1. Suppose the distribution of GRE scores satisfies the assumptions of normality with a mean score of $\mu=600$ and a standard deviation of $\sigma=80$.

a. What is the probability of a score less than 450 or greater than 750? **Answer:** .0608

**Solution:** Probability \{ score < 450 OR score > 750 \}

\[
\begin{align*}
&= \Pr(X < 450) + \Pr(X > 750) \\
&= \Pr\left(Z < \left(\frac{450-600}{80}\right)\right) + \Pr\left(Z > \left(\frac{750-600}{80}\right)\right) \\
&= \Pr(Z < -1.875) + \Pr(Z > +1.875) \\
&= 2\Pr(Z > +1.875) \\
&= 2(.0304) \\
&= .0608
\end{align*}
\]

b. What proportion of students have scores between 450 and 750? **Answer:** .9392

**Solution:** Proportion of students with scores between 450 and 750

\[
\begin{align*}
&= \Pr(450 < X < 750) \\
&=1 - \Pr[ x < 450 \text{ OR } X > 750] \\
&=1-.0608 \\
&=.9392
\end{align*}
\]

c. What score is equal to the 95th percentile? **Answer:** 731.2

**Solution:** For $Z \sim \text{Normal}(0,1)$

\[
\Pr[Z_{.95} < 1.645] = .95
\]

From $Z = \frac{X - \mu}{\sigma}$ substitute

\[
1.645 = \frac{X_{.95} - 600}{80}
\]
Thus, \( X_{.95} = \sigma Z_{.95} + \mu \)
\[ = (80)[1.645] + 600 \]
\[ = 731.6 \]

2. The Chapin Social Insight Test evaluates how accurately the subject appraises other people. In the reference population used to develop the test, scores is normally distributed with mean \( \mu = 25 \) and standard deviation \( \sigma = 5 \). The range of possible scores is 0 to 41.

a. What proportion of the population has scores below 20 on the Chapin test? Answer: .1587

Solution:
\[
pr(X < 20) = pr\left[ Z < \frac{20 - 25}{5} \right] = pr[Z < -1] = .15875
\]

b. What proportion has scores below 10? Answer: .0014

Solution:
\[
pr(X < 10) = pr\left[ Z < \frac{10 - 25}{5} \right] = pr[Z < -3] = .00145
\]

c. How high a score must you have in order to be in the top quarter of the population in social insight? Answer: 28.35

Solution:
\[
pr(Z > 0.6745) = .25
\]
Thus, \( X = \sigma Z + \mu \)
\[ = (5)[0.6745] + 25 \]
\[ = 28.3725 \]

3. A normal distribution has mean \( \mu = 100 \) and standard deviation \( \sigma = 15 \) (for example, IQ). Give limits, symmetric about the mean, within which 95% of the population would lie:

Solution: First obtain an interval for \( Z \sim \text{Normal}(0,1) \)

\[
pr\left[ -1.96 < Z < +1.96 \right] = .95
\]
Next, recall that the standard error, SE, of \( \bar{X} \), is related to \( \sigma \) via
\[
se[\bar{X}] = \frac{\sigma}{\sqrt{n}}
\]
And the mean of \( \bar{X} \) is
\[
E[\bar{X}] = \mu
\]
Thus, the standardization formula can be manipulated to yield a formula for \( \bar{X} \) in terms of \( Z \).
From $Z = \frac{X - E[X]}{SE[X]}$, solve for $\bar{X}$.

$$\bar{X} = \left\{se(\bar{X})\right\} Z + \mu$$

$$= \left(\frac{\sigma}{\sqrt{n}}\right) Z + \mu$$

Now we can make a little table

<table>
<thead>
<tr>
<th>Mean</th>
<th>SE</th>
<th>Lower limit</th>
<th>Upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>-1.96</td>
<td>+1.96</td>
</tr>
<tr>
<td>$X$</td>
<td>100</td>
<td>15</td>
<td>(15)(-1.96) + 100 = 70.6</td>
</tr>
<tr>
<td>$\bar{X}_{n=4}$</td>
<td>100</td>
<td>15/2</td>
<td>(15/2)(-1.96) + 100 = 85.3</td>
</tr>
<tr>
<td>$\bar{X}_{n=16}$</td>
<td>100</td>
<td>15/4</td>
<td>(15/4)(-1.96) + 100 = 92.65</td>
</tr>
<tr>
<td>$\bar{X}_{n=100}$</td>
<td>100</td>
<td>15/10</td>
<td>(15/10)(-1.96) + 100 = 97.06</td>
</tr>
</tbody>
</table>

a. Individual observations. **Answer:** 70.6, 129.4

b. Means of 4 observations. **Answer:** 85.3, 114.7

c. Means of 16 observations. **Answer:** 92.65, 114.7

d. Means of 100 observations. **Answer:** 97.06, 102.94

e. Write down an expression for the width of the limits symmetric about the mean, within which 95% of the population of means of samples of size $n$ would lie.

**Solution:**

Width of limits symmetric about the mean is therefore

$$= (\text{Upper endpoint}) - (\text{Lower endpoint})$$

$$= \left[ \frac{\sigma}{\sqrt{n}} (Z) + \mu \right] - \left[ \frac{\sigma}{\sqrt{n}} (-Z) + \mu \right]$$

$$= \frac{2\sigma Z}{\sqrt{n}}$$

Thus, the width gets smaller as the sample size $n$ gets larger. On to confidence intervals in unit 6!