Separation of Variables in Polar and Spherical Coordinates

Helmholtz/Poisson Equation in 2D: Polar Coordinates

Consider the 2D Helmholtz equation (if $\gamma = 0$, it is the Poisson equation):

$$\Delta u + \gamma u = 0,$$

(1)

where the function $u$ is defined in 2D plane in the region between two circles, the smaller one with the radius $r_1$, and the larger one, with the radius $r_2$ (see Fig. 1). The limiting cases $r_1 \to 0$ and $r_2 \to \infty$ are also included. The function $u$ is subject to certain boundary conditions at the boundaries of its domain of definition. In polar coordinates, $(\rho, \varphi)$, these conditions can be written as

$$u(\rho = r_1, \varphi) = f_1(\varphi), \quad u(\rho = r_2, \varphi) = f_2(\varphi).$$

(2)

The problem (1)-(2) can be solved by separating the variables. Looking at the 2D Laplace operator in 2D polar coordinates,

$$\Delta = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2},$$

(3)

we observe that the variable $\varphi$ enters the expression in the form of the operator $\partial^2/\partial \varphi^2$. This operator—1D Laplace operator with respect to the variable $\varphi$—is a Hermitian operator in the space of single-valued functions of the angle $\varphi$, because single-valuedness is equivalent to the $2\pi$-periodicity, and Laplace operator is Hermitian in the space of periodic functions. Hence, we have ONB of the eigenfunctions of the operator $\partial^2/\partial \varphi^2$:

$$\Phi_m(\varphi) = e^{im\varphi}/\sqrt{2\pi}, \quad m = 0, \pm 1, \pm 2, \ldots,$$

(4)

$$\frac{\partial^2 \Phi_m}{\partial \varphi^2} = -m^2 \Phi_m,$$

(5)

$$\int_0^{2\pi} \Phi_m(\varphi)^* \Phi_n(\varphi) d\varphi = \delta_{mn}.$$

(6)

We then expand our solution $u(\rho, \varphi)$ in the Fourier series with respect to our ONB:

$$u(\rho, \varphi) = \sum_{m=-\infty}^{\infty} P_m(\rho) \Phi_m(\varphi).$$

(7)
Here $P_m(\rho)$’s are Fourier coefficients (which naturally depend on $\rho$). Substituting $u$ in (1) with the right-hand side of (7) and taking into account (5), we get

$$
\sum_{m=-\infty}^{\infty} \Phi_m(\varphi) \left( \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} - \frac{m^2}{\rho^2} + \gamma \right) P_m(\rho) = 0 .
$$

(8)

Constructing inner products with the functions $\Phi_m(\varphi)$ we obtain the equation for each $P_m$:

$$
\rho^2 P_m'' + \rho P_m' + (\gamma \rho^2 - m^2) P_m = 0 .
$$

(9)

Eq. (9) is known as Bessel equation. It is a second-order ordinary differential equation, which implies that each $P_m$ is defined up to two unknown constants to be fixed by the boundary conditions:

$$
\sum_{m=-\infty}^{\infty} P_m(r_1) \Phi_m(\varphi) = f_1(\varphi) ,
$$

(10)

$$
\sum_{m=-\infty}^{\infty} P_m(r_2) \Phi_m(\varphi) = f_2(\varphi) ,
$$

(11)

from which we immediately have

$$
P_m(r_1) = \int_0^{2\pi} e^{-im\varphi} f_1(\varphi) d\varphi / \sqrt{2\pi} ,
$$

(12)
\[ P_m(r_2) = \int_0^{2\pi} e^{-im\varphi} f_2(\varphi) \, d\varphi / \sqrt{2\pi} . \]  

(13)

In the case \( r_1 \to 0 \) (or \( r_2 \to \infty \)) the corresponding boundary condition is replaced with the requirement that the function \( P_m \) be finite. The solution of Bessel equation is expressed in terms of so-called special functions (the Bessel function and functions associated to it). The properties of the special functions are well studied and described in textbooks. The case \( \gamma = 0 \) is especially simple. As it is easily checked, \( P_m \)'s now are just polynomials (below \( A \)'s and \( B \)'s are constants):

\[ P_m(\rho) = A_m \rho^m + B_m \rho^{-m}, \quad m \neq 0, \quad (\gamma = 0), \]  

(14)

\[ P_0(\rho) = A_0 + B_0 \ln \rho, \quad (\gamma = 0) . \]  

(15)

**Example.** Suppose we need to find the function \( u(\rho, \varphi) \) which satisfies the Poisson equation,

\[ \Delta u = 0 , \]  

(16)
everywhere in 2D plane, except for the two circles (Fig. 1), where it is subject to the boundary conditions

\[ u(\rho = r_1, \varphi) = \sin \varphi , \]  

(17)

\[ u(\rho = r_2, \varphi) = \cos \varphi , \]  

(18)
and tends to zero at \( \rho \to \infty \)).

**Solution.** First we note that there are 3 independent regions:

(a) \( \rho \in [0, r_1] \),

(b) \( \rho \in [r_1, r_2] \),

(c) \( \rho \in [r_2, \infty) \).

We thus have to separately solve the problem in each of the regions.

(a) Here we have the boundary condition (17) and the requirement that \( u \) be finite at \( \rho = 0 \). The latter means that we have to exclude the singular terms from the solution (7), (14)-(15), and write

\[ u = \sum_{m=-\infty}^{\infty} A_m \rho^{|m|} \Phi_m(\varphi) \equiv A_0 + \sum_{m=1}^{\infty} \rho^m (C_m \sin m\varphi + D_m \cos m\varphi) . \]  

(19)

The representation in terms of sines and cosines is more convenient here, because of the particular form of the boundary conditions.—We immediately
see that only one term of the series is non-zero. Namely, the one with \( \sin \varphi \); and the answer is

\[
\begin{align*}
  u &= (\rho/r_1) \sin \varphi \equiv y/r_1. 
\end{align*}
\]  

(20)

(c) This case is very similar to the previous one. Now we exclude the terms divergent in the limit of \( \rho \to \infty \) and write

\[
\begin{align*}
  u &= \sum_{m=1}^{\infty} \rho^{-m} (C_m \sin m\varphi + D_m \cos m\varphi). 
\end{align*}
\]  

(21)

Once again the sine-cosine representation is more convenient than the exponents. The relevant boundary condition is Eq. (18), from which we see that the only non-zero term is that with \( \cos \varphi \), and find

\[
\begin{align*}
  u &= (r_2/\rho) \cos \varphi. 
\end{align*}
\]  

(22)

(b) Here once again the sine-cosine representation is convenient:

\[
\begin{align*}
  u &= A_0 + B_0 \ln \rho + \sum_{m=1}^{\infty} \left\{ \left[ C_m \rho^m + \frac{\tilde{C}_m}{\rho^m} \right] \sin m\varphi + \left[ D_m \rho^m + \frac{\tilde{D}_m}{\rho^m} \right] \cos m\varphi \right\}. 
\end{align*}
\]  

(23)

We have only two terms non-orthogonal to the boundary conditions, the term with \( \sin \varphi \) and the term with \( \cos \varphi \). Corresponding coefficients are easily found from the boundary conditions [everything is seen by the naked eye, and we do not need to calculate the inner products]:

\[
\begin{align*}
  C_1 r_1 + \tilde{C}_1/r_1 &= 1, \\
  C_1 r_2 + \tilde{C}_1/r_2 &= 0, \\
  D_1 r_1 + \tilde{D}_1/r_1 &= 0, \\
  D_1 r_2 + \tilde{D}_1/r_2 &= 1. 
\end{align*}
\]  

(24)(25)

The final answer is:

\[
\begin{align*}
  u &= \frac{r_1 (r_2^2/\rho - \rho) \sin \varphi}{r_2^2 - r_1^2} + \frac{r_2 (\rho - r_2^2/\rho) \cos \varphi}{r_2^2 - r_1^2}. 
\end{align*}
\]  

(26)

**Helmholtz/Poisson Equation in 3D: Spherical Coordinates**

Generalization to the 3D case is quite straightforward. We have 3D Helmholtz (Poisson, if \( \gamma = 0 \)) equation:

\[
\begin{align*}
  \Delta u + \gamma u &= 0, 
\end{align*}
\]  

(27)
where the function $u$ is defined in 3D space in the region between two spheres, the smaller one with the radius $r_1$, and the larger one, with the radius $r_2$ (Fig. 1). The function $u \equiv u(r, \theta, \varphi)$ satisfies the boundary conditions:

\[ u(r = r_1, \theta, \varphi) = f_1(\theta, \varphi) , \quad u(r = r_2, \theta, \varphi) = f_2(\theta, \varphi) . \tag{28} \]

The problem (27)-(28) is solved by separating the variable $r$ from the angular variables. The 3D Laplace operator in the spherical coordinates is:

\[ \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} \hat{l}^2 , \tag{29} \]

where

\[ \hat{l}^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \tag{30} \]

is the square of the angular momentum operator. The operator $\hat{l}^2$ is a Hermitian operator in the space of single-valued functions of the angular variables $(\theta, \varphi)$. Hence, it features ONB of eigenfunctions (see, e.g., any serious textbook on Quantum Mechanics):

\[ \hat{l}^2 Y_{lm}(\theta, \varphi) = l(l+1) Y_{lm}(\theta, \varphi) , \tag{31} \]

\[ \langle Y_{l_1 m_1} | Y_{l_2 m_2} \rangle \equiv \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta Y_{l_1 m_1}^*(\theta, \varphi) Y_{l_2 m_2}(\theta, \varphi) = \delta_{l_1 l_2} \delta_{m_1 m_2} . \tag{32} \]

Now we expand the solution $u(r, \theta, \varphi)$ in the Fourier series with respect to our ONB:

\[ u(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l R_{lm}(r) Y_{lm}(\theta, \varphi) . \tag{33} \]

That is we treat $r$ in $u(r, \theta, \varphi)$ as a parameter, while $(\theta, \varphi)$ is a pair of variables—with respect to the basis $\{ Y_{lm}(\theta, \varphi) \}$. Substituting $u$ in (27) with the right-hand side of (33) and taking into account (30), we get

\[ \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\theta, \varphi) \left[ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{l(l+1)}{r^2} + \gamma \right] R_{lm}(r) = 0 , \tag{34} \]

which implies

\[ r^2 R_{lm}'' + 2r R_{lm}' + [\gamma r^2 - l(l+1)] R_{lm} = 0 . \tag{35} \]

Hence, $R_{lm}$’s are found by solving the second-order ordinary differential equation, the boundary conditions to which follow from

\[ u(r_1, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l R_{lm}(r_1) Y_{lm}(\theta, \varphi) , \tag{36} \]
\[ u(r_2, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_{lm}(r_2) Y_{lm}(\theta, \varphi), \quad (37) \]

which yields:

\[ R_{lm}(r_1) = \langle Y_{lm} | f_1 \rangle \equiv \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta Y_{lm}^*(\theta, \varphi) f_1(\theta, \varphi), \quad (38) \]

\[ R_{lm}(r_2) = \langle Y_{lm} | f_2 \rangle \equiv \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta Y_{lm}^*(\theta, \varphi) f_2(\theta, \varphi). \quad (39) \]

Introducing the new variable

\[ Q_{lm} = \sqrt{r} R_{lm}, \quad (40) \]

we reduce Eq. (35) to the Bessel equation

\[ r^2 Q''_{lm} + r Q'_{lm} + \left[ \gamma r^2 - (l + 1/2)^2 \right] Q_{lm} = 0. \quad (41) \]

As in 2D, the solution is especially simple at \( \gamma = 0 \) (A’s and B’s are constants):

\[ R_{lm}(r) = A_{lm} r^l + B_{lm} r^{-(l+1)} \quad (\gamma = 0). \quad (42) \]

**Diffusion and Wave Equations**

The Helmholtz equation considered above is time-independent. Typical examples of time-dependent linear PDEs are diffusion and wave equations. In both cases the separation of variables relies on the properties of Laplace operator, having little to do with the specific form of the time-derivative term. For definiteness, we consider the diffusion equation,

\[ u_t = \Delta u, \quad (43) \]

where the function \( u \) satisfies the conditions (2) in 2D, or (28) in 3D. These boundary conditions are not “convenient”, because the functions satisfying them do not form a linear space. This problem, however, is easily circumvented by taking advantage of the linearity of Eq. (43). If we subtract from the function \( u \) a particular *time independent* function \( \tilde{u} \) satisfying given boundary conditions and obeying the Poisson equation

\[ \Delta \tilde{u} = 0. \quad (44) \]

Then, for the difference \( w = u - \tilde{u} \) we will have the same diffusion equation

\[ w_t = \Delta w, \quad (45) \]
but with the condition \( w \equiv 0 \) at the boundaries (and shifted initial condition).

Since the solution of the Poisson equation is already discussed, we need to consider only the solution of the problem (45)-(46). The key idea is that with zero boundary conditions the Laplace operator is a Hermitian operator in the linear space of functions satisfying these boundary conditions. Correspondingly, one can introduce ONB of the eigenfunctions of the Laplace operator, \( \{e_s(r)\} \),

\[
\Delta e_s = -\lambda_s e_s ,
\]

and expand the solution in the Fourier series with respect to this basis (treating time variable in \( u(r, t) \) as just a parameter labelling some particular function of \( r \)). We thus write

\[
u(r, t) = \sum_s q_s(t)e_s(r) ,
\]

plug this into the diffusion equation, and get ordinary differential equations for \( q \)'s,

\[
\dot{q}_s = -\lambda_s q_s ,
\]

with the initial conditions

\[
q_s(0) = \int dr e_s^*(r) u(r, t = 0) .
\]

The real problem is how to find \( \{e_s(r)\} \). Eq. (46) is the Helmholtz equation, with zero boundary conditions in our case. In a general case, when the shape of the boundaries of the domain of definition of \( e_s(r) \) is arbitrary, the problem can be solved only numerically. The two simple cases when there are analytic solutions are (i) the case of rectangular geometry and (ii) the case of rotationally-symmetric geometry. Also simple is the case of cylindrical symmetry in 3D, which corresponds to rotational symmetry in the \( xy \)-plane, and rectangular geometry along the \( z \)-axis. The solution of the Helmholtz equation in polar and spherical coordinates has been discussed above. The only specifics is that now the parameter \( \gamma \) is a free parameter. Its possible values—eigenvalues of the problem—are defined in the process of solving Eq. (9)/Eq. (35) with the given linear boundary conditions: The solution exists only for special values of \( \gamma \); these values form the set \( \{\lambda_s\} \) of the eigenvalues of the Laplace operator.