

Unit 7 – Hypothesis Testing
Homework #11 (Unit 7 – Hypothesis Testing, part 1 of 2)

SOLUTIONS

1. **This exercise is asking for a hypothesis test of the equality of two means (continuous, normal distribution) in the setting of two independent groups. See, 7. Hypothesis Testing, pp 45-49.**

An independent testing agency was hired prior to the November 2012 election to study whether or not the work output is different for construction workers employed by the state and receiving prevailing wages versus construction workers in the private sector who are paid rates determined by the free market. A sample of 100 private sector workers reveals an average output of 74.3 parts per hour with a sample standard deviation of 16 parts per hour. A sample of 100 state workers reveals an average output of 69.7 parts per hour with a sample standard deviation of 18 parts per hour. In developing your answer, *you may assume that the unknown variances are equal.*

- (a) What is the achieved level of significance?

Hint – “Achieved level of significance” is the same thing as “p-value”

- (b) Is there evidence of a difference in productivity at the 0.10 level of significance?

Hint – This question is asking you to compare the p-value you obtained in “a” to the threshold 0.10

- (c) Is there evidence of a difference in productivity at the 0.05 level of significance?

Hint – This question is asking you to compare the p-value you obtained in “a” to the threshold 0.05

ANSWER

- a. The achieved level of significance is $p\text{-value} = 0.058$.
- b. The achieved significance ($p\text{-value} = .058$) < 0.1 . Thus, application of the null hypothesis model to the data has led us to an “extremely” unlikely outcome, relative to our threshold. → The null hypothesis is rejected. Conclude that these data provide statistically significant evidence of a difference in work output for employees employed by the state versus those in the private sector.
- c. Here, in contrast, the achieved significance ($p\text{-value} = .058$) $> .05$. Thus, with respect to the .05 “threshold”, application of the null hypothesis model to the data has *not* led us to an “extremely” unlikely outcome. → The null hypothesis is *not* rejected. According to a .05 test, conclude that these data do *not* provide statistically significant evidence of a difference in work output for employees employed by the state versus those in the private sector.

SOLUTION -

Research Question. Is the work output of state workers is different from that of workers in the private sector?

Assumptions.

Let subscript “1” reference the group of state employees, “2” the private sector employees.

\bar{X}_1 is distributed Normal ($\mu_1, \sigma^2/100$) and \bar{X}_2 is distributed Normal ($\mu_2, \sigma^2/100$)

H_0 and H_A .

$$H_0 : \mu_1 = \mu_2$$

$$H_A : \mu_1 \neq \mu_2$$

Test statistic is a t-score.

$$t_{\text{score}} = \left[\frac{(\bar{X}_1 - \bar{X}_2) - E[(\bar{X}_1 - \bar{X}_2) | H_0 \text{ true}]}{\hat{SE}[(\bar{X}_1 - \bar{X}_2) | H_0 \text{ true}]} \right]$$

It's okay to assume equality of unknown variances (because I said so. I called it ...)

$$\hat{SE}(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{S_{\text{pool}}^2}{n_1} + \frac{S_{\text{pool}}^2}{n_2}} \quad \text{where} \quad S_{\text{pool}}^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)}$$

For these data:

$$\hat{\sigma}^2 = S_{\text{pool}}^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)} = \frac{(100 - 1)18^2 + (100 - 1)16^2}{(100 - 1) + (100 - 1)} = 290$$

$$\hat{SE}(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{S_{\text{pool}}^2}{n_1} + \frac{S_{\text{pool}}^2}{n_2}} = \sqrt{\frac{290}{100} + \frac{290}{100}} = 2.4083$$

Degrees of freedom, $df = (n_1 - 1) + (n_2 - 1) = (100 - 1) + (100 - 1) = 198$.

“Evaluation” rule.

The likelihood of these findings or ones more extreme if H_0 is true is

$$p\text{-value} = \Pr\left[(\bar{X}_1 - \bar{X}_2) \geq |(69.7 - 74.3)| | H_0 \text{ true} \right].$$

Calculations.

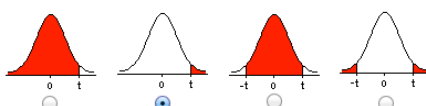
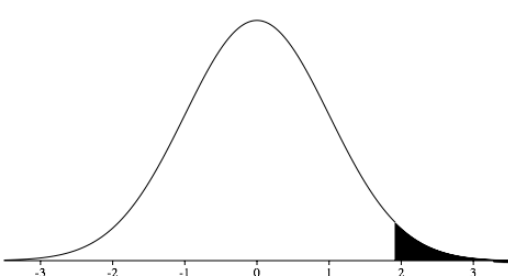
Note – Solving for the p-value yields the solution to exercise #1a:

$$p\text{-value} = (2)\Pr\left[(\bar{X}_1 - \bar{X}_2) \geq |(69.7 - 74.3)| \right] \text{ *Note – The (2) is in front because this is two sided*}$$

$$= 2 \Pr\left[\frac{(\bar{X}_1 - \bar{X}_2) - (0)}{\hat{SE}(\bar{X}_1 - \bar{X}_2)} \geq \frac{(69.7 - 74.3) - (0)}{2.4083} \right]$$

$$= (2)\Pr[t_{\text{score}} \geq 1.91] \text{ where degrees of freedom, } df = 198$$

$$= (2)(.028) = .056 \text{ *Note – Because the degrees of freedom = 198 is much larger than 30, I could have used the Normal(0,1) table and obtained the same answer. See below:*}$$

Pr[t-score _{DF=198} > 1.91] = .028	Pr[Normal(0,1) > 1.91] = .028
<p>SurfStat t-distribution calculator</p>  <div> <div>d.f. 198</div> <div>t value 1.91</div> <div>probability 0.0288</div> </div>	 <p> <input checked="" type="radio"/> Area from a value (Use to compute p from Z) <input type="radio"/> Value from an area (Use to compute Z for confidence intervals) </p> <p>Specify Parameters:</p> <p>Mean <input type="text" value="0"/></p> <p>SD <input type="text" value="1"/></p> <p> <input checked="" type="radio"/> Above <input type="text" value="1.91"/> <input type="radio"/> Below <input type="text"/> <input type="radio"/> Between <input type="text"/> and <input type="text"/> <input type="radio"/> Outside <input type="text"/> and <input type="text"/> </p> <p>Results:</p> <p>Area (probability) <input type="text" value="0.0281"/></p> <p><input type="button" value="Recalculate"/></p>
http://surfstat.anu.edu.au/surfstat-home/tables/t.php	http://davidmlane.com/hyperstat/z_table.html

“Evaluate”.

Under the null hypothesis H_0 (worker output is the same in both groups) the chances that the average work outputs differ by a magnitude greater than $|69.7 - 74.3|$ is about 6 in 100. This is a borderline suggestion that the two groups differ in their work output.

2. For the data in Exercise 1, what level of significance (**Hint – solve for the p-value**) is achieved by the data if the sample means and sample standard deviations are unchanged but the within group sample sizes are

- (a) both equal to 10
- (b) both equal to 200
- (c) Comment on the role of sample size in the probability of a type I error.
Recall – The probability of a type I error is the probability of rejecting the null when the null is true.

ANSWER

- a. p-value = .554**
- b. p-value = .007**
- c. All other things equal, a larger sample size reduces the probability of making a type I error.**

SOLUTION

The solution involves substitution of the new values of the sample sizes into the formulae shown in the solution for Exercise 1.

a. n=10 in each group

$$\hat{\sigma}^2 = S_{\text{pool}}^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{(n_1-1) + (n_2-1)} = \frac{(10-1)18^2 + (10-1)16^2}{(10-1) + (10-1)} = 290$$

$$SE(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{S_{\text{pool}}^2}{n_1} + \frac{S_{\text{pool}}^2}{n_2}} = \sqrt{\frac{290}{10} + \frac{290}{10}} = 7.6158$$

Degrees of freedom, $df = (n_1-1) + (n_2-1) = (10-1) + (10-1) = 18$.

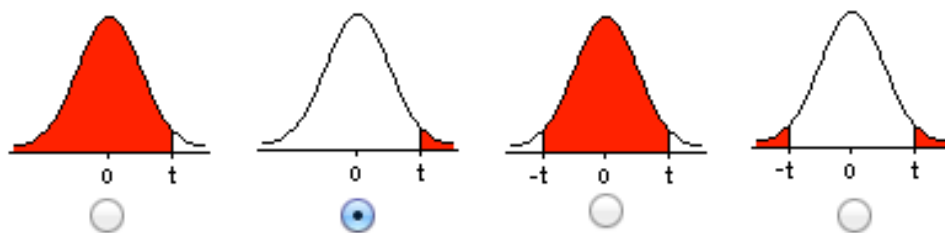
$$\text{p-value} = (2)\Pr\left[(\bar{X}_1 - \bar{X}_2) \geq |(69.7 - 74.3)|\right] \text{ **Note – The (2) is in front because this is two sided**}$$

$$= 2 \Pr \left[\frac{(\bar{X}_1 - \bar{X}_2) - (0)}{SE(\bar{X}_1 - \bar{X}_2)} \geq \frac{(69.7 - 74.3) - (0)}{7.6158} \right]$$

$$= (2) \Pr[t_{\text{score}} \geq 0.6040] \quad \text{where degrees of freedom, } df = 18$$

$$= (2) (.2767) = .55$$

SurfStat *t*-distribution calculator



d.f.	t value	----->	probability
18	.6040		0.2767
		<-----	

<http://surfstat.anu.edu.au/surfstat-home/tables/t.php>

b. n=200 in each group

$$\hat{\sigma}^2 = S_{\text{pool}}^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{(n_1-1) + (n_2-1)} = \frac{(200-1)18^2 + (200-1)16^2}{(200-1) + (200-1)} = 290$$

$$SE(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{S_{\text{pool}}^2}{n_1} + \frac{S_{\text{pool}}^2}{n_2}} = \sqrt{\frac{290}{200} + \frac{290}{200}} = 1.7029$$

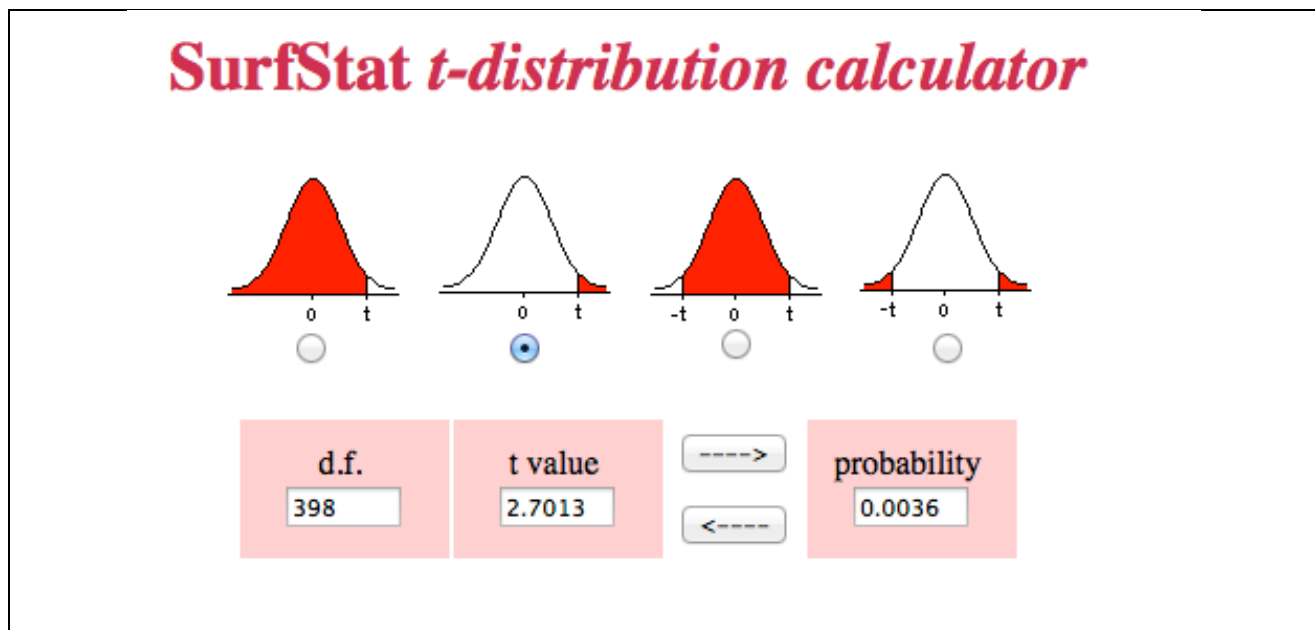
Degrees of freedom, $df = (n_1-1) + (n_2-1) = (200-1) + (200-1) = 398$.

p-value = $(2)\Pr[(\bar{X}_1 - \bar{X}_2) \geq |(69.7 - 74.3)|]$ *Note – The (2) is in front because this is two sided*

$$= 2\Pr\left[\frac{(\bar{X}_1 - \bar{X}_2) - (0)}{SE(\bar{X}_1 - \bar{X}_2)} \geq \frac{(69.7 - 74.3) - (0)}{1.7029}\right]$$

= $(2)\Pr[t_{\text{score}} \geq 2.7013]$ where degrees of freedom, $df = 398$

$$= (2) (.0036) = \mathbf{.0072}$$



<http://surfstat.anu.edu.au/surfstat-home/tables/t.php>