

Anisotropic Spin Diffusion in Trapped Boltzmann Gases

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Recent experiments in a mixture of two hyperfine states of trapped Bose gases show behavior analogous to a spin-1/2 system, including transverse spin waves and other familiar Leggett-Rice-type effects. We have derived the kinetic equations applicable to these systems, including the spin dependence of interparticle interactions in the collision integral, and have solved for spin-wave frequencies and longitudinal and transverse diffusion constants in the Boltzmann limit. We find that, while the transverse and longitudinal collision times for trapped Fermi gases are identical, the Bose gas shows diffusion anisotropy. Moreover, the lack of spin isotropy in the interactions leads to the non-conservation of transverse spin, which in turn has novel effects on the hydrodynamic modes.

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In recent JILA experiments,^{1,2} a mixture of two hyperfine states was found to segregate by species. The theoretical explanation³⁻⁸ for this behavior is based on the two states playing the role of a pseudo-spin-1/2 system, having transverse spin waves. The theory of these new effects is based on old ideas of the transport properties of polarized homogeneous quantum gases of real spins, such as ³He gas and solutions of ³He in liquid ⁴He,^{9,10} transcribed to the trapped gas pseudo-spin case.

Besides spin waves, the theory for homogeneous polarized fermions or bosons led to the prediction of anisotropic spin diffusion in the degenerate state.^{9,10} When a spin nonuniformity is longitudinal, that is, with a variation in the magnitude of the magnetization, the spin diffusion coefficient is D_{\parallel} . On the other hand, in a spin-echo experiment, the magnitude of the mag-

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netization is uniform but it varies spatially in direction. The corresponding diffusion coefficient, D_{\perp} is less than D_{\parallel} when the system is polarized and degenerate. Experimentally this feature has been seen, but was not always in reasonable accord with theory.⁹ Moreover, Fomin¹¹ has suggested the effect should not exist. However, a recent experiment⁹ has overcome several possible experimental objections and finds good agreement with theory. Moreover Mineev has very recently presented theoretical analysis that questions the validity of Fomin's approach.¹² Also see our own recent analysis.¹³

Thus it seems useful to see whether a similar difference between longitudinal and transverse diffusion in trapped gases might provide an alternative testing ground for this question. However, what we show here is that the physical possibility of having differing interaction parameters between up-up, down-down, and up-down states (interaction anisotropy) provides a new physical basis for anisotropic spin diffusion for bosons even in the Boltzmann limit. For longitudinal diffusion in the Boltzmann limit only up-down scattering contributes. However, in the transverse case, two spins at differing angles approach one another, and the scattering can be analyzed as being a superposition of, say, up-up and up-down scattering. In the fermion s-wave case, the up-up part gives no contribution, and, in the Boltzmann limit, the diffusion coefficients are identical. In that case one must go to the degenerate limit to see the anisotropy, which then is expected to arise because the density of scattering states differs in longitudinal and transverse cases.¹⁰ On the other hand, for bosons, for which both the up-up and down-down scattering rates do contribute, we find an anisotropy even in the Boltzmann limit, but only if the various scattering lengths differ.

The presence of interaction anisotropy provides another unusual effect, namely that transverse spin is not conserved.⁷ This leads to a decay of the transverse spin (a T_2 process) that seriously affects the hydrodynamic modes of the system. Below we first use the moments method to compute the spectra of the lowest-lying longitudinal and transverse modes. However, with that method we obtain a transverse decay rate γ_{\perp} that diverges as the collision relaxation time τ approaches zero, in contrast to the usual diffusive behavior where $\gamma_{\perp} \propto \tau$. In this case it is necessary to solve the *local* hydrodynamic equations to find the correct behavior, in which the hydrodynamic solutions are localized at the low-density regions at the edges of the cloud where the collision time is longer. The result is a much smaller decay rate than that obtained with the moments method.

In our previous work, Ref. 10, we derived an analog of the Landau-Silin equation for a 2×2 density operator \hat{n}_p (here acting in the pseudo-spin space), with effective mean-field single particle energy matrix $\hat{\epsilon}_p$. We can write the density and single-particle energy in a Pauli representation as

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$\hat{n}_p = \frac{1}{2} (f_p \hat{I} + \mathbf{m}_p \cdot \hat{\sigma})$ and $\hat{\varepsilon}_p = (e_p \hat{I} + \mathbf{h}_p \cdot \hat{\sigma})$ where $\hat{\sigma}$ is a Pauli matrix, $\frac{1}{2}(f_p \pm m_{pz})$ give the diagonal components of the density $n_{pi} = n_{pii}$, while \mathbf{m}_p represents the polarization, which in equilibrium is along the axis $\hat{\mathbf{z}}$. We find the following approximate equation for \mathbf{m}_p :

$$\frac{\partial \mathbf{m}_p}{\partial t} - \frac{2}{\hbar} \mathbf{h} \times \mathbf{m}_p + \sum_i \left[\frac{p_i}{m} \frac{\partial \mathbf{m}_p}{\partial r_i} - \frac{\partial U}{\partial r_i} \frac{\partial \mathbf{m}_p}{\partial p_i} \right] = \text{Tr} \{ \hat{\sigma} \hat{I}_p \} \quad (1)$$

with $m_{pz}(\mathbf{r}) = n_{p1} - n_{p2}$ and $n_{p12}(\mathbf{r}) = n_{p21}^* = \frac{1}{2} m_{p-}(\mathbf{r}) = \frac{1}{2} (m_{px} - im_{py})$. The 2×2 collision integral is \hat{I}_p . The effective mean magnetic field

$$\mathbf{h} = \frac{\hbar \Omega_0}{2} \hat{\mathbf{z}} + \eta \frac{t_{12}}{2} \mathbf{M} \quad (2)$$

where $\hbar \Omega_0 = V_1 - V_2 + [(t_{11} - t_{12}) n_1 - (t_{22} - t_{12}) n_2] (1 + \eta)$. In these η is 1 (-1) for bosons (fermions); $\mathbf{M}(\mathbf{r}) = \int d\mathbf{p}/h^3 \mathbf{m}_p(\mathbf{r})$; $n_i(\mathbf{r}) = \int d\mathbf{p}/h^3 n_p(\mathbf{r})$; V_i is the external field for species i ; $U = \frac{1}{2}(V_1 + V_2)$; and $M_z = n_1 - n_2$. The t 's can be evaluated in terms of the measured scattering lengths $a_{\alpha\beta}$ by using $t_{\alpha\beta} = 4\pi\hbar a_{\alpha\beta}/m$.

The equilibrium solution in the Boltzmann limit is $m_p^{(0)} = \mathcal{M}(\beta\hbar\bar{\omega})^3 \exp[-\beta(p^2/2m + U)]$ where N is the total number of particles, N_i is the number of species i , $\mathcal{M} = N_1 - N_2$ is the total magnetization, and $\bar{\omega} \equiv (\omega_x \omega_y \omega_z)^{1/3}$.

We have derived the collision integral for the Boltzmann case when the various interaction parameters differ. Our expression agrees with the same quantity derived in Refs. 7 and 8, and reduces properly to previous results if all the t 's are taken equal.^{10,14} We find

$$\begin{aligned} \langle \sigma | \hat{I}_p | \sigma' \rangle &= \frac{\pi}{\hbar} \int d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(\epsilon_{\mathbf{p}_1} + \epsilon_{\mathbf{p}_2} - \epsilon_{\mathbf{p}_3} - \epsilon_{\mathbf{p}_4}) \\ &\sum_{\sigma_2} \left\{ -t_{\sigma\sigma_2}^2 [(n_{\mathbf{p}_1})_{\sigma\sigma'} (n_{\mathbf{p}_2})_{\sigma_2\sigma_2} + \eta (n_{\mathbf{p}_2})_{\sigma\sigma_2} (n_{\mathbf{p}_1})_{\sigma_2\sigma'}] \right. \\ &\quad - t_{\sigma'\sigma_2}^2 [(n_{\mathbf{p}_1})_{\sigma\sigma'} (n_{\mathbf{p}_2})_{\sigma_2\sigma_2} + \eta (n_{\mathbf{p}_1})_{\sigma\sigma_2} (n_{\mathbf{p}_2})_{\sigma_2\sigma'}] \\ &\quad \left. + 2t_{\sigma\sigma_2} t_{\sigma'\sigma_2} [(n_{\mathbf{p}_3})_{\sigma\sigma'} (n_{\mathbf{p}_4})_{\sigma_2\sigma_2} + \eta (n_{\mathbf{p}_3})_{\sigma\sigma_2} (n_{\mathbf{p}_4})_{\sigma_2\sigma'}] \right\} \quad (3) \end{aligned}$$

We will linearize the kinetic equation for \mathbf{m}_p around the global equilibrium value $m_p^{(0)} \hat{\mathbf{z}}$ and use a moment approach to compute the spin wave and diffusive damping just as done previously.^{5,7} As in Ref. 5 we set the effective longitudinal field Ω_0 to zero. Experimentally this adjustment may not be feasible; we will avoid it in a more complete treatment elsewhere. The linearized longitudinal and transverse equations are

$$\frac{\partial \delta m_{pe}}{\partial t} + \sum_i \left[\frac{p_i}{m} \frac{\partial \delta m_{pe}}{\partial r_i} - \frac{\partial U}{\partial r_i} \frac{\partial \delta m_{pe}}{\partial p_i} \right] = \sum_{\sigma} \sigma \langle \sigma | \hat{I}_p | \sigma \rangle \quad (4)$$

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and

$$\begin{aligned} & \frac{\partial \delta m_{p+}}{\partial t} + i\eta t_{12} \left(m_p^{(0)} \delta M_+ - M_0 \delta m_{p+} \right) \\ & + \sum_i \left[\frac{p_i}{m} \frac{\partial \delta m_{p+}}{\partial r_i} - \frac{\partial U}{\partial r_i} \frac{\partial \delta m_{p+}}{\partial p_i} \right] = 2(2|\hat{L}_p|1). \end{aligned} \quad (5)$$

where \hat{L}_p is the linearized form of \hat{I}_p

In the following, for brevity, we compute only results for the monopole and dipole modes although experiments have detected the quadrupole modes. Similar arguments hold for the quadrupole case, which we will present in a longer publication.

Longitudinal case: We use a variational function of the form

$$\delta m_{pz} = (a_0 + a_1 z + a_2 p_z) m_p^{(0)} \quad (6)$$

and take the 1, z , and p_z moments of the kinetic equation in both the longitudinal and transverse cases. The results for the longitudinal case, if we assume a time dependence of $\exp(i\omega t)$ for a_1 and a_2 , are

$$da_0/dt = 0 \quad (7)$$

$$i\omega a_1 - \omega_z a_2 = 0 \quad (8)$$

$$i\omega a_2 + \omega_z a_1 = -\gamma_{\parallel} a_2 \quad (9)$$

with $\gamma_{\parallel} = 4\gamma_0/3$ where $\gamma_0 = \pi\beta m^3 \bar{\omega}^3 t_{12}^2 N/h^4$ comes from integrating the collision integral. Eq. (7) indicates that the monopole mode does not decay in the longitudinal case, which is consistent with the conservation of magnetization. The second line is the magnetization equation of continuity. The relaxation rate γ_{\parallel} agrees with that derived in Ref. 5. If we define $\tau_{\parallel} \equiv 1/\gamma_{\parallel}$, the *spatially averaged* collision time, then the dipole spectrum for small τ_{\parallel} is

$$\omega_{\parallel} = i\omega_z^2 \tau_{\parallel}, \quad (10)$$

which has the form of the lowest-order solution of a diffusion equation in a harmonic potential.

Transverse case: We again use the form of Eq. (6). Taking 1, z , and p_z moments of Eq. (5) yields the results

$$da_0/dt = -\gamma_T a_0 \quad (11)$$

$$i\omega a_1 - \omega_z a_2 = -\frac{1}{2}\gamma_T a_1 \quad (12)$$

$$i(\omega - \omega_M) a_2 + \omega_z a_1 = -\gamma_{\perp} a_2 \quad (13)$$

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where $\gamma_T = \gamma_0(1 + \eta) \sum_{\sigma} \left(\frac{t_{\sigma\sigma} - t_{12}}{t_{12}} \right)^2 f_{\sigma}$ with $f_{\sigma} = N_{\sigma}/N$, and

$$\gamma_{\perp} = \gamma_{\parallel} \left[\frac{7R - 3S}{8t_{12}^2} \right] \quad (14)$$

with $R = (1 + \eta) \sum_{\sigma} t_{\sigma\sigma}^2 f_{\sigma} + (1 - \eta)t_{12}^2$ and $S = 2t_{12}[(1 + \eta) \sum_{\sigma} t_{\sigma\sigma} f_{\sigma} - \eta t_{12}]$, and the mean-field frequency is

$$\omega_M = \eta \frac{t_{12} \mathcal{M}}{\hbar} \left(\frac{\beta \hbar \bar{\omega}}{\sqrt{2} \lambda} \right)^3 \quad (15)$$

Comments:

1) If the interactions parameters t_{ij} are all equal, we have $\gamma_T = 0$, $R = S = 2t^2$ so that $\gamma_{\perp} = \gamma_{\parallel}$. Eqs. (11)-(13) then reduce to those of Ref. 5 and the longitudinal and transverse relaxation rates are the same, which agrees with the standard result for a homogeneous real spin system in the Boltzmann limit.

2) For fermions, we have $\eta = -1$, so that, even if the t 's are *not* equal, $\gamma_T = 0$ and $\gamma_{\perp} = \gamma_{\parallel}$.

3) For bosons with unequal t 's, the spatial averaged transverse relaxation rate is not generally the same as the longitudinal. Moreover, we have a T_2 -type relaxation rate for a_0 and in the equation of continuity (12). The interaction anisotropy behaves something like a dipole-dipole interaction allowing relaxation of the transverse spin, an effect noted previously in Ref. 7.

If, for now, we take $\gamma_T = 0$, then the lowest mode in the hydrodynamic limit takes the form

$$\omega_{\perp} = \frac{\omega_z^2 (i - \mu \mathcal{M}) \tau_{\perp}}{[1 + (\mu \mathcal{M})^2]}. \quad (16)$$

where $\tau_{\perp} \equiv 1/\gamma_{\perp}$ and the so-called ‘‘spin-rotation parameter’’ $\mu = \omega_M \tau_{\perp}$. The form of Eq. (16) is the hydrodynamic frequency as modified by spin rotation.^{9,10,14} The first term is the effective diffusion frequency while the second is the dipole-mode pseudo-spin-wave frequency.

The effect of non-zero γ_T is to allow a T_2 relaxation of the transverse spins. If we write $1/\tau_{\perp} = \gamma_{\parallel}$ then, in the small τ_{\perp} limit, one no longer has the hydrodynamic decay rate approaching zero, but instead it diverges at the origin because $\text{Im}(\omega) \approx (\omega_z^2 \tau_{\perp} + \gamma_T)$, and $\gamma_T \sim 1/\tau_{\perp}$. However, the moments method, while suitable for finite $\omega\tau$, is inadequate in the hydrodynamic limit. The moments method fails because the simple forms assumed for spatial dependence are unable to adjust to the spatially dependent relaxation rates. One needs to solve local equations numerically for the spatial behavior.

For $\gamma_T = 0$ the numerical hydrodynamic spectrum differs little from that of the moments method calculation. However, for $\gamma_T > 0$ the hydrodynamic

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dipole and monopole modes are found to decay, not $\sim 1/\tau_{\perp}$, but instead at a slower rate $\sim \sqrt{\log(1/\omega_z\tau_{\perp})}$. In fact, at small enough $\omega_z\tau_{\perp}$ the T_2 decay of the magnetization at the center of the trap causes the monopole and dipole modes to coalesce into spin-waves localized on the lower density regions on the left and right sides of the trap. We will give more details in a future publication.

In experiments on Rb, the interaction anisotropy is very small. To test the novel effects predicted here one might use Na,¹⁵ which has a difference in interaction parameters. Numerically we estimate that for ^{23}Na γ_{\perp} can differ from γ_{\parallel} by as much as 14% with $\gamma_T/\gamma_{\perp} \approx 0.04$. Unfortunately large differences in interaction parameters can lead to short lifetimes of trapped samples. Interaction differences might also be induced by using Feshbach resonance methods.

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