Green Function of a Time-Dependent Linear PDE

Consider one-dimensional linear PDE of the form

$$\gamma u_t = u_{xx},$$  \hspace{1cm} (1)

where $u = u(x,t), \ x \in (-\infty, \infty), \ u_t = \partial u/\partial t, \ u_{xx} = \partial^2 u/\partial x^2$. We want to find $u(x,t)$, provided we know $u(x,t = 0)$. Suppose for a while, that $x \in [-a/2, a/2]$, and we then take the limit of $a \to \infty$. For finite $a$ the problem has the generic form of the differential equation in a Hilbert space:

$$\left( \frac{d}{dt} \right) |u(t)\rangle = L |u(t)\rangle,$$  \hspace{1cm} (2)

where $L$ is a self-adjoint operator. One looks for $|u(t)\rangle$, provided $|u(t = 0)\rangle$ is known. The generic solution of (2) can be written as a Fourier series over the orthonormal basis of the eigenfunctions of the operator $L$:

$$|u(t)\rangle = \sum_s C_s e^{\lambda_s t/\gamma} |e_s\rangle,$$  \hspace{1cm} (3)

$$L |e_s\rangle = \lambda_s |e_s\rangle.$$  \hspace{1cm} (4)

To satisfy the initial condition we choose $C_s$’s as

$$C_s = \langle e_s | u(t = 0) \rangle.$$  \hspace{1cm} (5)

Translating into the language of functions, where the inner product is understood as an integral, we see that

$$u(x,t) = \int G(x, x', t) u(x', t = 0) \, dx',$$  \hspace{1cm} (6)

where

$$G(x, x', t) = \sum_s e^{\lambda_s t/\gamma} e_s(x) e^*_s(x').$$  \hspace{1cm} (7)

The function $G$ is called Green function. If we know the Green function, we can immediately restore $u(x,t)$ from $u(x,0)$.

A note is in order here. When obtaining (6) we swapped the integration and the summation which might be not legitimate within the space of ordinary functions. This means that in certain cases $G$ should be understood as a generalized function, which is even better from the practical viewpoint.—The integrals with the generalized functions are easily taken!
How can we find $G$ explicitly? And why do we expect $G$ to exist in the limit of $a \to \infty$? We start from the second question. Physically we understand that if $a$ is large enough and our $u(x, t)$ is well-behaved in the sense that it is practically zero already at $|x| \ll a$, then the evolution of $u$ should be independent of $a$, which implies that $G$ reaches some $a$-independent limit, corresponding to the case $x \in (-\infty, \infty)$.

To find $G$, we act with the operator $\hat{O} = \gamma \partial / \partial t - \partial^2 / \partial x^2$ on both sides of Eq. (6) to obtain (subscript $x$ means that the coordinate part of the operator $\hat{O}$ acts on the coordinate $x$ rather than $x'$)

$$\int \hat{O}_x G(x, x', t) u(x', t = 0) \, dx' = 0 .$$  

(8)

Since (8) is supposed to be valid for any function $u(x', t = 0)$, that is the function $\hat{O}_x G(x, x', t)$ (of the argument $x'$, while $x$ and $t$ are treated as parameters) is orthogonal—in the vector sense—to any function of the Hilbert space, we conclude that

$$\hat{O}_x G(x, x', t) = 0 .$$  

(9)

Setting in (6) $t = 0$ we see that for any function $g(x)$:

$$g(x) = \int G(x, x', 0) g(x') \, dx' ,$$  

(10)

which means that

$$G(x, x', 0) = \delta(x - x') .$$  

(11)

Now we take into account that in the limit $a \to \infty$ the problem becomes translationally invariant, and thus $G(x, x', t) \equiv G(x - x', t)$. Hence we need to find such function $G(x, t)$ that

$$\gamma G_t = G_{xx} ,$$  

(12)

$$G(x, 0) = \delta(x) .$$  

(13)

Below we will do it by revealing the self-similarity of the function $G$.

**Scaling transformation. Similarity. Self-similarity.** Let $u(x, t)$ be a solution of the equation (1). Consider the scaling transformation

$$u(x, t) \to \tilde{u}(x, t) = \lambda u(\lambda^\alpha x, \lambda^\beta t) ,$$  

(14)

where $\lambda$, $\alpha$, and $\beta$ are real numbers. By a direct check we make sure that

$$\gamma \tilde{u}_t = \lambda^{\beta - 2\alpha} \tilde{u}_{xx} ,$$  

(15)
which means that if we take $\beta = 2\alpha$, the function $\tilde{u}(x,t)$ satisfies Eq. (1). We thus arrive at the similarity transformation

$$ u(x,t) \to \tilde{u}(x,t) = \lambda u(\lambda^\alpha x, \lambda^{2\alpha} t) , \quad (16) $$

that, given some solution $u(x,t)$, produces a continuum of other, similar, solutions. In Eq. (16), $\lambda$ and $\lambda^\alpha$ are actually two independent parameters. The reason why we write them in such a form becomes clear when we perform the similarity transformation with the function $G$: $G(x,t) \to \tilde{G}(x,t) = \lambda G(\lambda^\alpha x, \lambda^{2\alpha} t)$ and consider

$$ \tilde{G}(x,0) = \lambda G(\lambda^\alpha x, 0) = \lambda \delta(\lambda^\alpha x) = \lambda^{1-\alpha} \delta(x) . \quad (17) $$

We see that at $\alpha = 1$, the function $\tilde{G}$ satisfies both the equation (12) and the initial condition (13), which means that it is nothing else than the function $G$. Hence, we have proven the self-similarity of the function $G$:

$$ \lambda G(\lambda x, \lambda^2 t) \equiv G(x,t) \quad (18) $$

This property implies some special form of the function $G$. Indeed, without loss of generality one may introduce the new variable $\xi = x^2/t$ instead of $x$ and write $G(x,t) \equiv f(\xi,t)$. With the new variables, Eq. (18) reads

$$ \lambda f(\xi, \lambda^2 t) \equiv f(\xi, t) . \quad (19) $$

Lemma. If any function $f(t)$ satisfies

$$ \lambda f(\lambda^\alpha t) \equiv f(t) \quad (20) $$

for some given $\alpha$ and arbitrary $\lambda$, then

$$ f(t) \propto t^{-1/\alpha} . \quad (21) $$

Proof. Differentiate (20) with respect to $\lambda$ and set $\lambda = 1$. Solve the resulting first-order differential equation.

With this lemma and (19) we have

$$ f(\xi,t) = \frac{g(\xi)}{\sqrt{t}} , \quad (22) $$

where $g(\xi)$ is some function of $\xi$. Substituting then $g(x^2/t)/\sqrt{t}$ for $G$ in (12), we get

$$ \left( \frac{1}{2} + \xi \frac{d}{d\xi} \right) \left[ 4g'(\xi) + \gamma g(\xi) \right] = 0 . \quad (23) $$
This is the second-order ordinary differential equation. Its general solution contains two free constants, which will be fixed by the two conditions: (i) \( g(\xi) \) should remain at least finite at \( \xi \to \infty \) and (ii) it should be consistent with (13), which implies

\[
\int t^{-1/2} g(x^2/t) \, dx = \int g(y^2) \, dy = 1 .
\]  

Eq. (23) is solved as follows. First one finds a solution of the supplementary equation

\[
\left( \frac{1}{2} + \xi \frac{d}{d\xi} \right) \chi(\xi) = 0
\]  

and then finds \( g(\xi) \) from

\[
4g'(\xi) + \gamma g(\xi) = \chi(\xi) .
\]  

It turns out that in our case the relevant solution of Eq. (25) is just \( \chi(\xi) \equiv 0 \). Indeed, solving (26) with \( \chi \equiv 0 \) we find

\[
g(\xi) = Ae^{-\gamma \xi/4} ,
\]  

where \( A \) is a constant. This solution does satisfy the requirement (i), and we need only to find \( A \) from (24), which yields \( A = (1/2)\sqrt{\gamma/\pi} \). The answer for the Green function thus is

\[
G(x, x', t) = \frac{1}{2} \sqrt{\frac{\pi}{\gamma}} e^{-\gamma(x-x')^2/4} .
\]  

**Problem 29.** The initial temperature profile is given by the function \( u(x) = u_0 e^{-x^2/\lambda_0^2} , x \in (-\infty, \infty) \). Use the Green function (28) to find \( u(x, t) \).

So far \( \gamma \) was real. What about \( \gamma = -i|\gamma| \) (Schrödinger equation: \( |\gamma| = 2m/\hbar \))? The procedure is exactly the same up to Eq. (27), inclusively. A slight difference arises when we find \( A \). Basically, we need to perform the integral

\[
\int_{-\infty}^{\infty} e^{iy^2} \, dy = \sqrt{\pi} e^{i\pi/4} .
\]  

The result for \( G \) then is

\[
G(x, x', t) = \frac{e^{-i\pi/4}}{2} \sqrt{\frac{|\gamma|}{\pi t}} e^{i|\gamma|(x-x')^2/4} \quad (\gamma = -i|\gamma|) .
\]
Problem 30. The initial state of the one-dimensional quantum particle is given by the wavefunction \( \psi(x) = Ae^{-(x/l_0)^2 + ikx}, \ x \in (-\infty, \infty), \ A \) is the normalization constant. Find \( \psi(x,t) \) with the Green function (30).