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COGNITIVE MICROANALYSIS: AN APPROACH TO ANALYZING INTUITIVE MATHEMATICAL REASONING PROCESSES

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This paper is divided into five main sections: an introduction to the methodology of cognitive microanalysis, observations from the protocol of a third-grader's word problem solution, a detailed model of the subject's cognitive processes, a revised model, and conclusions and implications. A point of departure for the study is Piaget's theory of cognitive functioning based on action-oriented schemes, assimilation, accommodation, disequilibrium, and symbolization processes. Because of the magnitude of the task of constructing a general theory of intellectual development, Piaget tended to focus his attention on long-term developmental processes rather than on the details of mental functioning during problem solving in everyday contexts. The present paper represents an attempt to narrow this gap by illustrating an approach to the study of cognition which I will call cognitive microanalysis. A major purpose of the paper is to assemble an adequate set of concepts as well as a diagramming system for describing certain intuitive reasoning processes. To do this it will be necessary to refine the meaning of the Piagetian concepts mentioned above and to draw on other theoretical concepts as well, such as hill-climbing, recursion, and internalized actions on images. The analysis will be restricted to a single protocol in order to give as detailed an example as possible. Additional protocol data which are analyzed using the same theoretical approach are described in Clement (1977).

The protocol analyzed here is that of an eight-year-old student working on a division word problem about sharing some objects. The analysis models the child's reasoning in terms of action-oriented cognitive structures that remain active in parallel over a period of time. Reasoning processes in this model do not take the form of manipulation of internal

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In Wagner, S., & Geeslin, W. (Eds.), Modeling Mathematical Cognitive Development, p. 53-89. In ERIC Clearinghouse.

statements according to the rules of a formal logic. Reasoning takes place when schemes coordinate to form action sequences that were not specified by a predetermined procedure. A method of diagramming is used that allows the tracking of such processes as they occur over time. The analysis is therefore an extension of Piaget's attempt to provide a theory of thinking based primarily on the coordination of actions and only secondarily on the manipulation of verbal symbols. However, the concepts used here are related to concepts found in both Piagetian and information processing theories of cognition. Thus the paper also suggests a framework within which these theories might fruitfully interact.

Still another purpose of this paper is to develop models of intuitive reasoning processes that are grounded in clinical observations of behavior, rather than in a prior analysis of the subject area. It is assumed here that the logical exposition of a certain area of mathematics is not necessarily identical in form to the knowledge structures children can develop most easily. If the teacher's role is to facilitate a process of knowledge construction that takes into account the ideas children bring to school, then it becomes important for educators to know something about the intuitive conceptions children construct. Children's intuitive understanding may be concrete, practical, or inconsistent, where the discipline is abstract, logical, and consistent. Constructivists assume that certain ideas children construct may never have been identified before. This assumption stems from a recognition of each individual child's creative potential and from the Piagetian position that children construct ideas partially on their own. The exploratory clinical interview is then a search for the authentic ideas of the child, whether or not those ideas fit into the mold of standard mathematics. The present analysis will identify practical, action-oriented conceptions used by the subject and will raise the question of whether such intuitive conceptions might be tapped as starting points for building mathematical ideas in the classroom.

Methodology of Cognitive Microanalysis

Methodology and diagramming techniques related to cognitive microanalysis have been discussed by Clement (1977, 1979), Driver (1973), Easley (1974), Knifong (1971), and Witz and Easley (1978). A related, but somewhat different, approach to protocol analysis is described by Newell and Simon (1972).

Characteristics of cognitive microanalysis. Cognitive microanalysis is marked by several characteristics:

(a) The basic experimental tool is the taped clinical interview in which subjects are encouraged to think aloud while solving a problem, giving an explanation, or playing spontaneously;

(b) The investigator avoids specifying predefined response categories and avoids experimental situations which greatly restrict the range of possible responses; the subject is encouraged to give creative and natural responses in relatively unstructured problem-solving interviews; observations of unorthodox responses are valued as clues to the structure of intuitive conceptions;

(c) The investigator avoids making prior assumptions about the form or functioning of a child's cognitive structures; instead, the investigator attempts to construct a model of structures during intensive observation of the child's spontaneous behavior;

(d) The investigator strives in this way to map out conceptions as they exist in the child rather than to test the degree to which the child's conceptions conform to those of an adult.

Many scientific theories attain a significant part of their explanatory power from the use of visualizable models such as molecules, waves, fluids in circuits, etc. (Hesse, 1966). In cognitive microanalysis, diagrams are an important tool for representing visualizable models of cognitive processes. Diagrams will be used in this paper to:

(a) Model the cognitive structures used by the child;

(b) Model the child's reasoning by mapping the interaction of cognitive structures during the interview;

(c) Exhibit explicit ties between theorized cognitive structures and the protocol observations they account for.

The protocol analysis comprising the main body of this paper is divided into two sections: a section describing observations derived from the protocol and a section describing a model of cognitive processes which can account for these observations. These two separate sections reflect another important characteristic of the method, the attempt to separate observations from theory as clearly as possible, i.e., the attempt to separate descriptions of external behavior from models of internal cognitive events.

Observations from a Problem-Solving Protocol

This protocol is from an eight-year-old student (referred to here as David) who solved a word problem about sharing some objects. At the time of the interview David was in the third month of third grade and lived in a working class district of a middle-sized town in midwestern United States. His teacher characterized his general level of mathematical performance as well above average. He had not yet studied multiplication or division in school.

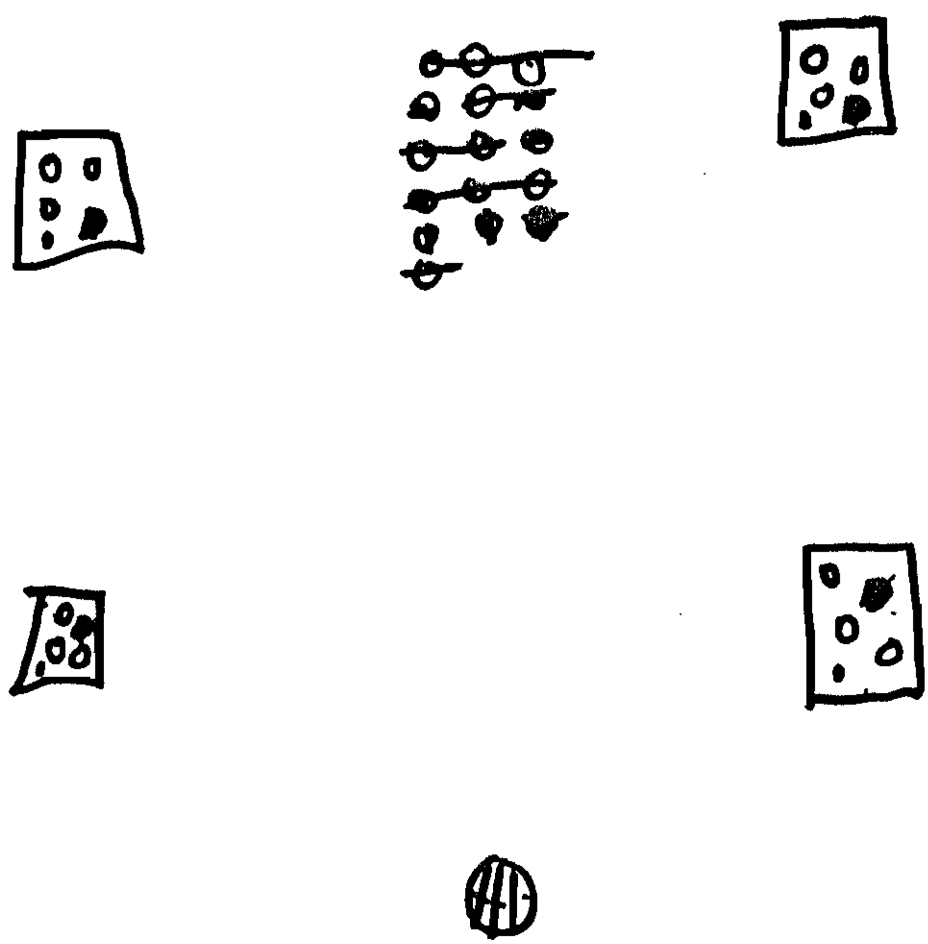


Figure 1. David's completed drawing.

David's Protocol. The subject's completed drawing is shown in Figure 1. Arrows indicate points in the transcript corresponding to stages in the drawing.

Section A

1. David: (Reads the problem) "Jim and his 4 friends found a green paper bag about 2 feet from a rabbit hole.

2. Inside they found 15 green stones.

3. They want to share them equally.

4. How many green stones should each one get?"

5. Oh no --

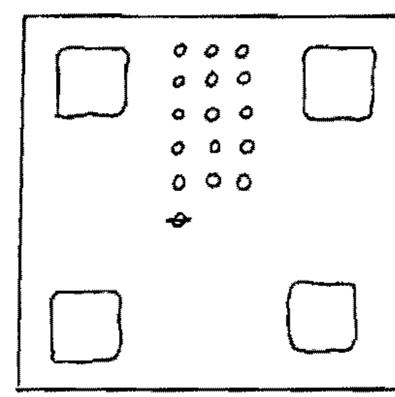
6. Investigator: Tough?

7. D: Uh-huh

8. I: How can we start on it?

9. D: 15 green stones--(draws 15 circles in rows of 3, and a 16th, recounts them and crosses out the 16th).

10. OK, now we want to divide it by 4.



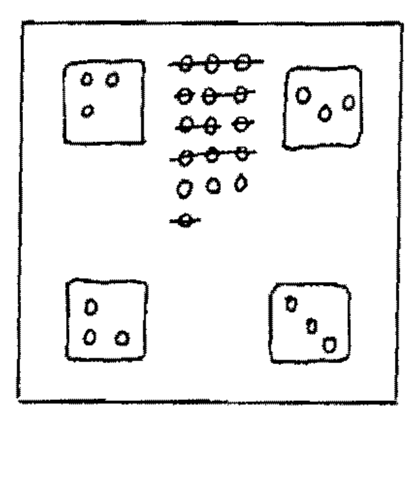
11. I: What does that mean?

12. D: Here's one sack--little can (draws a square), another, another, another (draws 3 more squares).

13. OK, one for each--1,2,3,4, (draws small circle in each square).

14. OK, 4 are gone (crosses off 4 circles in center group).

15. Now--we divide 4 more--1, oops, 2,3,4 (draws small circle in each square).



16. (Crosses off 4 more circles in center).

17. Now we divide this by 4 more (adds circle to each square).

18. Everybody's got 3.

19. (Crosses off 4 circles in center).

Section B

20. D: And there's 3 more! [concerned tone]

21. I: What's wrong?

22. D: Cut one in half, put it in here and here (draws a circle in 2 of the 4 squares).

23. I: And this is another half?

24. D: Cut this in half [referring to second circle in fifth row of central group] and here, and here (draws a circle in each of the remaining 2 squares).

25. I: Now, what are those you just put here [the last piece of stone put in each box], are these whole stones?

26. D: Half--half stones.

27. I: Let's blacken those in so we know they're halves.

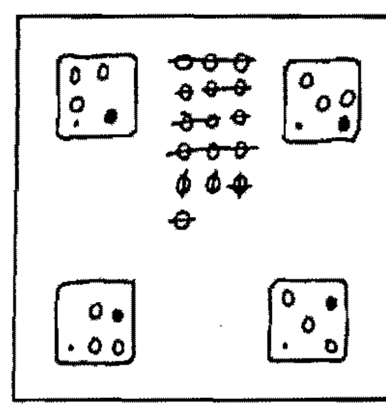
28. Are there any more?

29. D: (Blackens half circles in the 4 squares).

30. (Draws vertical lines through 2 of the 3 circles remaining uncrossed in center).

31. There's just one more.

32. So we'll put little chunks of that one in each box (puts a dot in each square, puts two crossed lines on last circle in center).



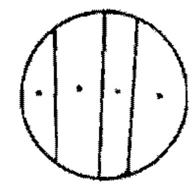
Section C

33. I: OK, how big are the little chunks?

34. D: Little--like chunk, chunk, chunk.

35. I: Could you draw that last stone down at the bottom--make a big --great big thing for the last stone--show me how you--

36. D: 1,--2,3,--1,2,3,--4 chunks-- divided (draws a large circle, divides it into 4 parts with vertical lines, puts a dot in each part).



37. I: What can we call those chunks?

38. D: I don't know.

39. I: A half of a stone?

40. D: Uh-huh--half of half of half of a stone.

41. I: A half of a half of a stone? What does that mean?

42. D: I don't know. Half of a half of a half of a stone.

43. I: Half of a half of a half of a stone--is that what they get?

44. D: I don't know.

45. I: Is there any way to write what you did with numbers?

46. D: I don't know.

47. I: That was a rough one, huh?

48. D: Yeah, I think I needed bigger cans.

David's solution was precocious in the sense that he solved a story problem ordinarily thought of as a division problem even though he had not had multiplication or division in school. David's intuitive solution illustrates an important finding: <u>In solving story problems, students do not always formulate an arithmetic problem to be solved</u>. David, for example, seemed to "act out" the solution instead.

The problem David was given contains some extra information, stating that the stones to be shared were found "about 2 feet away from a rabbit hole." David successfully ignored this information, but, as might be expected, he interpreted the text as designating a total of 4 people sharing the stones instead of the 5 people described in the problem. It is not clear why he used 4 people, but one explanation could be that David may have been in the habit of always using the printed numerals as they appear in story problems in school. This demonstrates that a story problem cannot be assumed to be a neutral, standard stimulus for all subjects. The subject's perception of the problem will depend on the form of the structures in the subject that assimilate the problem. Analysis can proceed, however, on the assumption that David was solving a problem involving four people.

Overview of the protocol. In Section A of the transcript, David read the problem and immediately drew a group of 15 circles, then saw his error and crossed out the last one. This group of circles will be referred to as the source group. He then drew 4 squares which he called "sacks" or "cans" and transferred (by drawing) 12 of the circles from the source group to the squares. He did this in lots of 4 circles, drawing one in each square and then crossing off 4 circles in the source group before distributing the next 4 circles.

In Section B David distributed the 3 circles that remained. He cut 2 of the circles in half and distributed a half stone to each square. For the single remaining stone he said, "We'll put little chunks of that one in each box." Some children would have been content to leave 3 objects as an unused portion or let one person go short by one, but David found a more interesting solution. Thus he shifted spontaneously to a new method when the initial method of repeatedly giving one stone to each became inapplicable.

In Section C the interviewer probed for a more detailed description of the "little chunks" from the last stone. David was uncertain about their size but said they could be called a "half of a half of a half of a stone."

The protocol raises a number of interesting theoretical questions, such as: If David was not using an arithmetic operation, what method was he using? How should his concept of sharing be modeled? His concept of cutting in half? His concept of cutting in chunks? What kind of mental reasoning process tied these concepts together? Did he use heuristics? It does not occur to many children to cut the stones; what triggered this idea in David? How can his reasoning, "half of a half of a half of a stone," be modeled? The analysis which deals with these questions begins with some general protocol observations.

Observation 1. David acted out the problem situation relatively explicitly. Had he made a more realistic drawing, or found some real stones to use, we would say that he was even more explicit. Conversely, if he had mentioned only numbers and number operations we would say that he had not explicitly acted out the situation.

Observation 2. He did not refer to any arithmetic problems. A possible exception appears in line 10, "OK, now we want to divide it by 4." However, it appears from the transcript that the antecedent of "it" was not a number but the group of stones. Thus, that statement probably was not an expression of an arithmetic problem. David may have been trying to make his comments "sound mathematical" by using the word divide but there is no evidence that he was thinking about dividing one number by another number.

Observation 3. David constructed a drawing as part of his solution. In this case, it was a "skeleton" drawing, with only selected aspects of the story represented.

Observation 4. David changed the drawing as he solved the problem. He verbally related aspects of the story to different parts of his drawing and to changes he made in the drawing.

Observation 5. Several sections of transcript can be identified which show that David repeatedly referred to or acted on several distinct groups of objects in his drawing. These include Groups 1-6 shown in Figure 2. Groups 1,2,3,4 and 5 are drawn with their members spatially contiguous. Group 1's members are drawn sequentially in rows. Groups 2,3,4, and 5 were referred to when David said, "Everybody's got 3." Groups 1 and 6 were also referred to verbally. (This observation provides evidence that David attended mentally to various specific groups of objects at specific times.)

Observation 6. David referred to the squares differently at different times during the interview, calling them sacks, cans, and boxes and apparently associating them with people in the statement, "Everybody's got 3." This behavior indicates that the abstract figures in his drawing are flexible to a certain extent as symbols for imaginal variations on the story.

Observation 7. In several places, David described actions he was about to perform before he manipulated (made a change in) the drawing. These include, for example, line 13, "OK, one for each (draws small circle in each square)"; line 32, "So we'll put little chunks of that one in each box (puts a dot in each square)"; as well as lines 10, 15, 17, 22 and 24. (This phenomenon will be interpreted as anticipations which occurred internally before he represented them on the drawing.)

Observation 8. David exhibited several repeated behavior patterns. (a) In Section A, David drew a circle in each of the 4 boxes and then crossed off 4 circles in the center group. This behavior pattern was repeated 3 times. (b) When 3 circles were left in the central group, David referred to cutting one in half, and put a circle in 2 squares. This pattern was repeated once more and each time he crossed off only 2 circles in the central group. (c) There was a more general behavior pattern of repeatedly transferring identical objects to each of the 4 squares. The 4 squares were each assigned one circle, then 2, then 3, then 3-1/2, and finally 3-1/2 and a "little chunk." These actions together form another behavior pattern that was executed 5 times--each time David drew an identical object in each of the 4 squares and then crossed off one or more circles in the central group. This last behavior pattern is shown more explicitly in Table I

It should be noted that these behavior categories were not defined before the interview. They were formulated from the child's behavior by the analyst as he viewed the tape.

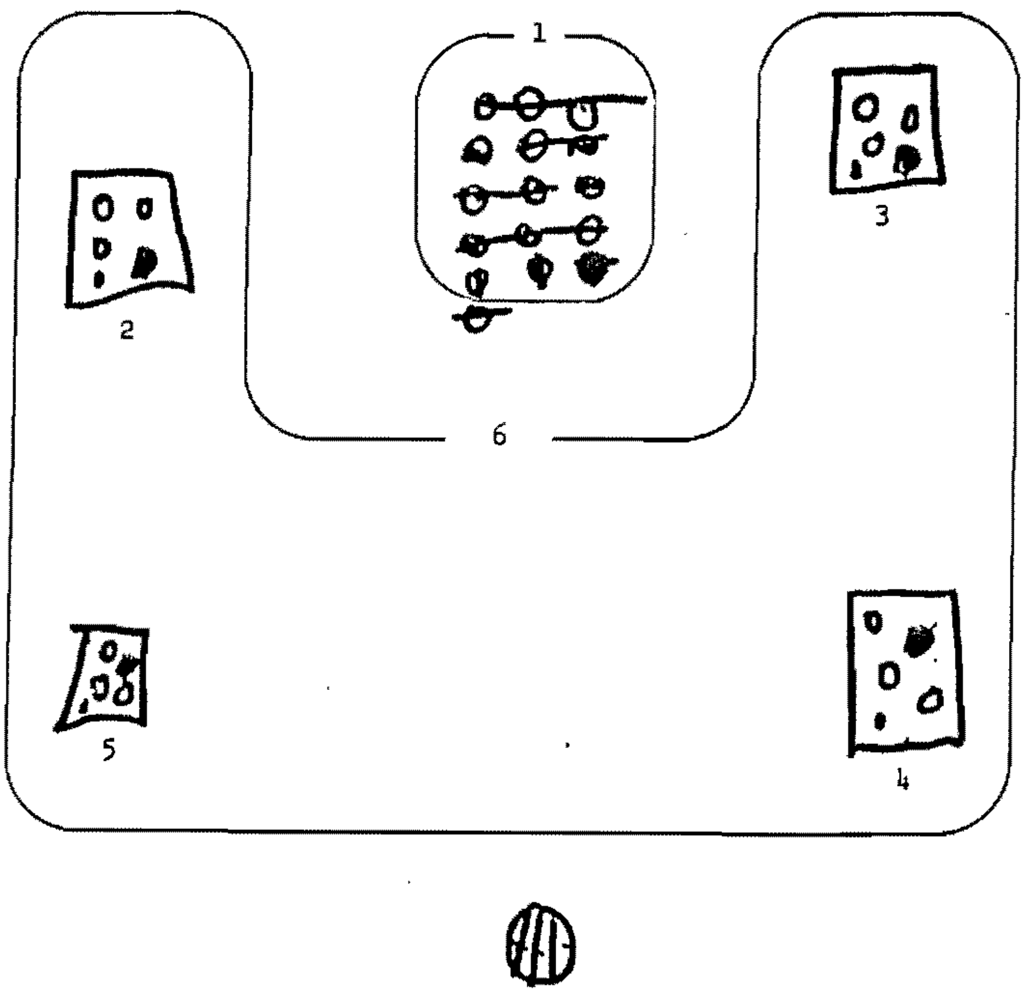


Figure 2. Groups David refers to.

TABLE I

Behavior Pattern Chart

General Behavior Pattern[a]	Line	Excerpt from Protocol
X	13	Draws small circle in each square.
Y	14	Crosses off 4 circles in center group.
	15	"Now we divide 4 more."
X	15	Draws small circle in each square.
Y	16	Crosses off 4 more circles in center.
	17	"Now we divide this by 4 more."
X	17	Adds circle to each square.
	18	"Everybody's got 3."
Y	19	Crosses off 4 circles in center.
	22	"Cut one in half."
	22	Draws a circle in 2 of the 4 squares.
X	24	"Cut this in half."
	24	Draws a circle in each of the remaining 2 squares.
Y	30	Draws vertical lines through 2 circles in center.
	32	"We'll put little chunks of that one in each box."
X	32	Puts a dot in each square.
Y	32	Puts 2 crossed lines on last circle in center.

[a]Key: X - put an indentical object in each square.
Y - crossed off circles in center group.

Cognitive Process Model

In this section the above observations will be interpreted by constructing a model of the cognitive processes going on in David during the interview. Observations 1-4 concern acting out the problem without arithmetic operations and suggest that the model should involve knowledge structures for basic practical actions such as sharing and cutting in half, rather than knowledge structures for arithmetic operations. The observed behavior pattern of distributing four objects to the squares five times can be used to describe David's basic solution method as solving the problem in parts by distributing manageable portions of the source group to the four squares. This method contrasts with a single-step solution of dividing the number 4 into 15 to obtain 3-3/4.

However, as an example of solving a problem in parts, David's approach was of a particular kind. He did not give evidence of going through a preliminary process of defining all of the subproblems before beginning to solve each part. Rather, he seemed to "slice off" a new piece of the problem as he disposed of the previous piece. An additional characteristic of his approach was that each act of sharing small groups of objects contributed to the overall goal of using up the source group of stones. A name used for this type of approach is hill-climbing. This metaphor refers to the simplest strategy for finding one's way through a forest to the top of a mountain by simply taking each step in a direction that goes uphill, each step being thought of as a piece of the solution. Hill-climbing is a well-known problem-solving heuristic (Wickelgren, 1974). It can be described more precisely as a cycle with the following form:

(a) The current situation is viewed and an action-oriented structure is activated that contributes directly toward the goal;

(b) The action is performed within the story situation constraints;

(c) Steps (a) and (b) are repeated until the solution is completed.

David's spontaneous solution had this cyclic characteristic even though he was probably not conscious of it as a general strategy; his solution process thus included an intuitive heuristic. The cycle coincides with behavior pattern X-Y in Table I. Each time he distributed a single object to each person, he moved directly toward the goal of using up the source group. In the model to be developed it will not be assumed that this cycle was produced directly by a general cognitive structure for hill-climbing. Rather, the cycle will be described as a property of David's processing that emerged from the recurring assimilatory activity of an action-oriented structure for sharing objects.

Enabling actions. David cut two objects in half in order to enable him to share them. This cutting action did not contribute directly toward the goal of depleting the source group. Instead, the cutting-in-half structure acted "in the service of" the sharing structure to generate the four objects of equal size that enabled the sharing structure to act again. For this reason, cutting in half will be called an enabling action. Cutting in chunks was also an enabling action in this same sense. These enabling actions allowed David to fill in a missing precondition for the operation of the sharing structures used in the main hill-climbing sequence of actions. This type of enabling action appears to be a fundamental reasoning process. Hill-climbing actions and enabling actions, then, were the two basic components of David's solution process.[1]

David's structure for sharing. Having identified these important components of David's solution process, the second step in the modeling task is to specify a cognitive mechanism that can account for the basic hill-climbing cycle. The observed pattern of behavior in which David repeatedly transferred identical objects to each of the four squares suggests that the initial reading of the problem activated a cognitive structure in David that embodied the idea of sharing some things fairly by giving each person an identical piece.

Following Witz and Easley (1978), a cognitive structure will be defined as a unit of knowledge which can assimilate certain aspects of the environment and provide an interpretation or a response to them. Structures which are activated can play a role in the current thinking process and remain activated on their own for a short period of time. A structure constitutes a stable unit of knowing that can be remembered; it is presumably realized as a neural pattern of activity which, if activated, can be repeated even after months of disuse. In this model, learning, then, would involve permanent changes in cognitive structures.

There are, however, a variety of ways to think about sharing (cutting, dealing, transferring, etc.) and the particular way in which David thought about it should be specified. The data here indicate that the basic concept David used is extremely simple, namely, the idea that equal portions should be transferred to each person from a source. David performed this basic act repeatedly during his solution as he gave one object to each person five times.

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[1]These processes are related to the theory of means-ends analysis proposed by Newell, Shaw, and Simon (1959), but they are developed here as emergent properties of groups of autonomous, action-oriented schemes.

There is a competing hypothesis that passing out the first 12 stones in groups of 4 was governed by an established procedure for "dealing" which assimilated the entire source group. Such a procedure would contain some automatic looping mechanism that caused it to repeat the action of distributing the stones. Although this procedural model does offer an alternative interpretation, a model involving a less sophisticated structure which simply gives one portion to each person is preferable for several reasons. It is important here to rely on the protocol for direction rather than on intuition about how a general method for sharing might be defined. David's behavior contrasts to that of children who deal out 12 or more of the stones in a circle continuously without a break. He passed out groups of 4 stones at a time in action episodes that were clearly separated by crossing off 4 stones each time. He also began each of these episodes with a punctuating expression such as, "Now, we're going to divide 4 more," apparently indicating the start of a new task. His later acts of passing out pieces of stones also involved 4 objects (one-half or one-quarter to each). Thus, the action of passing out one object to each person appears to be a coherent and independent unit of action in the protocol. This observation will be modeled by showing a simple give-one-portion-to-each-person sharing structure acting five separate times in the diagrammed model.

Initial diagram model. Figure 3 shows a simplified model of David's cognitive structure activity during the interview. In this diagram time runs from left to right. Roughly, what is going on inside the subject's head appears above the wavy line and what is going on externally appears below the wavy line. The diagram as a whole reads somewhat like a musical score, with different instruments (cognitive structures) coming in at different points and playing roles for varying amounts of time.

More precisely, the investigator's observations of events during the interview are shown below the wavy line. These observations include statements by the interviewer and by the subject, actions performed by the subject (written in parentheses), and aspects of the environment (shown in enclosed regions). The investigator's model of the subject's mental activity is shown above the wavy line. The structures shown as being active at various points in this diagram would correspond roughly to structures currently in short-term memory in an information-processing model.

The activity of the sharing structure can account for the repeated actions of passing out stones. Figure 3 shows the structure operating five times. Each time the action-oriented substructure labeled give one portion to each person assimilated a source group of 4 stones and distributed them to 4 people.

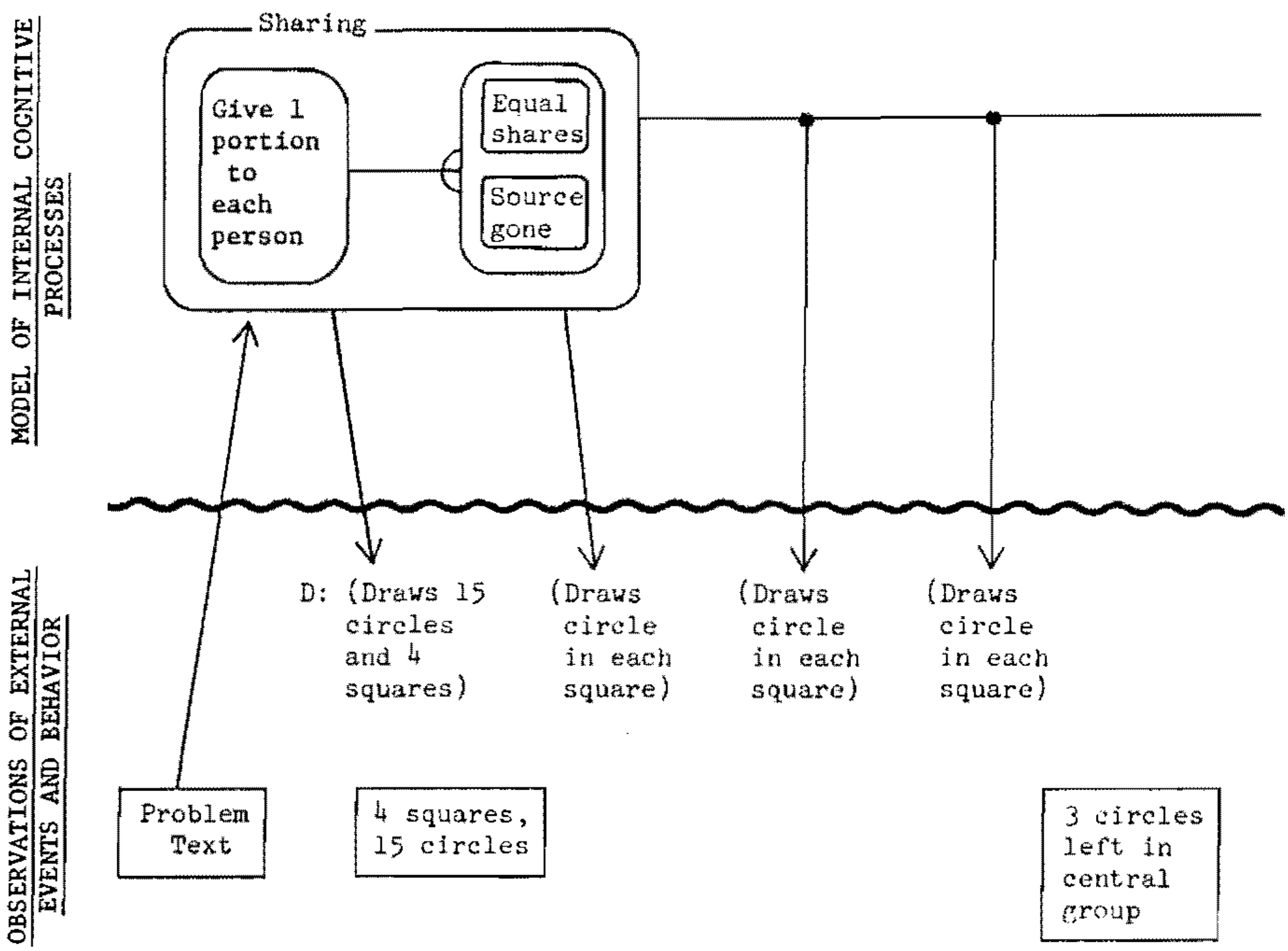


Figure 3. Initial diagram - David's solution process

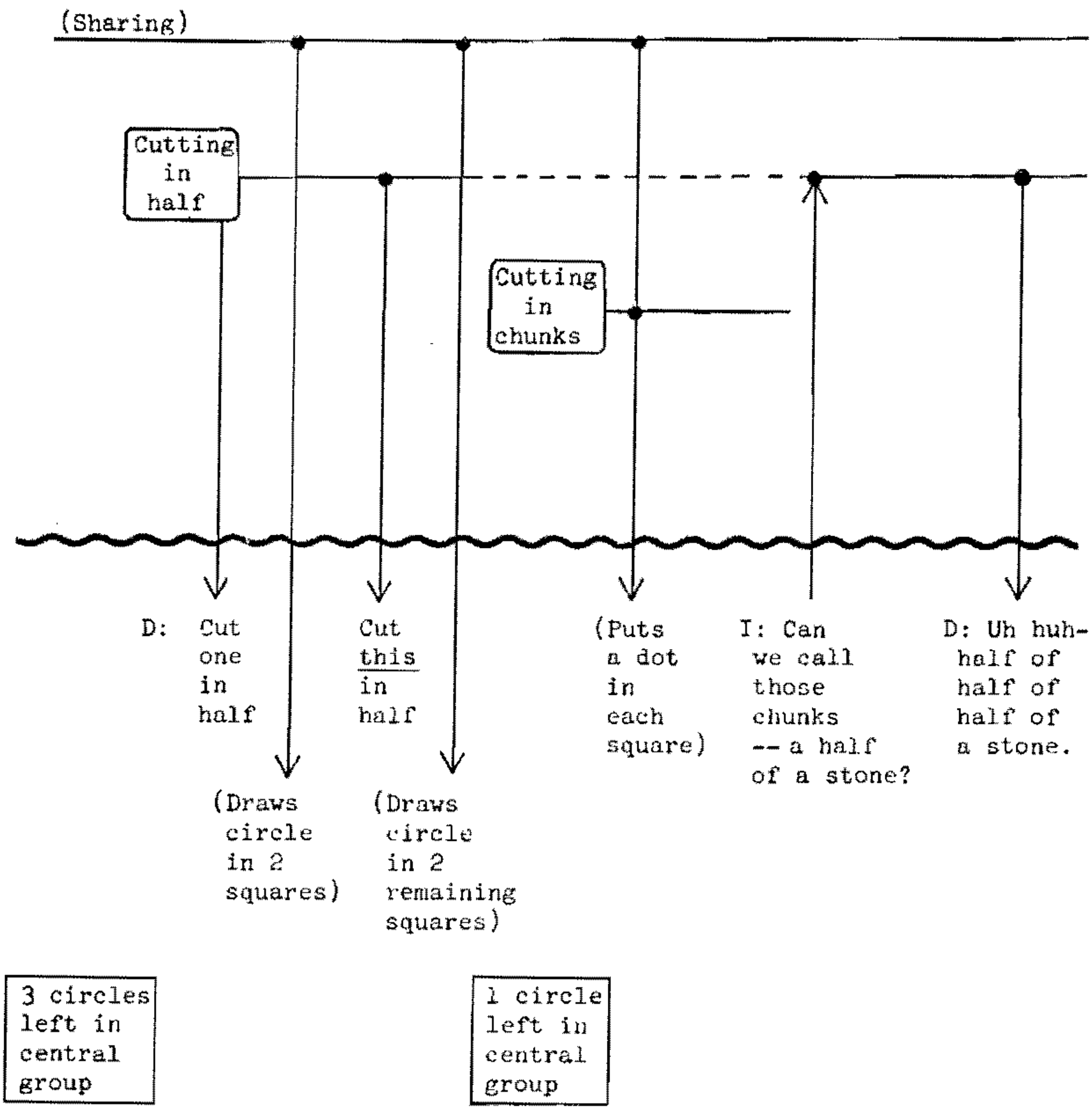


Figure 3 (cont.)

When the sharing structure could not assimilate the last 3 stones, the structure labeled cutting in half became active in parallel with the sharing structure as two of the circles were cut in half and four half-circles were passed out, two at a time. (How the cutting-in-half structure was activated will be the focus of a later discussion.) Another structure labeled cutting in chunks accounts for the way David handled the last remaining stone as he said, "So we'll put little chunks of that one in each box." The cutting-in-chunks structure was related to, but less differentiated than, the cutting-in-half structure, and it anticipated the size of each of the resulting chunks with less precision. When asked about the size of the "chunks," David's cutting-in-half structure appeared to operate recursively, causing him to describe the chunks as a "half of a half of a half of a stone." Lower level perceptual and motor output structures can be assumed to have been operating as well but are not shown explicitly in the diagram. Only the higher level "mediating process" is represented.

The sharing structure is represented by a closed region above the wavy line, labeled sharing. The horizontal activity trace line extending to the right from this structure indicates its extended, continuous activity, stretching almost to the end of the protocol. Vertical lines connect this structure to the aspects of observed behavior that it accounts for below the wavy line. Thus, beginning from the left side of Figure 3, the sharing structure was activated as David read the problem, and it played a part in producing his behavior throughout most of the solution process. The model of action-oriented structures used here implies that if David were to pass out 4 real objects to 4 real people one would see the behavioral output of this same structure. It can be assumed that, when he put a circle in each square in the drawing, this structure was operating in the same way, except that instead of feeding low level motor commands for moving objects the structure fed perceptual motor routines for producing drawings.

The sharing structure can be viewed as a scheme in Piagetian terms. (I use the Piagetian spelling for action-oriented structures here, although others may use schema.) The sharing structure is considered a scheme when thought of as a unit of knowledge, a unit that controlled David's sharing behavior and monitored the status of several groups--a group of people, the source group, and the groups of material each person received. The sharing structure is treated as a process when it remains active over a period of time and controls behavior or participates in reasoning activity.

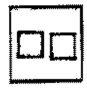
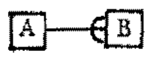
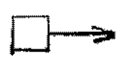
The diagram indicates an important link between the give-one-portion-to-each-person substructure of the sharing structure and the other two substructures which create the expectations that each person should have an equal amount and that the source group should be used up. The *action-expectation form* of this model for scheme structures is indicated by the notation $a \in (B)$, meaning: Do *a*, then expect *B*. Structures of this form have been shown by Knifong (1971) and Witz (1976) to account for the spontaneous behavior of 3- to 5-year-olds manipulating simple pieces of apparatus such as a hook balance. The term *expectation* is used here to mean that certain perceptual substructures are activated ("warmed up"), ready to assimilate an external event. There is a kind of tension condition set up within the *B* substructure, and this tension condition is relaxed after the act, if the expectation is fulfilled, that is, if the expected event is assimilated to the waiting substructure.

It has not been assumed that any kind of external or internal verbal activity was necessary on David's part in order for a structure to be active or in order for David to act or think about acting. A special effort has been made to avoid thinking of the sharing structure or any other structure as a piece of static information or some kind of verbal statement. Instead, a structure should be thought of as a stable, action-oriented unit of functioning in the child. As a unit of knowledge, it is closer to "knowing how" than to "knowing that."

Detailed diagram model. Figure 4 includes a number of features missing from the model in Figure 3. Two separate levels of cognitive activity are included: action-oriented structures and perceptual structures. Also, ties between cognitive structures and behavior are shown in greater detail by vertical lines; these multiple ties provide empirical support for the model constructed above the wavy line. Arrows pointing downward indicate those places where David's observable actions or statements are initiated and controlled by one or more cognitive structures.

Arrows pointing upward indicate *external assimilation*; they show aspects of the environment below the wavy line that are assimilated by cognitive structures above the wavy line. In assimilating an external object or event, a structure orients to the object and provides an interpretation for it. The assimilation of a group of four stones, for example, is then a temporary relationship wherein the structure interprets, attends to, and keeps track of the group over a period of time (in this case about ten seconds). It is assumed in this model that a similar relationship can also occur between internal structures, such as the sharing structure and a perceptual structure, and the symbol $\perp$ indicates an *internal assimilation*. It should be noted that assimilation is not a process whereby there is a one-way causal link from an object to an internal structure. The form of the assimilating structure will determine which objects are

Key to Figure 4

	Cognitive structure and substructures
	Doing A leads to expectation B
	Disequilibrium
	A assimilates B internally
	C assimilates D externally
	Activity trace
	Structure in tension
	Low level activity

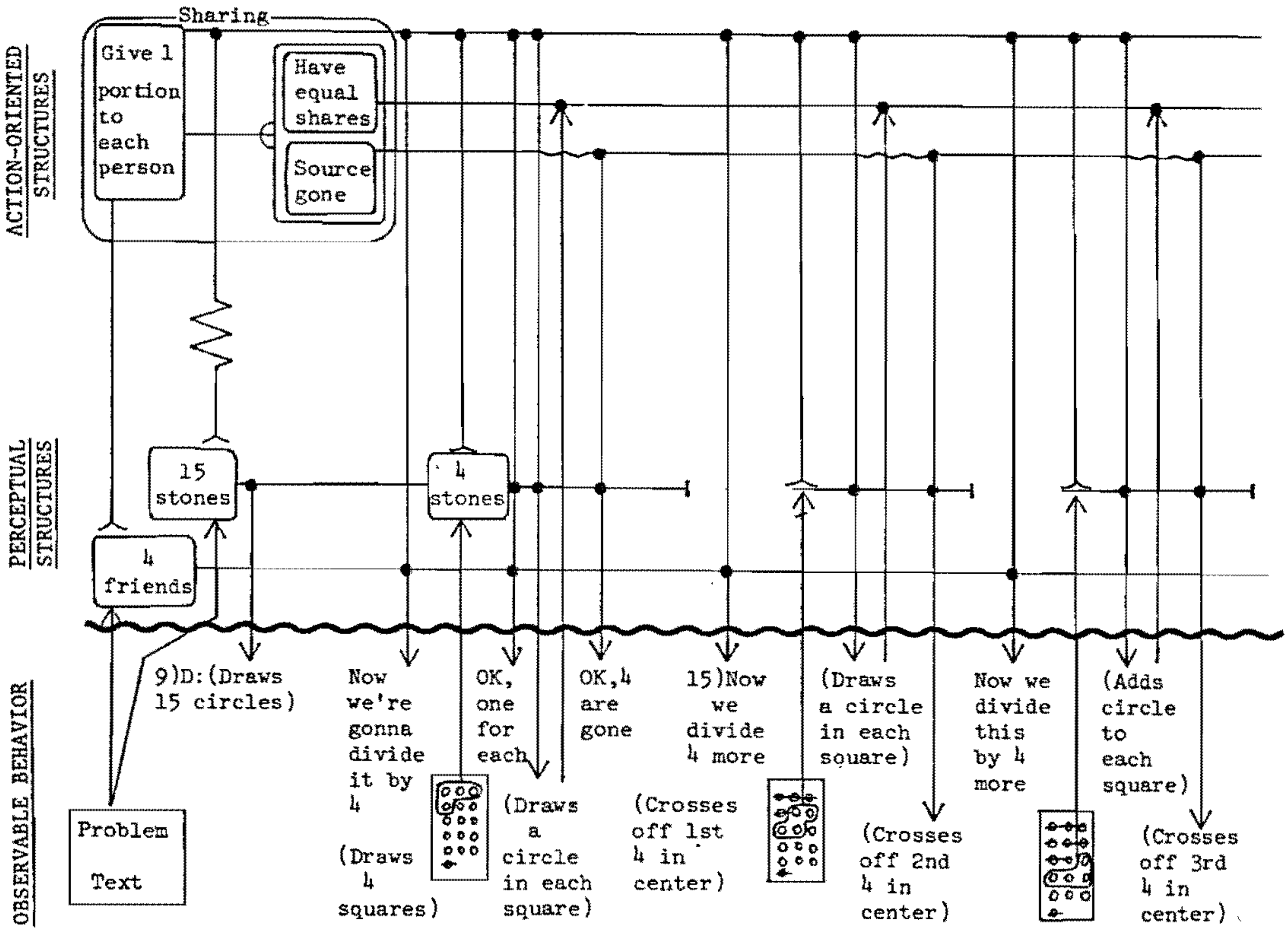


Figure 4. Detailed diagram - David's solution process

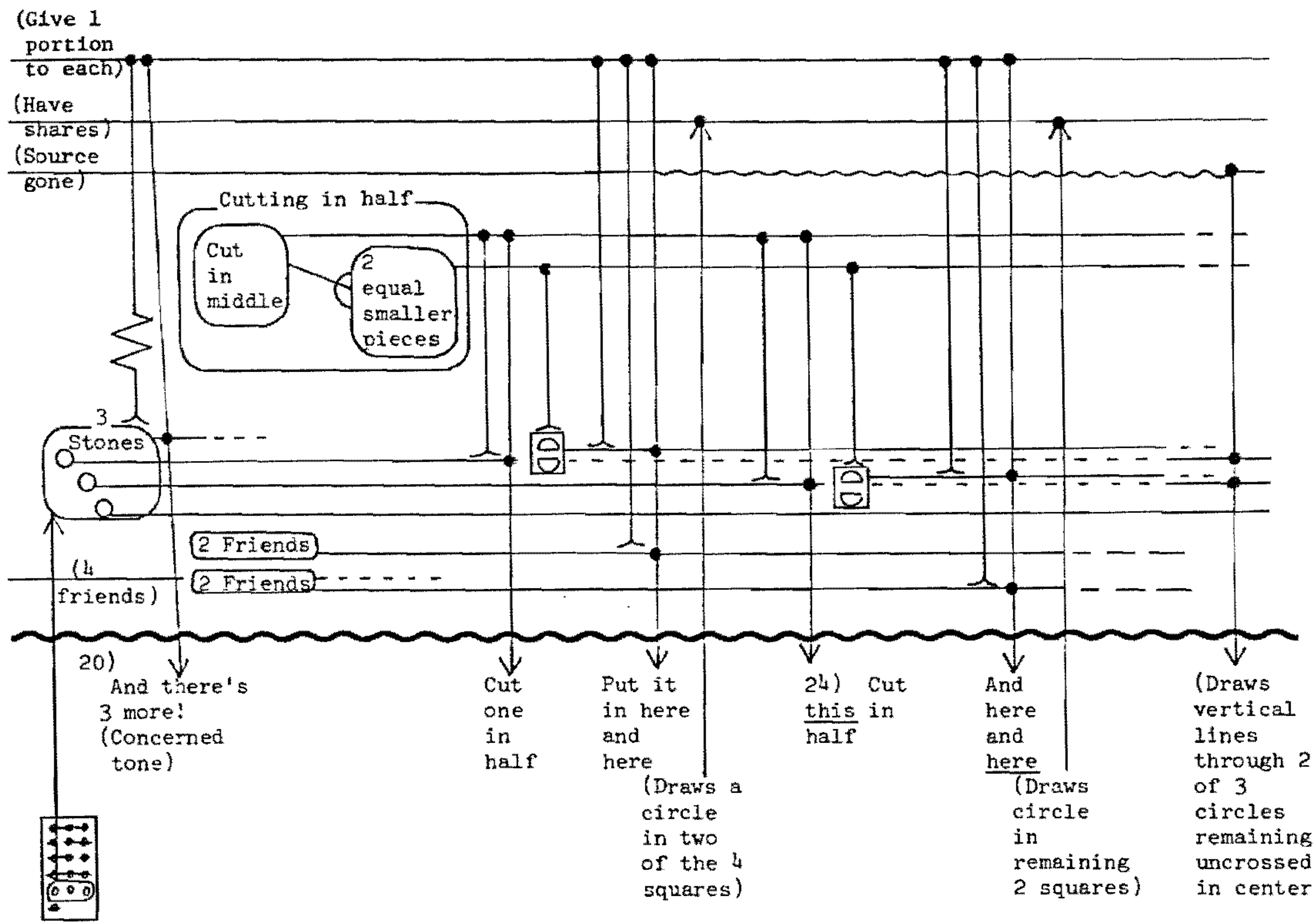


Figure 4 (cont.)

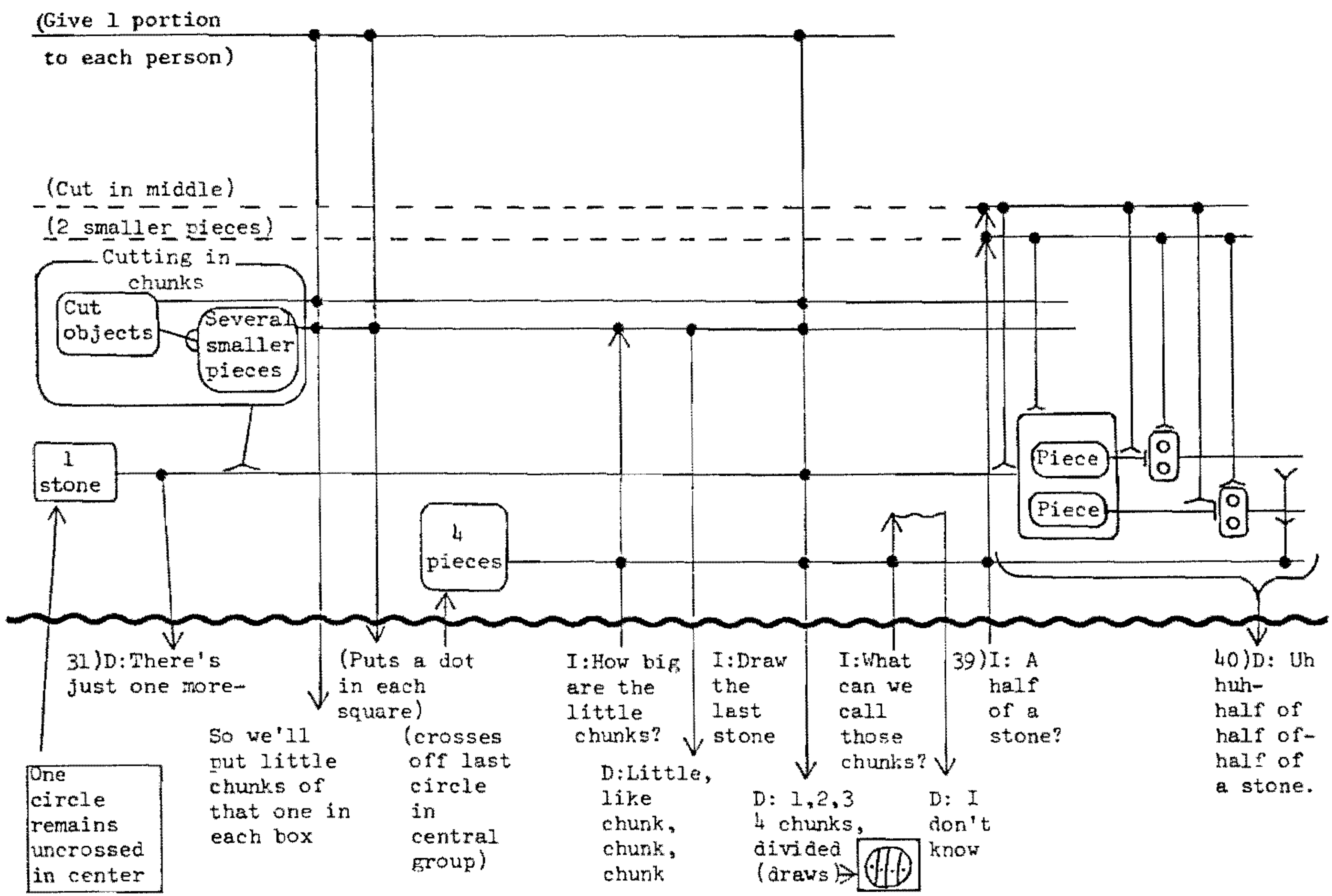


Figure 4 (cont.)

assimilated and how they are interpreted. Thus, external assimilation, for example, is indicated by a unidirectional, upward-pointing arrow only for reasons of notational simplification.

To account for the way in which David began the basic hill-climbing approach, the detailed diagram in Figure 4 first shows the 15-stones structure being assimilated to the give-one-portion-to-each-person substructure. But, there was no evidence of any action being taken on the source group as a whole. Instead, as the diagram shows, the give-one-portion-to-each-person substructure assimilated a group of 4 stones, signifying that David stopped attending to the 15 stones and focused on only the first 4 stones as a source group of manageable size.

The give-one-portion-to-each-person substructure reassimilated new groups of 4 stones and distributed them until only 3 were left, at which point it could no longer assimilate a group compatible with the 4-friends structure. This kind of situation, in which a structure attempts to make an assimilation but cannot do so, creates a type of disequilibrium condition that I will call vertical disequilibrium, as distinct from horizontal disequilibrium, in which two higher-order structures compete for the assimilation of a lower-order structure. The vertical disequilibrium condition shown at the left of Figure 4b presumably encouraged another structure to become active and resolve the difficulty, namely, the cutting-in-half structure became active and resolved the disequilibrium condition by providing enough half stones for the sharing structure to distribute. How the cutting-in-half structure was activated is still to be discussed.

Action-expectation activity over time. Figure 5 is an elaboration of a portion of Figure 4b and shows a method of diagramming the activity of cutting in half, a typical action-expectation structure, over a time period of 20 seconds or so. The substructure labeled object embodies the knowledge of a precondition, that one must first focus on an object (perhaps with specific properties) to be cut. (Such preconditions were omitted from Figures 3 and 4 to conserve space.) Two equal, smaller pieces embodies the knowledge of what one expects to see after the cutting takes place. The tension condition that exists until this expectation is fulfilled is indicated by the oscillating horizontal line emanating from the structure. In Figure 5 the structure is shown performing a real action. However, it is hypothesized that the same internal processes could take place in solving a word problem when no real actions take place, as will be discussed further. It is not assumed here that preconditions are best modeled as precise sets of discrete features. Some preconditions may be more Gestalt-like and flexible. In the case of physical actions some preconditions are embedded in the proprioceptive

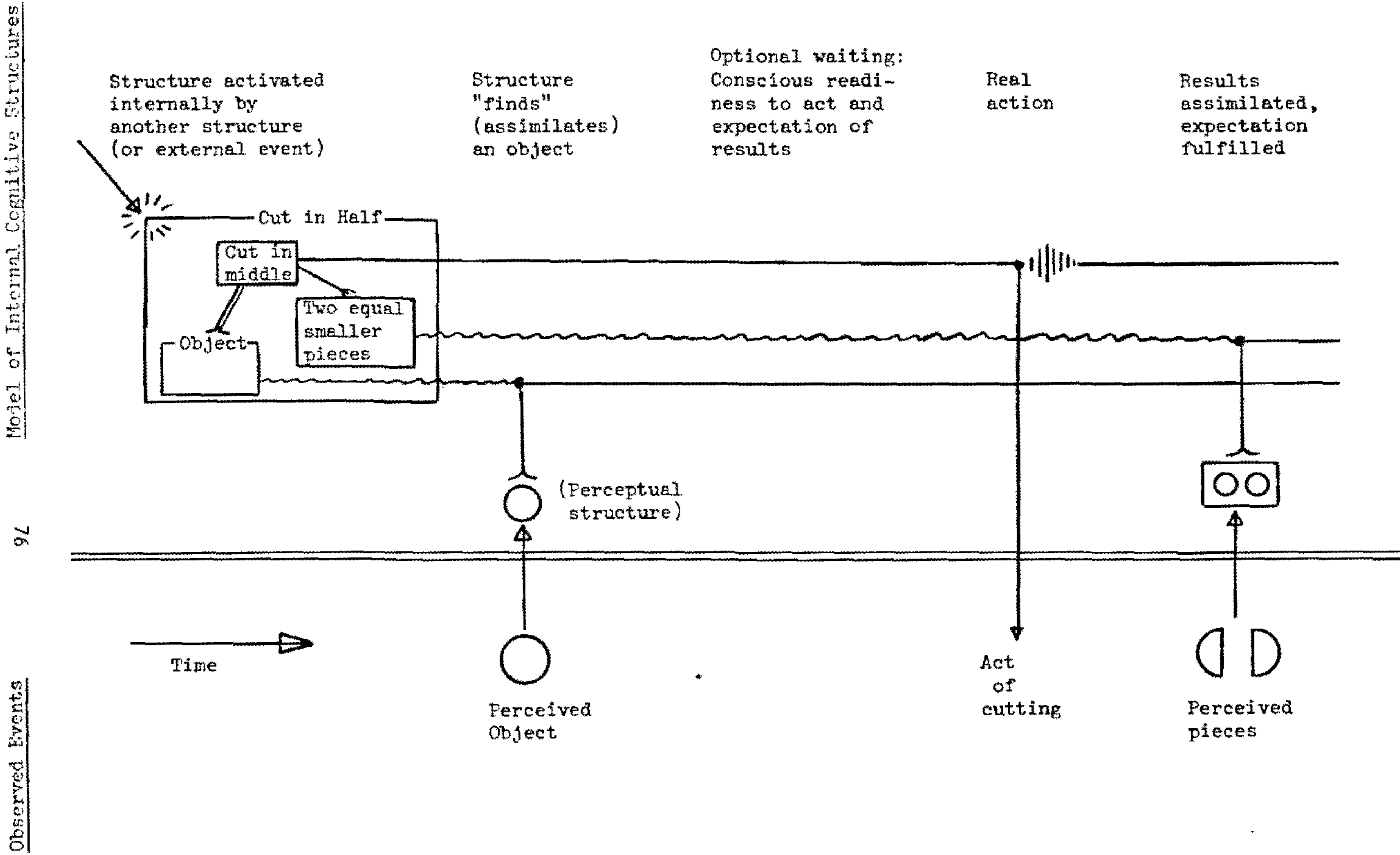


Figure 5. Activity of an Action-Expectation Structure over Time

orienting activities that constitute the first stages of the action itself and are not easily modeled as discrete features symbolized internally in some static form.

Anticipation and internalized action. The fact that David could actually anticipate what would happen when an object was cut in half (leading to his use of the idea) can be explained by assuming that David went through the internalized action of cutting the stone in half. In Figure 4 the diagramming technique developed so far is powerful enough to show some aspects of this internalized action in detail. It is assumed that, in thinking about small numbers of objects or groups (approximately 1-5), David was capable of holding active a separate perceptual structure for each object or group. For example, the structures represented by two semi-circles inside a box in Figure 4b were responsible for the perceptual expectation of having two smaller pieces as a result of cutting one stone in half. The same perceptual structure that would have been active if David had actually been viewing a small stone is assumed to have been active here, even though there were no real stones present. That is, David imagined the presence of a stone. Similarly, he imagined cutting the stone in half when the cutting-in-half structure was active without actually producing cutting movements in output mode. It is assumed that these internally activated perceptual structures were what enabled him to make a drawing (in conjunction with appropriate hand movement structures not modeled here). More importantly, they helped him anticipate the beneficial results of cutting in half as an enabling action that would allow the sharing structure to continue operating.

The interpretation represented in the diagram is that the ability to perform a mental action on an image is basically a nonverbal, perceptual-motor anticipation. David knew how to cut something in half, and his cutting-in-half scheme activated his perceptual structures in a top-down manner to assimilate two new, smaller objects of equal size even before the actual cutting occurred. That is, he was already imagining the two halves when he drew them. This interpretation contrasts with the idea that David might have been using a memorized arithmetic fact in a verbal form like "1 divided by 1/2 is 2." I am instead inferring that David's mental activity in this case was very similar to what it would have been had he actually been sharing and cutting real objects.

This model is consistent with the Piagetian view that there is a basic "logic of actions" level of reasoning that is more fruitfully modeled, not as manipulations of verbal symbols or abstract propositions, but rather as the coordination of internalized actions. While aspects of

David's behavior might be accounted for by a model which uses only verbal representations of abstract features, a reason for taking the internalized-actions point of view here is the smoothness and ease with which David constructed and interacted with his visual drawing. The fact that David focused on his drawing during the entire interview and worked so closely with it is consistent with the hypothesis that the drawing, as a spontaneous mode of symbolization, was tied very closely to the internal imagery processes he was using and was, for him, a fairly direct symbolization of those processes. Of course, it is clear that there were eventually more items represented in the drawing than he could hold in mind at once, and that is why the drawing was useful.

On the other hand, it might be argued that the drawing itself was the representation David was acting on and that there is no need for positing internal image structures. However, anticipations via internalized actions on images are included in the model in order to account for David's insight that cutting two objects in half was the right thing to do and to account for his description of the last distribution involving "half of a half of a half of a stone." Although confirmation of this interpretation will require much more research on the theory of knowledge representations, it appears to be the most plausible interpretation in the case of the present protocol.

An important task for future research is to determine the limitations of this internalized-actions-on-images system. The conjecture that the system can only operate effectively on less than six objects, or groups of objects, at a time is consistent with research on children's <u>subitizing</u> ability--the ability to enumerate groups of less than six objects very quickly without counting (Klahr, 1973). For example, David could not have been expected to keep track of changes in the source group of 15 stones without using a drawing, but he might have been expected to handle 6 stones mentally by focusing first on 4 stones and then on 2. It is further conjectured that objects are not imagined in detail--that only gross characteristics are imagined. These conjectures point to some important questions for future research.

<u>Goals</u>. The model developed here to explain how goals are set up and maintained is a vertical disequilibrium, or <u>tension</u>, model. In Figure 4 the oscillating portions of the horizontal line to the right of the <u>source-gone</u> structure indicate a state of tension. The activation of the sharing structure by the task creates the expectation that the source group must be used up in order to arrive at a solution to the problem. In general, we assume that a perceptual structure S_1 can act as a goal when it is "held active" by some basic (possibly chemical) drive or by some other continually

activated structure S_2 for which there is an expectation that the satisfaction of S_1 will lead to the satisfaction of S_2. Since the source-gone substructure cannot assimilate an empty source group at the beginning of the solution, the structure is initially in tension. The tension caused by this "assimilation gap" is reduced each time an action brings the number in the source group closer to zero.

A tension condition is hypothesized to have two effects: It makes the structure a strong competitor for attention in the organism; and it motivates the subject to act so as to satisfy the structure by trying out various possible actions mentally. When he can imagine one of these actions contributing to the goal, this action becomes the dominant focus of attention. Thus, the model explains goal-oriented behavior in terms of high activations maintained in specific knowledge structures, rather than as the transfer of a static symbol to a "place" labeled "the current goal." A consequence of this model is that it is natural to imagine the possibility of several competing goals, at different tension levels, with the strongest tension dominating at any moment.

Recursion. As described earlier, the last single "stone" remaining in David's source group is assumed to have been assimilated by a cutting-in-chunks structure. The phrase half of a half of a half of a stone, used by David in response to a question about the size of the resulting "little chunks," has several possible interpretations. It appears that David was applying the cutting-in-half structure recursively. Roughly, this means that the structure was applied to its own output. More precisely, recursion refers to the activity of a structure $\underline{S}$ assimilating a perceptual situation $\underline{P}_1$ and activating the expectation of another perceptual situation $\underline{P}_2$ (see Figure 6). When $\underline{S}$ is reapplied by assimilating $\underline{P}_2$ to its own action component, then $\underline{S}$ is said to be applied recursively.

David's situation was slightly more complicated than that shown in Figure 6 because the first expected effect of cutting in half the single stone was to produce two objects. Each of these could have been assimilated by the same cutting-in-half structure to yield four equal pieces of smaller size. However, it is not clear that David was able to imagine this with precision. He indicated his uncertainty (line 42) by saying, "I don't know," before saying "half of a half of a half of a stone." Yet, there is a certain definiteness to his response, countering as it does the interviewer's probing (line 41), "a half of a half of a stone? What does that mean?" (indicating only two "halvings"). Two possible interpretations of David's statement are as follows:

(a) He had an appreciation for the possibility of generating four equal pieces from one stone by

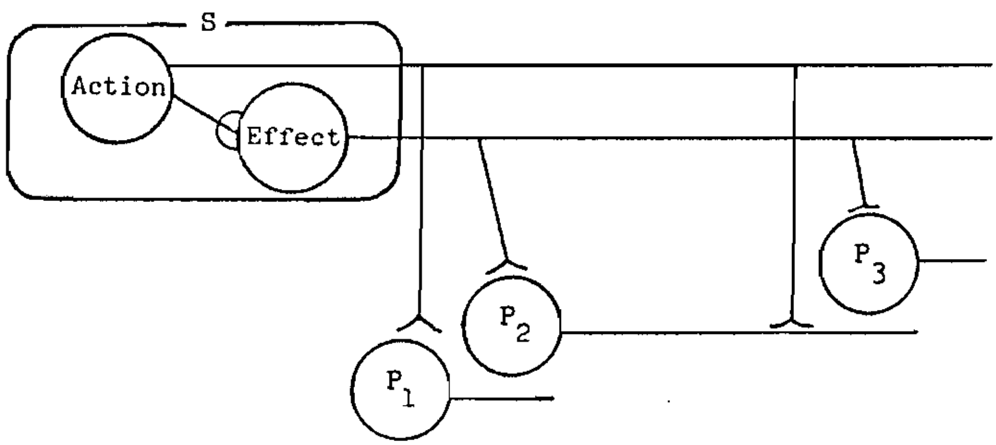


Figure 6. Recursion

applying the cutting-in-half structure recursively, but the exact sequence and number of halvings required were unclear to him;

(b) He comprehended the chain of actions required, but described them linguistically in a nonstandard format; there were three acts of halving required to generate the four pieces he drew, so he said "half of" three times.

It is the latter interpretation that is represented in Figure 4c. This interpretation is consistent with an assumed tendency on David's part to focus on the act of cutting a piece in half as opposed to focusing on the resulting half-pieces. In either case, the fact that David appreciated the possibility of recursively applying the cutting-in-half structure to produce four identical pieces is an impressive example of intuitive reasoning.

Improving the Model

An important characteristic of structure-interaction diagrams is that they are criticizable as theories. Figure 4, for example, is the end result of many cycles of constructing a model, checking it against the exact sequence of events in the tape, and modifying it. However, the present map can still be criticized for leaving certain protocol data unexplained. Several improvements can be made to produce a slightly more complex model that would describe David's mental activity in the following sequence (asterisks indicate new steps):

*(a) A general, frame-like structure activated the goal of sharing 15 objects among 4 people;

(b) David realized that he did not know how to do this in a single step;

(c) A specific sharing structure repeatedly oriented to sharing 4 objects among 4 people to account for the basic hill-climbing cycle;

(d) The first 12 stones were distributed;

*(e) The general frame-like sharing structure also weakly activated the cutting-in-half and cutting-in-chunks structures as actions which are related to a sharing context;

*(f) David planned the action of cutting 2 objects in half in order to produce 4 objects to satisfy a precondition for the sharing structure that required 4 objects for distribution;

(g) He executed this plan in two steps, giving a piece to each person after cutting each stone in half;

(h) Cutting in chunks operated on the last stone in order to produce 4 more pieces for distribution by the sharing structure;

(i) Cutting in half operated recursively on the last stone to provide a more precise solution.

These changes in the model involve three issues: first, accounting for the goals in the hill-climbing cycle via two separate sharing structures at different levels of generality; second, the extent to which planning occurs via internalized actions; and third, choosing between an autonomous-schemes model and a more structured, frame-oriented model to account for the initial activation of structures. These issues will be discussed in turn below.

Levels of structure. In addition to leaving certain data unexplained, the model shown in Figure 4 does not account for the goal that motivated the hill-climbing cycle which David followed in repeatedly distributing a single object to each of four people. One might theorize that this goal was embodied in the continuing tension in the source-gone expectation in the sharing structure, but, as David's focus switched from the original source group to the first subgroup of four objects, the source-gone expectation would have been fulfilled after distributing four objects; there would have been no source of tension to drive further distributions. This difficulty can be resolved by theorizing that there were two levels of sharing structures (see Figure 7). The first structure to be activated, shown in the upper left corner of the diagram, was a generalized conception of the gross qualitative features of a sharing episode, where fairness is desired, while the second structure embodied a specific method for sharing, where fairness is guaranteed. (The latter is equivalent to the sharing structure modeled in Figure 4.) Related models of action-oriented structures at differing levels of generality have been discussed by Newell, Shaw, and Simon (1959) and by Witz (1973).

The hill-climbing cycle that David followed can be explained by positing that the general sharing structure assimilated the entire source group of 15 stones. A tension or goal-seeking condition persisted in the source-gone expectation of this structure during the entire problem solution. Any action which transferred stones from the source group to the people reduced this tension. Such an action was provided by the second, more specific sharing structure shown in Figure 7. This structure repeatedly assimilated only four stones at a time and gave one to each person.

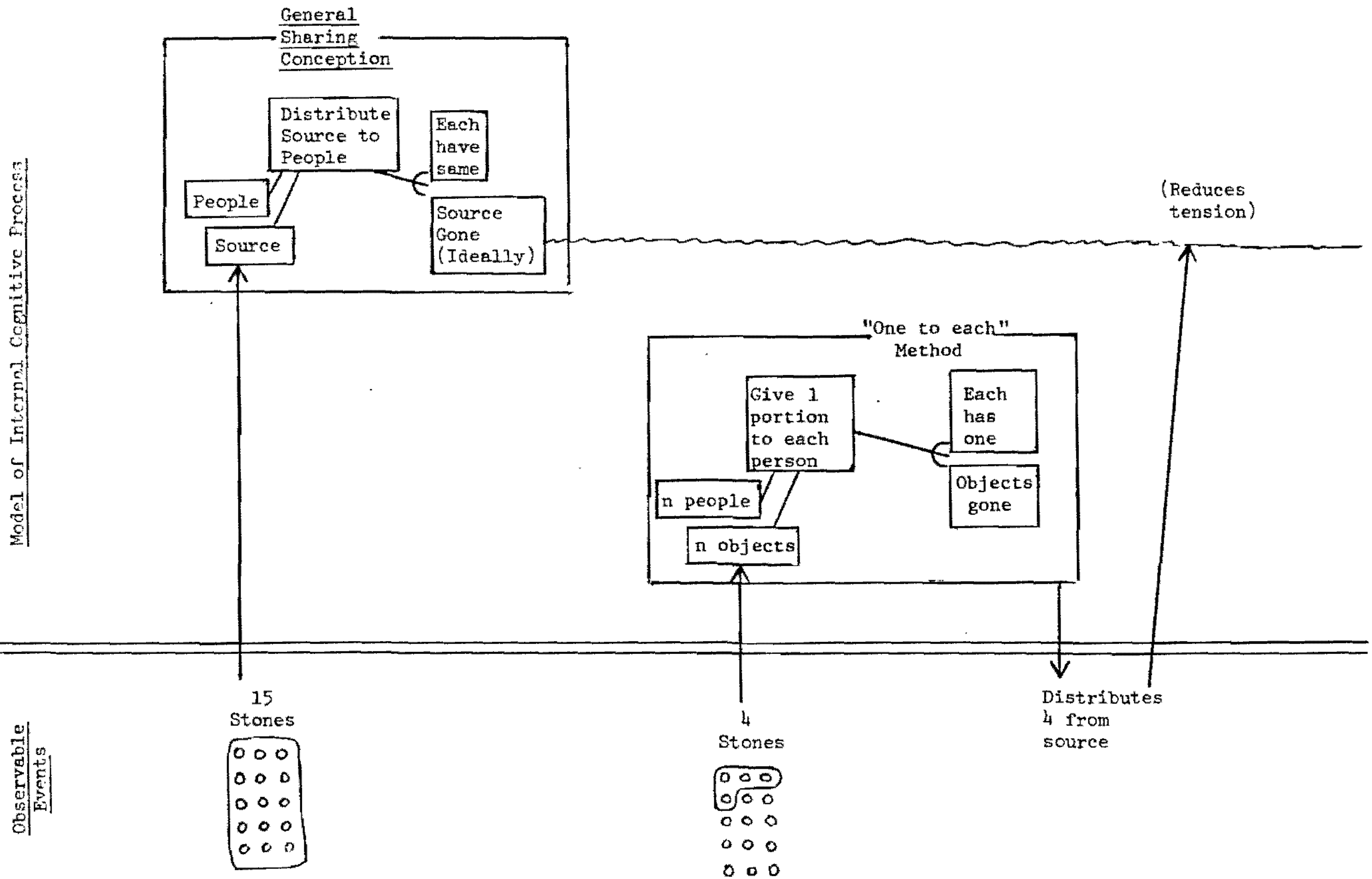


Figure 7. Generalized and Specific Sharing Structures

An interesting feature of this tension-reduction model is that it eliminates the need for an "executive procedure" which organizes the problem-solving process in a centrally controlled way. Such a procedure might have been used to model a general hill-climbing or difference-reduction strategy explicitly and sequentially carried out by the problem solver. In the present model the tension-reduction mechanisms are thought of as being built into the "hardware" of the system in a more distributed way, that is, built into the "drive-to-assimilate" property of the relatively autonomous cognitive structures themselves. The hill-climbing cycle then emerges as a systemic property of a distributed system rather than being determined locally by an established procedure with controlled looping.

Chaining of structures. The use of cutting in half as an enabling action can also be thought of as "working backwards" or chaining backwards from a subgoal. If finding four objects was the goal when David had three stones left, then cutting the stones in half constituted working backwards from the goal of sharing four objects to the idea of finding four objects to the idea of cutting in half. Internalized actions offer a mechanism for explaining how such chains of action could be planned mentally ahead of time by the subject. If a goal tension condition exists in a perceptual structure P, an action-expectation structure A with an expectation substructure identical or similar to P will tend to be activated and tried out internally. If the precondition substructure in A is uninstantiated, it will in turn become a new primary goal and this explains how chaining can occur spontaneously. An intriguing question to ask at each point in the protocol is: How much did David plan ahead of time--how far ahead could he look?

It is not possible to answer this question definitively in the absence of announced predictions or plans on the part of the subject. The position represented in the diagram in Figure 4 is that most results of action chains were not anticipated beyond the current drawing cycle, a drawing cycle being a relatively well-defined burst of drawing activity in the protocol, separated by pauses and spontaneous verbalization on the part of the subject.

One probable exception to this limitation was the sequence in which David cut two stones in half to distribute to four people. It has already been inferred that he anticipated that cutting one stone in half would provide equal pieces for two people. How much more did he anticipate here--that cutting two stones in half would give just enough for the four people? The diagram as drawn in Figure 4b does not reflect this anticipation, but it could easily be changed to model it by compressing and shifting to the left the cognitive activity associated with each cutting action, thereby implying more internalized actions on images before the drawing of "half stones."

The major Piagetian theme is that, given an appropriate object, active schemes will operate autonomously on the object to produce an expected result, even when there is no special reason to do so. This tendency of structures to apply themselves autonomously is most apparent in the relatively goal-free play behavior of children, who are always "trying out" things with (applying action schemes to) different objects. The present model assumes that when a scheme like cutting in half was activated, it automatically tried itself out internally on the images of the circles already drawn. Thus the beneficial results (four equal objects to share) were anticipated. This sequence outlines a mechanism by which planning can occur in a spontaneous manner, as the result of active assimilations on the part of autonomous structures.

Activation of structures. There remains the question of how the cutting-in-half and cutting-in-chunks structures were activated--the memory access, or scheme activation, problem. Successful problem solving involves at least two considerations, namely, determining what actions are relevant to the problem and determining what sequence of these actions will work. The discussion so far has focused largely on the latter consideration, but the question of determining actions relevant to a problem should not be ignored, as it too often is in psychological research. Several possible answers to this question will be described below.

At one extreme, it might be that structures call other structures directly, as is done in many everyday computer programs. An established, higher-order procedure would determine the order in which these structures come into play using direct subroutine calls. A weakness in such a highly structured format is the difficulty of obtaining robust procedures that adapt flexibly, as David seemed to do, to new problem-solving situations.

At the opposite extreme, there might be many autonomous structures (variously called schemes, demons, or productions) each of which determines its own relevance to the current situation and competes for attention if the relevance is high. The search for a match between structure and situation would presumably have to occur in parallel, given the vast number of structures in the organism. Models along these lines have been proposed by Newell (1973), Selfridge (1959), and Witz and Easley (1978).

Between these extremes is a scenario in which structures related to the same context are activated together, but their sequence of deployment is not determined ahead of time. Models of this last type have been proposed by Witz (1973), who models "framework" structures in young children from a Piagetian perspective, and Minsky (1975), who argues for the use of "frame" structures in an artificial intelligence system.

Action-oriented structures might be activated in any of these ways. Their effects, or short chains of several actions and effects, could be anticipated internally and evaluated on the basis of current goals. In David's case the initial sharing structure was probably activated directly via the linguistic process involved in reading the problem text. It is unlikely that the cutting-in-half and cutting-in-chunks structures were called at a certain point by an established procedure for solving sharing problems, because of the flexibility with which David coordinated sharing and cutting in half in parallel and the novelty of his solution. If this had been his approach, then in one sense he could not have been considered to be engaged in problem solving, since he would have already had an established procedure for doing the task. The model favored here is the framework model in which the cutting-in-half and cutting-in-chunks structures were initially activated by a general sharing structure which included associations to relevant enabling actions. However, the short length of the protocol precludes a definitive judgment on this issue. Interviewing strategies that will make such judgments possible need to be developed. One strategy might be to examine a series of solutions to related problems by the same subject.

Conclusions and Summary

The proposed model of David's reasoning posits relatively little high-level structuring, that is, no general macro-procedure which would specify how sharing objects, cutting in half, and cutting in chunks are to be sequenced in a given situation. This lack of structuring is made up for by a mechanism of spontaneous reasoning interactions. Reasoning takes place when schemes coordinate to form action sequences that are not specified by an established, predetermined procedure. A combination of limited knowledge structures and spontaneous reasoning potential might be a more powerful (adaptive, flexible, general) configuration for problem solving than a set of highly structured procedures oriented to specific tasks.

David's behavior was accounted for via three practical, action-expectation structures: sharing, cutting in half, and cutting in chunks. Each of these structures has basically the same form, consisting of an action substructure connected to a perceptual substructure comprising an expected effect.

Internalized actions involve the activity of these structures in the absence of external output. David's overall approach was described in terms of a hill-climbing cycle, each cycle corresponding to an act whereby the sharing structure reassimilated a new group of identical objects and distributed them. It was hypothesized that internalized actions can be chained to provide for planning and that a structure can function recursively by reassimilating its own effects.

This model of David's thought processes is not based on stored arithmetic facts or other passive structures but on action-oriented cognitive structures that remain active in parallel for varying lengths of time during a problem-solving episode. Reasoning processes in this model do not take the form of manipulation of internal statements according to the rules of a formal logic but, rather, consist of cooperation between structures during internal assimilation and conflict between structures during disequilibrium. The modeling of both knowledge structures and their interactions as dynamic processes preserves and reflects one of Piaget's most important insights into cognition. Many concepts discussed in this paper are related to concepts found in both Piagetian and information-processing theories of cognition and provide points at which these theories might fruitfully interact.

By intensively analyzing a single short protocol, some aspects of a model of the child's cognitive processes can be supported, while other aspects remain conjectural. The most firmly supported cognitive processes in the present model, for example, are those that have been tied to several different observations from the transcript, as shown explicitly in Figure 4 by the vertical lines connecting structures to observable behavior. The primary objective here, however, has been to raise key questions and to illustrate a variety of techniques and concepts that might be applied to a richer data base.[2] One promising direction for future research is to take a single subject and analyze several related problem solutions. Such case studies should provide more data for deciding between alternative models.

Educational implications. The existence of action-oriented approaches like the one observed here provides evidence for the assertion that children have intuitive conceptions about mathematical situations that are more basic than the four arithmetic operations. It is suggested here that, if traditional arithmetic operations are to have meaning for children, it must be in terms of similar types of

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[2]Additional data which motivated the development of the concepts used in this paper are provided in Clement (1977).

underlying intuitive conceptions. It is also suggested that these intuitions are the natural source of the semantic interpretations for number operations needed to accompany the syntactic rules for symbol manipulation that are so heavily emphasized in school.

More exploratory clinical interviews are needed to map intuitive mathematical structures and to study the relationship between formal and intuitive mathematics in the classroom. For example, the author has found that many third graders are able to solve practical story problems involving multiplication and division with small numbers before studying these operations in school (Clement, 1977). These students use a variety of methods such as skip counting, drawings, concrete materials, etc. Because some children have more difficulty than others in using intuitive methods, it would seem to make sense to have children strengthen their intuitive conceptions with problems like the one discussed here before they learn the operations as facts and algorithms. This intuitive approach could be a step toward making arithmetic more meaningful for children and a step toward remedying the difficulties many students have in applying their knowledge of arithmetic to practical problems.

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