## Unit 6 – Estimation
## Homework #9 (Unit 6 –Estimation, part 1 of 2)

### *SOLUTIONS*

1. The results of IQ tests are known to be normally distributed. Suppose that in 2014, the distribution of IQ test scores for persons aged 18-35 years has a variance $\sigma^2 = 225$. A simple random sample of 9 persons take the IQ test. The sample mean score is 115. Calculate the 50%, 75%, 90% and 95% confidence interval estimates of the unknown population mean IQ score.

**Answer:**

| | |
|---|---|
| **50% CI** | **(111.6 , 118.4)** |
| **75% CI** | **(109.2, , 120.8)** |
| **90% CI** | **(106.8 , 123.2)** |
| **95% CI** | **(105.2 , 124.8)** |

**Solution:**
Let the random variable X = IQ test result assumed normal with:

$\mu$ unknown
$\sigma^2 = 225$, known
$\sigma = 15$, known

A confidence interval estimate of the unknown mean is given by:

estimate $\pm$ { confidence coefficient} { se of estimate}

where,

estimate = observed sample mean = $\bar{X} = 115$
confidence coefficient = $(1 - \alpha/2)100$th percentile Normal(0,1)
se of estimate = standard error of sample mean = $SE(\bar{X}) = \sqrt{\sigma^2/n}$

$$= \sqrt{225/9}$$
$$= 15/3$$
$$= 5$$

For 50% confidence interval estimate:

$1 - \alpha = (1 - 0.50) = 0.50$

$\alpha/2 = 0.50 / 2 = 0.25$

Therefore want ( 1 - .25)100th or 75th percentile of the Normal(0,1) distribution.
If you launch the David Lane calculator, this is seen to be = 0.6745
The required confidence interval estimate is thus,

estimate $\pm$ { confidence coefficient} { se of estimate}

$= \bar{X} \pm z_{.75} \sqrt{\sigma^2/n}$

$= 115 + \{ 0.6745 \} \{ 5 \}$

**= ( 111.6 , 118.4 )**

For 75% confidence interval estimate:

$1 - \alpha = (1 - 0.25) = 0.75$

$\alpha/2 = 0.25 / 2 = 0.125$

Therefore want ( 1 - .125)100th or 87.5th percentile of the Normal(0,1) distribution.
Now if you launch the David Lane Calculator this is seen to be = 1.1505
The required confidence interval estimate is thus,

estimate $\pm$ { confidence coefficient} { se of estimate}

$= \bar{X} \pm z_{.875} \sqrt{\sigma^2/n}$

$= 115 \pm \{ 1.1505 \} \{ 5 \}$

**= ( 109.2 , 120.8 )**

For 90% confidence interval estimate:

$1 - \alpha = (1 - 0.10) = 0.90$

$\alpha/2 = 0.10 / 2 = 0.05$

Therefore want ( 1 - .05)100th or 95th percentile of the Normal(0,1) distribution.
Again, using the David Lane calculator, this is seen to be = 1.645
The required confidence interval estimate is thus,

estimate $\pm$ { confidence coefficient} { se of estimate}

$= \bar{X} \pm z_{.95} \sqrt{\sigma^2/n}$

$= 115 \pm \{ 1.645 \} \{ 5 \}$

**= ( 106.8 , 123.2 )**

For 95% confidence interval estimate:

$1 - \alpha = (1 - 0.05) = 0.95$

$\alpha/2 = 0.05 / 2 = 0.025$

Therefore want ( 1 - .025)100th or 97.5th percentile of the Normal(0,1) distribution.
Here the David Lane calculator tells us this value = 1.96
The required confidence interval estimate is thus,

estimate $\pm$ { confidence coefficient} { se of estimate}

$= \bar{X} \pm z_{.975} \sqrt{\sigma^2/n}$

$= 115 \pm \{ 1.96 \} \{ 5 \}$

**= ( 105.2 , 124.8 )**

2. What trade-offs are involved in reporting one interval estimate over another?

**Answer:**
For a given probability distribution with a known variance and a fixed sample size,

(i) Increasing the confidence coefficient is at the price of a wider confidence interval.

(ii) Decreasing the width of a confidence interval estimate is at the price of a lower confidence coefficient.

This is apparent in the following summary of the solution to Exercise #1.

| Desired Confidence | Lower Limit | Upper Limit | Width = [Upper limit – Lower limit] |
|---|---|---|---|
| .50 | 111.6 | 118.4 | 6.8 |
| .75 | 109.2 | 120.8 | 11.6 |
| .90 | 106.8 | 123.2 | 16.4 |
| .95 | 105.2 | 124.8 | 19.6 |

3. If it is known that the population mean IQ score is $\mu = 105$ and $\sigma^2=225$, what proportion of samples of size 6 will result in sample mean values in the interval [135,150]?

**Answer: << .0001**

**Solution:**
Solve for Prob [ $135 < \bar{X}_{n=6} < 150$ ] using the standardization formula.
Note that $\bar{X}_{n=6}$ is normally distributed with:

$$\mu_{\bar{X}} = 105$$

$$\sigma^2_{\bar{X}} = \frac{\sigma^2}{n} = \frac{225}{6} = 37.5$$
$$SE_{\bar{X}} = \sqrt{\sigma^2_{\bar{X}}} = \sqrt{37.5} = 6.1238$$

Thus,

$$\text{Probability [ } 135 < \bar{X}_{n=6} < 150 \text{ ]} = \text{Probability [ } \frac{135\text{-}105}{6.1238} < \frac{\bar{X}_{n=6} - \mu_{\bar{X}}}{SE_{\bar{X}}} < \frac{150\text{-}105}{6.1238} \text{ ]}$$
$$= \text{Probability [ } 4.89 < \text{Z-score} < 7.34 \text{]} << .0001$$

4. An entomologist samples a field for egg masses of a harmful insect by placing a yard-square frame at random locations and carefully examining the ground within the frame. A simple random sample of 75 locations selected from a county's pasture land found egg masses in 13 locations. Compute a 95 confidence interval estimate of all possible locations that are infested.

**Answer: (.0876, .2590)**
**Solution:**
The setting is estimation of a binomial proportion $\pi$. In this exercise, the number of trials is N=75. Since this is sufficiently large, we can obtain a confidence interval using $\hat{\pi} \pm (z_{1\text{-}\alpha/2})\hat{SE}(\hat{\pi})$ using the standard error formula "(3)" that appears on page 60. Thus, the calculations are

$$\bar{X} = \frac{X}{N} = \frac{13}{75} = .1733$$
$$\hat{\pi} = \bar{X} = .1733$$
$$\hat{SE} = \sqrt{\frac{\bar{X}(1\text{-}\bar{X})}{N}} = \sqrt{\frac{(.1733)(.8267)}{75}} = .0437$$
$$z_{1\text{-}\alpha/2} = z_{.975} = 1.96$$
$$\hat{\pi} \pm (z_{1\text{-}\alpha/2})\hat{SE}(\hat{\pi}) = .1733 \pm (1.96)(.0437) = \underline{\mathbf{(.0876,\ .2590)}}$$