in your equation.

Don't forget to check by replacing letters with numbers.

Page 6:

Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel." Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered.

Don't forget to check.

Page 7:

Given the following statement: "At the last football game, for every seven people who bought sandwiches, there were nine who bought hamburgers." Write an equation which represents the above statement, using S for the number of people who bought sandwiches and H for the number of people who bought hamburgers.