The results of IQ tests are known to be normally distributed. Suppose that in 2007, the distribution of IQ test scores for persons aged 18-35 years has a variance $\sigma^2 = 225$. A random sample of 9 persons take the IQ test. The sample mean score is 115.

1. Calculate the 50%, 75%, 90% and 95% confidence interval estimates of the unknown population mean IQ score.

**Answer:**

<table>
<thead>
<tr>
<th>CI %</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>50% CI</td>
<td>(111.6, 118.4)</td>
</tr>
<tr>
<td>75% CI</td>
<td>(109.2, 120.8)</td>
</tr>
<tr>
<td>90% CI</td>
<td>(106.8, 123.2)</td>
</tr>
<tr>
<td>95% CI</td>
<td>(105.2, 124.8)</td>
</tr>
</tbody>
</table>

**Solution:**

Let the random variable $X =$ IQ test result assumed normal with:

- $\mu$ unknown
- $\sigma^2 = 225$, known
- $\sigma = 15$, known

Confidence interval estimate of the unknown mean is given by:

$$\text{estimate} \pm \{ \text{critical value} \} \{ \text{se of estimate} \}$$

where,

- estimate $= \text{observed sample mean} = 115$
- critical value $= (1 - \alpha / 2)100\text{th percentile Normal}(0,1)$
- se of estimate $= \text{standard error of sample mean}$
  $$= \sqrt{\frac{225}{9}}$$
  $$= \frac{15}{3}$$
  $$= 5$$
For 50% confidence interval estimate:

\[ 1 - \alpha = (1 - 0.50) = 0.50 \]
\[ \alpha/2 = 0.50 / 2 = 0.25 \]

Therefore want (1 - .25)100th or 75th percentile = 0.6745
The required confidence interval estimate is thus,

\[ \text{estimate} \pm \{ \text{critical value} \} \{ \text{se of estimate} \} \]
\[ = 115 \pm \{ 0.6745 \} \{ 5 \} \]
\[ = (111.6, 118.4) \]

For 75% confidence interval estimate:

\[ 1 - \alpha = (1 - 0.25) = 0.75 \]
\[ \alpha/2 = 0.25 / 2 = 0.125 \]

Therefore want (1 - .125)100th or 87.5th percentile = 1.1505
The required confidence interval estimate is thus,

\[ \text{estimate} \pm \{ \text{critical value} \} \{ \text{se of estimate} \} \]
\[ = 115 \pm \{ 1.1505 \} \{ 5 \} \]
\[ = (109.2, 120.8) \]

For 90% confidence interval estimate:

\[ 1 - \alpha = (1 - 0.10) = 0.90 \]
\[ \alpha/2 = 0.10 / 2 = 0.05 \]

Therefore want (1 - .05)100th or 95th percentile = 1.645
The required confidence interval estimate is thus,

\[ \text{estimate} \pm \{ \text{critical value} \} \{ \text{se of estimate} \} \]
\[ = 115 \pm \{ 1.645 \} \{ 5 \} \]
\[ = (106.8, 123.2) \]

For 95% confidence interval estimate:

\[ 1 - \alpha = (1 - 0.05) = 0.95 \]
\[ \alpha/2 = 0.05 / 2 = 0.025 \]

Therefore want (1 - .025)100th or 97.5th percentile = 1.96
The required confidence interval estimate is thus,

\[ \text{estimate} \pm \{ \text{critical value} \} \{ \text{se of estimate} \} \]
\[ = 115 \pm \{ 1.96 \} \{ 5 \} \]
\[ = (105.2, 124.8) \]
2. What trade-offs are involved in reporting one interval estimate over another?

**Answer:**
For a given probability distribution with a known variance and a fixed sample size,

(i) Increasing the confidence coefficient is at the price of a wider confidence interval.

(ii) Decreasing the width of a confidence interval estimate is at the price of a lower confidence coefficient.

This is apparent in the following summary of the solution to Exercise #1.

<table>
<thead>
<tr>
<th>Confidence Coefficient</th>
<th>Lower Limit</th>
<th>Upper Limit</th>
<th>Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>.50</td>
<td>111.6</td>
<td>118.4</td>
<td>6.8</td>
</tr>
<tr>
<td>.75</td>
<td>109.2</td>
<td>120.8</td>
<td>11.6</td>
</tr>
<tr>
<td>.90</td>
<td>106.8</td>
<td>123.2</td>
<td>16.4</td>
</tr>
<tr>
<td>.95</td>
<td>105.2</td>
<td>124.8</td>
<td>19.6</td>
</tr>
</tbody>
</table>

3. If it is known that the population mean IQ score is $\mu = 105$, what proportion of samples of size 6 will result in sample mean values in the interval $[135,150]$?

**Answer: << .0001**

**Solution:**
Solve for $\Pr [ 135 < \bar{X}_{n=6} < 150 ]$ using the standardization formula.

Note that $\bar{X}_{n=6}$ is normally distributed with:

$\mu_X = 105$

$\sigma_X^2 = \frac{225}{6} = 37.5$

$SE_X = \sqrt{\sigma_X^2} = \sqrt{37.5} = 6.1238$

Thus,

$\Pr [ 135 < \bar{X}_{n=6} < 150 ] = \Pr [ \frac{135-105}{6.1238} < \frac{\bar{X}_{n=6}-\mu_X}{SE_X} < \frac{150-105}{6.1238} ]$

$= \Pr [ 4.89 < Z\text{-score} < 7.34 ] << .0001$