Problem 35. \[ u''(x) - x^2 u(x) = f(x), \quad x \in [0, 1) \]
\[ u(0) = 0, \quad u'(1) = 0 \]

(a) \[ G(x, x_0) = \begin{cases} c \cdot V(x_0) U(x), & x < x_0 \\ c \cdot U(x_0) V(x), & x > x_0 \end{cases} \]
\[ c = \frac{1}{u' v - v u'} \]

\[
\begin{align*}
U'' - x^2 U &= 0 \\
U(0) &= 0 \\
U(x) &= Ae^{rx} + Be^{-rx}
\end{align*}
\]
\[ U(0) = 0 \Rightarrow A + B = 0 \]
\[ a \quad 1 \text{ then } A = -1 \]
\[ U(x) = e^{rx} - e^{-rx} \]

\[
\begin{align*}
V'' - x^2 V &= 0 \\
V'(1) &= 0 \\
V(x) &= Ee^{-rx} + Be^{rx}
\end{align*}
\]
\[ V'(1) = 0 \Rightarrow -Ee^{-r} + De^{r} = 0 \]
\[ a \quad d = e^{r} \text{ then } E = e^{r} \]
\[ V(x) = e^{r(x-1)} + e^{r(x-1)} \]

\[
c^{-1} = (e^{rx} - e^{-rx})(-8e^{x(1-x)} + 8e^{x(2-x)}) = \\
- (e^{x(1-x)} + e^{x(x-1)})(re^{rx} + re^{-rx}) = \\
-8re^{x} + 8e^{x(1-x)} + re^{x(1-x)} - re^{x(2-x)} \\
- re^{x(2-x)} - re^{x} - re^{x} \\
\]

\[ c = -\frac{1}{2r(e^{r} + e^{-r})} \]

(b) \( f(x) = x \)

\[
U(x) = \int_0^x G(x, x_0) f(x_0) \, dx_0 = \int_0^x G(x, x_0) f(x_0) \, dx_0 + \int_0^x G(x, x_0) f(x_0) \, dx =
\]

\[
e^{-V(x)} \int_0^x U(x_0) f(x_0) \, dx_0 + e^{-V(x)} \int_0^x U(x_0) f(x_0) \, dx_0
\]

\[
\int_0^x U(x_0) f(x_0) \, dx_0 = \int_0^x (e^{\sigma x_0} - e^{-\sigma x_0}) x_0 \, dx_0
\]

\[
\int_0^x V(x_0) f(x_0) \, dx_0 = \int_0^x (e^{\sigma (1-x)} + e^{\sigma (x-1)}) x_0 \, dx_0
\]

By parts:

\[
\int e^{\sigma x} x \, dx = -\frac{1}{\sigma} e^{\sigma x} x - \frac{1}{\sigma} \int e^{\sigma x} \, dx =
\]

\[
e^{\sigma x} \left( \frac{1}{\sigma} x - \frac{1}{\sigma^2} e^{\sigma x} \right)
\]

\[
\int e^{\sigma x_0} x_0 \, dx_0 = \frac{1}{\sigma} e^{\sigma x} x_0 - \frac{1}{\sigma^2} \left[ e^{\sigma x} (\sigma x_0 - 1) \right]_0^x = \frac{1}{\sigma^2} \left[ e^{\sigma x} (\sigma x_0 - 1) + 1 \right]
\]

\[
\int e^{-\sigma x_0} x_0 \, dx_0 = \frac{1}{\sigma} e^{-\sigma x} x_0 - \frac{1}{\sigma^2} \left[ e^{-\sigma x} (-\sigma x_0 - 1) \right] = \frac{1}{\sigma^2} \left[ 1 - e^{-\sigma x} (\sigma x_0 + 1) \right]
\]

\[
\int e^{\sigma (1-x_0)} x_0 \, dx_0 = e^{\sigma x} \left[ e^{-\sigma x_0} x_0 \right]_0^x = e^{\sigma x} e^{-\sigma x_0} \left[ x_0 \right]_0^x =
\]

\[
= \frac{1}{\sigma^2} e^{\sigma (1-x)} = \frac{1}{\sigma^2} (\sigma x + 1) - \frac{1}{\sigma^2} (\sigma + 1)
\]
\[ \int e^{\gamma x_0} x_0 \, dx_0 = e^{-\gamma} \frac{1}{x} \left. e^{\gamma x_0} x_0 \, dx_0 \right|_x^1 = e^{-\gamma} \frac{1}{\gamma^2} e^{\gamma x_0} (\gamma x_0 - 1) \mid_x^1 = \frac{1}{\gamma^2} (\gamma - 1) - \frac{1}{\gamma^2} e^{\gamma (\gamma - 1)} (\gamma x_0 - 1) \]

\[ u(x) \cdot e^{-\gamma} = \frac{1}{\gamma^2} \left[ e^{\gamma (1-x)} + e^{\gamma (x-1)} \right] \left[ e^{\gamma x_0} (\gamma x_0 - 1) + \frac{e^{-\gamma x}}{\gamma} + \frac{e^{\gamma x}}{\gamma} + \frac{e^{\gamma (1-2x)}}{\gamma} \right] + \]

\[ + \frac{1}{\gamma^2} \left[ e^{\gamma x} - e^{-\gamma x} \right] \left[ e^{\gamma (1-x)} - \frac{e^{-\gamma x}}{\gamma} + \frac{e^{\gamma x}}{\gamma} - e^{\gamma (x-1)} \right] \]

\[ u(x) \cdot e^{-\gamma} = e^{\gamma (\gamma x_0 - 1)} + e^{\gamma (x-1)} + e^{\gamma (1-2x)} + e^{\gamma (1-2x)} - e^{\gamma (2x-1)} - e^{\gamma (x+1)} - e^{\gamma (1-2x)} - e^{\gamma (x-1)} + 2 \left( e^{-\gamma x} - e^{\gamma x} \right) = \]

\[ = 2 \gamma x (e^{\gamma} + e^{-\gamma}) + 2 \left( e^{-\gamma x} - e^{\gamma x} \right) \]

\[ u(x) = - \frac{x}{\gamma^2} + \frac{e^{\gamma x} - e^{-\gamma x}}{\gamma^3 (e^{\gamma} + e^{-\gamma})} \]
Check

\[ u'(x) = -\frac{1}{x^2} + \frac{e^x + e^{-x}}{x^2(e^x + e^{-x})} \]

\[ u(0) = 0 \quad V \]
\[ u'(1) = 0 \quad V \]

\[ u''(x) = \frac{e^x - e^{-x}}{x(e^x + e^{-x})} \]

\[ u'' - \frac{1}{x^2}u = \frac{e^x - e^{-x}}{x(e^x + e^{-x})} + x - \frac{e^x - e^{-x}}{x(e^x + e^{-x})} = x \quad V \]
Problem 34.

\[ u(x,t) = \int g(x,x_0,t) u(x_0,0) \, dx_0 = \]

\[ = u_0 \frac{1}{2} \sqrt{\frac{\pi}{t}} \int e^{- \frac{(x-x_0)^2}{4t}} \, dx_0 \]

Re-scaling: \( x = e_0 \cdot x', \quad x_0 = e_0 \cdot y', \quad u = \frac{u_0}{\sqrt{\pi}} u', \quad t = 8 e_0^2 \cdot t'/4 \)

\[ u'(x',t') = \frac{1}{\sqrt{t'}} \int e^{- \frac{(x'-y)^2}{4t'}} \, dy \]

below we omit primes:

\[ u = \frac{1}{\sqrt{t}} \int e^{- \frac{1}{t} (x-y)^2 - y^2} \, dy \]

\[ \frac{1}{t} (x-y)^2 + y^2 = \frac{1}{t} (x^2 + y^2 - 2xy) + y^2 = \frac{x^2}{t} + (1+\frac{1}{t})y^2 - \frac{2x}{t}y \]

\[ u = \frac{1}{\sqrt{t}} \int e^{- \frac{1}{t} (1+\frac{1}{t})y^2 + \frac{2x}{t}y} \, dy = \int e^{- \frac{y}{t - \frac{x^2}{2(1+\frac{1}{t})}}} \, dy \]

\[ = \sqrt{\frac{t}{1+t}} e^{- \frac{x^2}{4(1+t)\frac{1}{2(1+t)}}} \]

\[ = \sqrt{\frac{t}{1+t}} e^{- \frac{x^2}{4(1+t)}} \quad \text{and} \quad \sqrt{\frac{t}{1+t}} e^{- \frac{x^2}{1+t}} \]
Returning to the original variables:

\[ u \to u \frac{u_0}{\sqrt{4t}} \]

\[ x \to x/e_0 \]

\[ t \to 4t/\delta e_0^2 \]

\[
u(x,t) = \frac{u_0}{\sqrt{1 + \frac{4t}{\delta e_0^2}}} e^{-\frac{(x/e_0)^2}{1 + \frac{4t}{\delta e_0^2}}} \]

At \( t \gg \frac{\delta e_0^2}{4} \), \( u(x,t) \approx \frac{u_0 e_0}{2} \sqrt{\frac{5}{t}} e^{-\frac{5x^2}{4t}} \)
Problem 35.

\[-i \mu U_t = U_{xx} \quad (1)\]

Re-define \( t = \gamma t' \).
Below we omit primes.

\[-i U_t = U_{xx} \]

\[ U(x, y) = \int g(x, y, x_0) U(x_0, 0) \, dx_0 \]

\[ G(x, x_0, t) = G(x-x_0, t) \quad \text{because of the translational invariance.} \]

For \( G(x,t) \) we have:

\[-i G_t = G_{xx} \quad (2)\]

Initial condition: \( \forall q(x), \lim_{t \to 0} \int G(x,t) q(x) \, dx = q(0) \).

Our Eq. (1) differs from the heat equation by the coefficient \(-i\) in the l.h.s.
This is not important for scaling transformation and we immediately conclude that \( \forall \lambda \),

\[ \lambda G(x, \lambda^2 t) = G(x, t) \]

and thus

\[ G(x, t) = \frac{q(x^2/t)}{\sqrt{t}} \]
and \[ \int_0^t g(y^2) \, dy = 1. \]

derivatives:
\[ G_x = \frac{1}{\sqrt{t}} \, g'(x^2/t) \cdot 2x/\sqrt{t} \]
\[ G_{xx} = \frac{1}{\sqrt{t}} \, g'(x^2/t) \cdot 2/t + \frac{1}{\sqrt{t}} \, g''(x^2/t) \cdot 4x^2/t^2 \]
\[ G_t = -\frac{1}{t} \frac{1}{\sqrt{12}} \, g'(x^2/t) + \frac{1}{\sqrt{t}} \, g'(x^2/t) \cdot x^2 (-t^2) \]

Plug this into (2), introduce \( \xi = x^2/t \).

\[ -i \left[ -\frac{1}{2} \frac{1}{\sqrt{12}} \, g = \frac{i}{\sqrt{12}} \, g^1 \cdot \xi \right] = 2 \frac{i}{\sqrt{12}} \, g^1 + 4 \frac{i}{\sqrt{12}} \, g^\prime \cdot \xi \]

\[ \xi \xi \dot{g}^\prime + 2 \xi \dot{g} - i \xi g^\prime - \frac{i}{2} \dot{g} = 0 \]

\[ (\xi \frac{1}{\sqrt{12}} + \frac{1}{2})(\xi g^\prime - ig^1) = 0 \]

\[ \dot{g} = \xi g^\prime - ig^1 \]

\[ (\xi \frac{1}{\sqrt{12}} + \frac{1}{2}) \dot{X} = 0 \]

Let us try \( X = 0 \). Then \( g(\xi) = Ae^{\xi^2/4} \).

Our guess was right: \( g(\xi) \) remains finite at \( \xi \to \infty \).
\[ t = \int_0^\infty g(y^2) \, dy = A \int_0^\infty e^{iy^2/4} \, dy = 2A \int_0^\infty e^{iy^2} \, dy = \\
= 2A \sqrt{\pi} e^{-i\pi/4} \Rightarrow A = \frac{e^{-i\pi/4}}{2\sqrt{\pi}} \]

\[ G(x,t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \ e^{i x^2/4t} \]

In the original variables:

\[ G(x-x_0,t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \sqrt{\frac{t}{\pi}} \ e^{i \frac{r(x-x_0)^2}{2t}} \]
Problem 36.

\[ y(x,t) = A \int \frac{e^{-i\pi/4}}{2\sqrt{\pi}} \sqrt{\frac{t}{x}} e^{\frac{i}{2} \frac{(x-x_0)^2}{t}} e^{-(x_0/c_0)^2 + ikx_0} \, dx_0. \]

Re-scaling: \( x = c_0 x', \quad x_0 = c_0 y, \)

\[ y = \frac{A e^{i\pi/4}}{\sqrt{\pi}} y', \quad t = 8c_0^2 t'/4, \quad k = k'/c_0. \]

Below we omit primes:

\[ y = \frac{1}{\sqrt{t}} \int e^{\frac{i}{t} \frac{(x-y)^2}{s}} - y^2 + iky \, dy. \]

\[ \frac{i}{t} \frac{(x+y^2-2xy)}{s} - y^2 + iky = \]

\[ = \frac{i}{t} \frac{x^2}{s} - \frac{y^2 (1 - i/t)}{s} + \frac{i (k - 2x/t)}{s} y, \]

\[ y = \frac{1}{\sqrt{t}} e^{\frac{i}{t} \frac{x^2}{s}} \int e^{-\frac{(1-i/t)y^2 + i(k-2x/t)y}{s}} \, dy = \]

\[ = \frac{e^{i x^2/4}}{\sqrt{t} \sqrt{1-i/t}} \cdot \frac{\sqrt{\pi}}{\sqrt{1-i/t}} e^{- \frac{(k-2x/t)^2}{4(1-i/t)}} \]
\[
\frac{(k - 2x/t)^2}{4(t^2 + 1)} = \frac{t(x - (k/2)t)^2}{4(t^2 + 1)} + \frac{t(t+i)(kt-2x)^2/t^2}{4(t^2+1)}
\]

\[
= \frac{(t+i)(x - (k/2)t)^2}{(t^2 + 1)t} = \frac{[x - (k/2)t]^2}{-t^2 + 1} + \]

\[+ \frac{i}{(t^2 + 1)t} \left[ x^2 - kxt + \frac{k^2}{4} t^2 \right] \]

\[
\frac{i x^2}{t} = \frac{(k - 2x/t)^2}{4(1-i/t)} = -\frac{(x - k t/2)^2}{t^2 + 1} + \]

\[+ \frac{i}{(t^2 + 1)t} \left[ x^2(t^2+1) - x^2 + kxt - \frac{k^2 + 2}{4} \right] =
\]

\[= -\frac{(x - k t/2)^2}{t^2 + 1} + i \frac{x^2 t + kxt - kt/4}{t^2 + 1} \]

\[
\psi(x, t) = \frac{\sqrt{\pi}}{\sqrt{t-i}} \ e^{-\frac{(x - k t/2)^2}{t^2 + 1}} \ e^{i \frac{x^2 t + kxt - kt/4}{t^2 + 1}}
\]

\[
|\psi(x, t)|^2 = \frac{\pi}{\sqrt{t^2 + 1}} e^{-\frac{2(x - k t/2)^2}{t^2 + 1}}
\]
\[ 14 (x + 1) = \frac{1}{t^2 + 1} e^{-2 \left( \frac{x - x_x(t)}{t^2 + 1} \right)^2} \]

where \( x_x(t) = k t / 2 \) is the coordinate of the maximum of \( |f|^2 \).

In the original units:

- \( x_x \rightarrow x_x / \ell_0 \)
- \( k \rightarrow k \ell_0 \)
- \( \ell \rightarrow 4 \ell / \ell_0^2 \)

\[ x_x(t) / \ell_0 = \frac{1}{2} \frac{k \ell_0}{\ell} \frac{4t}{\ell_0^2} \]

\[ x_x(t) = \frac{2k}{\ell} \frac{t}{\ell_0} = \frac{k}{m} \frac{t}{\ell_0} \]

Hence, the velocity of the wavepacket is

\[ v = \frac{k}{m} \]

The width of the packet is given by

\[ e^{-2 \frac{(\Delta x)^2}{t^2 + 1}} \approx 1 \]

\[ \Delta x \approx \ell_0 \sqrt{t^2 + 1} \]

In the original units:

\[ \Delta x \approx \ell_0 \sqrt{1 + \frac{16t^2}{\ell_0^2 \ell_0^2}} \]