# CCP Estimation of Dynamic Discrete/Continuous-Choice Models with Generalized Finite Dependence and Correlated Unobserved Heterogeneity

Wayne-Roy Gayle*

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**Abstract**

This paper investigates conditional choice probability estimation of dynamic structural discrete- and continuous-choice models. I extend the concept of finite dependence in a way that accommodates non-stationary, irreducible transition probabilities. I show that under this new definition of finite dependence, one-period dependence is obtainable in a larger class of dynamic structural models than previously investigated. This finite-dependence property also provides a convenient and computationally cheap representation of the optimality conditions for the continuous-choice variables. I allow for discrete-valued permanent unobserved heterogeneity in utilities and production functions. The unobserved heterogeneity may be correlated with the observable state variables. I propose sufficient conditions for identification of the utility functions and the distribution of the unobserved heterogeneity. I show the estimator is root-n–asymptotically normal. I develop a new and computationally cheap algorithm to compute the estimator, and analyze the finite-sample properties of this estimator via Monte Carlo techniques. I apply the proposed method to estimate a model of education and labor-supply choices to investigate the effect of race and parent income on the distribution of returns to education, using data from the National Longitudinal Survey of Youth 1979.

*Department of Resource Economics, University of Massachusetts, 80 Campus Way, Amherst, MA 01003, E-mail: wayneroygayle@gmail.com. The author is grateful to Victor Aguirregabiria, George-Levi Gayle, Shakeeb Khan, Arthur Lewbel, Lala Zun Ma, Holger Sieg, and Steven Stern for insightful comments. I also thank the seminar participants at the University of Virginia and the University of North Carolina, Chapel Hill. All errors are my own.

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# 1 Introduction

In this paper, I investigate conditional choice probability (CCP) estimation of dynamic structural discrete/continuous-choice models with unobserved individual heterogeneity. I show that an extension to the definition of finite dependence proposed in Altug and Miller (1998) and Arcidiacono and Miller (2011) accommodates general non-stationary and irreducible transition functions, as well as a general form of correlated, permanent unobserved heterogeneity in the utility and production functions. I propose sufficient conditions for identification of the period-specific payoff functions and the distribution of the unobserved heterogeneity, and I propose a generalized method of moments (GMM) estimator for the structural parameters of the model and derive their asymptotic distributions. I also propose an algorithm to implement the estimator. I investigate the finite-sample properties of the estimator by way of Monte Carlo analysis and implement this method to estimate a model of education and labor-supply choices to investigate the distribution of returns to education, using data from the National Longitudinal Survey of Youth 1979 (NLSY79).

Since its introduction by Hotz and Miller (1993), CCP estimation of dynamic structural models has flourished in empirical labor economics and industrial organization, largely because of its potential for material reduction in computational costs relative to the more traditional backward recursive- and contraction mapping-based full maximum likelihood estimation pioneered by Rust (1987), referred to as the nested fixed-point algorithm (NFXP). The CCP estimator circumvents having to solve the dynamic programming problem for each trial value of the structural parameters, by making use of a one-to-one mapping between the normalized value functions and the CCPs established in Hotz and Miller (1993). Therefore, nonparametric estimates of the CCPs can be inverted to obtain estimates of the normalized value functions, which can then be used to estimate the structural parameters.

Empirical application of the early formulation of CCP estimation had important limitations relative to the NFXP method. The emerging literature has focused on separate but related drawbacks. The first is that nonparametric estimation of the CCPs results in less efficient estimates of the structural parameters, as well as relatively poor finite-sample performance. The second is the difficulty of accounting for unobserved individual heterogeneity, mainly due to having to estimate the CCPs by nonparametric methods. A limitation of both the CCP and NFXP approaches to estimation is that they are largely restricted to discrete-choice, discrete-states models.

Aguirregabiria and Mira (2002) propose a solution to the issue of efficiency and finite-sample performance of the CCP estimator relative to the NFXP estimator. They show that for a given value of the preference parameters, the fixed-point problem in the value function space can be transformed into a fixed-point problem in the probability space. The authors suggest swapping the nesting of the NFXP and show the resulting estimator is asymptotically equivalent to the NFXP estimator. Furthermore, they show in simulation studies that their method produces estimates 5 to 15 times faster than NFXP. The method they propose is restricted to discrete-choice models in stationary environments and is not designed to account for unobserved individual heterogeneity.

Recent developments in accounting for unobserved heterogeneity in CCP estimators include Aguirregabiria and Mira (2007) and Arcidiacono and Miller (2011). Aguirregabiria and Mira (2007) allow for permanent unobserved heterogeneity in stationary, dynamic discrete games. Their method requires multiple inversion of potentially large dimensional matrices. Arcidiacono and Miller (2011) propose a more general method for incorporating time-specific or time-invariant unobserved heterogeneity into CCP estimation of discrete dynamic models.

Altug and Miller (1998) propose an approach that allows for continuous choices in the CCP framework. By assuming complete markets, estimates of individual effects and aggregate shocks are obtained, which are then used in the second stage to form (now) observationally equivalent individuals. These observationally equivalent individuals are used to compute counterfactual continuous choices. Bajari, Benkard, and Levin (2007) modify the methods of Hotz and Miller (1993) and Hotz, Miller, et al. (1994) to estimate dynamic games. They consider models of pure discrete choice or pure continuous choice, but not both.

The finite-dependence property – when two different policies associated with different initial choices lead to the same distribution of states after a few periods – is critical for the computational feasibility and finite-sample performance of CCP estimators. Finite dependence combined with the invertibility result of Hotz and Miller (1993) results in a significant reduction in the computational cost of estimating dynamic structural models. Essentially, the smaller the order of dependence, the faster and more precise the estimator, because fewer future choice probabilities have to be estimated or updated, depending of the method of estimation. The concept of finite dependence was first introduced by Hotz and Miller (1993), extended by Altug and Miller (1998), and further by Arcidiacono and Miller (2011). Despite these generalizations, the concept of finite dependence is largely restricted to discrete-choice models with stationary transitions and models with the renewal or exchangeability properties.

This paper makes three separate but closely related contributions to the literature on CCP estimation of dynamic structural models. I extend the concept of finite dependence to allow for general non-stationary and irreducible transition probabilities. Although its definition is precise and well understood, the strategy to construct finite dependence in dynamic structural models has been largely ad hoc and often achieved by relying on assumptions that are either theoretically unjustified or by significantly restricting the data. Altug and Miller (1998), Gayle and Miller (2003), and Gayle (2006) rely on complete markets and degenerate transition probability assumptions to form counterfactual strategies that obtain finite dependence. A key insight of Arcidiacono and Miller (2011) is that "the expected value of future utilities from optimal decision making can always be expressed as functions of the flow payoffs and CCPs for *any* sequence of future choices, optimal or not." This insight is the basis of our extension of the finite-dependence property. I show the expected value of future utilities from optimal decision-making can be expressed as *any linear combination* of flow payoffs and conditional CCPs, as long as the weights sum to one. This insight converts the difficult problem of finding one pair of sequences of choices that obtains finite dependence to a potential set of finite dependencies from which to choose.

Given that I am now able to choose from a class of finite-dependence representations, the question becomes whether a choice of weights exists that obtains one-period finite dependence. Indeed, one-period finite dependence is achievable regardless of the form of the transition functions, as long as they are non-degenerate. The resulting form of the conditional value function provides a simple method for accommodating continuous choices.

The approach taken to model continuous choices may be considered a dynamic version of the Roy (1951) model, and parallels the method for estimating discrete/continuous static structural models of Dubin and McFadden (1984) and Hanemann (1984). Particularly, in each period and for each discrete alternative, the agent observes the period-specific shocks and solves for the associated conditional continuous choices (henceforth CCCs) that maximizes the corresponding alternative-specific value of the discrete choice. The agent then chooses the alternative with the highest (maximized) value. This dynamic selection on unobservables implies the distribution of observed CCCs differs from the distribution of optimal CCCs, rendering first-stage estimation of optimal CCCs biased without additional restrictions, such as the Pareto optimality condition imposed by Altug and Miller (1998) and subsequent authors. The alternative sequence of the revelation of the shocks assumed in Blevins (2014) also circumvents this dynamic selection on unobservables problem, but at the cost of restricting the sources of selection effects.

Along with individual-time-specific shocks, the model developed in this paper allows for discrete-valued, permanent unobserved heterogeneity in the utility functions and production functions. The distribution of these unobserved random variables may be correlated with observable covariates of the model. I provide sufficient conditions for identification of the utility functions and the distribution of the unobserved heterogeneity. The identification strategy taken in this paper is closely related to Magnac and Thesmar (2002), Blevins (2014), and Arcidiacono and Miller (2020).[1] Magnac and Thesmar (2002) investigate the potential for restrictions such as absorbing states, additive separability, and terminal conditions to deliver identification of the period-specific payoff functions in stationary environments. Blevins (2014) investigates identification of the parameters of models with discrete and continuous choices, also in stationary environments. Although the timing restriction of Blevins (2014) – the discrete-choice shocks are observed before the continuous-choice shocks – is an effective restriction for identification of the period-specific utilities and continuous-choice shocks, it restricts the scope of selectivity in dynamic structural models. This paper assumes the continuous-choice shocks are observed before, or simultaneously with, the discrete-choice shocks, but also assumes the distribution of the continuous-choice shocks are known.

As discussed by Arcidiacono and Miller (2020), identification becomes more difficult in nonstationary environments, particularly when the final period of the decision process is unobservable to the investigator. Arcidiacono and Miller (2020) discuss identification of models of this type under the assumption that single-action $\rho$-dependence holds, meaning the $\rho$-period-ahead transition probabilities of the observed state variables given any action in the current period and the normalizing actions (the action for which the period-specific utilities are known) in all $\rho$ periods thereafter coincide. The identification strategy in this

[1] Blevins (2014) and Arcidiacono and Miller (2020) provide a comprehensive review of the literature on identification of dynamic structural models.

paper imposes different restrictions on state transition probabilities of the observed state variables, which can obtained by index restrictions or exclusion restrictions.

I propose a GMM estimator for these parameters and an iterative algorithm to compute them. Relative to maximum likelihood-based estimators, the GMM estimator has the advantage of not requiring specification of the distribution of measurement errors, which is of particular concern in this framework because continuous outcome variables are often measured with errors. Also, the GMM estimator is robust to the initial-conditions problem in that consistent estimation of the parameters does not require observing the initial state variables or estimating the initial conditions. A consequence of opting for the GMM approach to estimation is that the method developed in Arcidiacono and Miller (2011) to account for unobserved heterogeneity is no longer available. I address this deficiency by developing an iterative Bayes method that uses information from the CCPs alone.

I investigate the finite-sample properties of the proposed estimator by way of Monte Carlo methods. I consider two environments. The first is a discrete-continuous choice model in which the performance of the proposed estimator is compared with the full-solution method of obtaining the CCPs and CCCs. The second compares the approach proposed in this paper with the one proposed in Arcidiacono and Miller (2011) in estimating a model that exhibits the renewal property. The results show the proposed estimator performs well in both environments.

I apply the methods developed in this paper to estimate a model of educational attainment and labor supply to investigate properties of the distribution of the returns to education, using data from the NLSY79. Key features of the model are as follows: (1) I allow for individuals to choose to simultaneously participate in the labor market and enroll in school; (2) I treat hours worked as a continuous-choice variable and allow for it to affect the probability of completing the grade level enrolled in; (3) I allow for psychic costs of school attendance and labor market activities; and (4) returns to education is modeled as a random coefficient with a finite-mixture distribution and mixing probabilities depending on racial, parental income, and on Armed Forces Qualification Test score (AFQT) categories. The model estimated in this paper is closely related to those estimated in Keane and Wolpin (1997; 2000; 2001) and Eckstein and Wolpin (1999). A key distinction between the theoretical model presented in these cited papers and the one in this paper is this paper specifies unobserved heterogeneity as a random coefficient on the level of education.

The results show the distribution of returns to education varies significantly with AFQT, with the distribution for individuals with AFQT above the median first-order stochastically dominating the distribution for individuals with AFQT below the median. Racial differences in the distribution of returns to education are greatly reduced when AFQT is accounted for. However, parental income remains a significant determinant of the distribution of returns to education, with the distribution for individuals with high parental income stochastically dominating the distribution for individuals with low parental income for all AFQT and race categories, but marginally so for blacks with high AFQT. I also find the black-white gap in constants of the log wage function is decreasing in higher parental income, suggesting this gap reflects greater access to higher-paying jobs by individuals with richer parents (higher permanent income) rather than a skill gap.

The results suggest that, net of differences in scholastic endowments as measured by AFQT, no economically significant racial variation exists in the barriers to school enrollment, nor in the likelihood of completing a grade level given enrollment. I also find inclusion of AFQT significantly reduces the effect of parental income on the barriers to school enrollment and grade-completion rates. This result is consistent with Cameron and Heckman (2001) and Carneiro and Heckman (2003), who argue the effect of parental income on educational attainment is primarily a result of it being a proxy for permanent income, which influences early childhood development of scholastic abilities. An important caveat to these results pertains to individuals who work while attending school, for whom I find the nonpecuniary costs of working while enrolled in school is higher for blacks. I also find these costs to be increasing in parental income, which is consistent with the existence of borrowing constraints found in Keane and Wolpin (2001).

I find that nonpecuniary costs of labor force participation are decreasing in parental income and does not vary by race. These results are unchanged with the inclusion of AFQT, which itself reduces the nonpecuniary costs of employment. These results suggest AFQT also measures valuable labor market abilities, which are developed during early childhood, and that individuals with richer parents possess greater access to the labor market.

The rest of the paper proceeds as follows. Section 2 outlines the class of dynamic structural models investigated in this paper and presents the new alternative representation of the value functions that I use to obtain finite dependence. Section 3 then defines generalized finite dependence, shows one-period finite dependence can be obtained in my class of models, defines first-order optimality conditions for optimal choices, and outlines my approach to incorporating correlated unobserved heterogeneity in the model. I provide sufficient conditions for identification of the parameters of the model in section 4. Section 5 proposes a GMM estimator for the parameters. Section 6 outlines the algorithm I propose to compute the estimator, and section 7 presents the limiting variance of my estimator. The Monte Carlo analysis of the finite-sample properties of the proposed estimator is presented in section 8, and in section 9, I implement my method to estimate a model of educational attainment and labor supply. Section 10 concludes. The Appendix contains the proofs and the tables reporting the estimation results from my empirical application in section 9.

# 2 Model

## 2.1 General framework

This section outlines the class of dynamic structural discrete/continuous-choice models that I consider and corresponding alternative representation. This framework only modifies that of Arcidiacono and Miller (2011) to include the CCCs, and I maintain the notation of notation of Arcidiacono and Miller (2011) where feasible for consistency. The inclusion of continuous choices follows closely the framework of the static discrete/continuous-choice models presented in Dubin and McFadden (1984) and Hanemann (1984).

In each period, $t$, an individual chooses among $J$ discrete, mutually exclusive, and exhaustive alternatives. Let $d_{jt}$ be 1 if the discrete action $j \in \{1, \cdots, J\}$ is taken in period $t$, and 0 otherwise, and define $d_t = (d_{1t}, \cdots, d_{Jt})$. Associated with each discrete alternative, $j$, the individual chooses $L_j$ continuous alternatives. Let $c_{l_j t} \in \mathcal{C}_{l_j t} \subseteq \Re_+$, $l_j \in 1, \cdots, L_j$, be the continuous actions associated with alternative $j$, with $c_{l_j t} > 0$ if $d_{jt} = 1$. Define $c_{jt} = (c_{1t}, \cdots, c_{L_j t}) \in \mathcal{C}_{jt}$, where $\mathcal{C}_{jt} = \times_{l_j=1}^{L_j} \mathcal{C}_{l_j t}$ and $c_t = (c_{1t}, \cdots, c_{jt}) \in \mathcal{C}_t$, where $\mathcal{C}_t = \times_{j=1}^{J} \mathcal{C}_{l_j t} \subseteq \Re_+^L$, and $L = \sum_{j=1}^{J} L_j$. Also, let $(j, c_{jt})$ be the vector of discrete and continuous actions associated with alternative $j$. The current-period payoff associated with action $(j, c_{jt})$ depends on the observed state $x_t \in \mathcal{X} \subseteq \Re^{D_x}$, where $D_x$ is the dimension of $x_t$, the unobserved (to the investigator) state $s_t \in \mathcal{S} \subseteq \Re^{D_s}$, where $D_s$ is the dimension of $s_t$, the unidimensional discrete-choice–specific shock $\varepsilon_{jt} \in \Re$, and the $L_j$-dimensional vector of continuous-choice–

specific shocks $r_{jt} = (r_{1t}, \cdots, r_{L_j t}) \in \Re^{L_j}$. Let $z_t = (x_t, s_t) \in \mathcal{Z} \subseteq \Re^{D_x + D_s}$, $e_{jt} = (\varepsilon_{jt}, r_{jt})$, $e_t = (e_{1t}, \cdots, e_{jt})$, and $r_t = (r_{1t}, \cdots, r_{jt}) \in \Re^L$.

With respect to the empirical application, the individual chooses among four $(J = 4)$ discrete alternatives, which are, stay home ($j = 1$), not attend school and work ($j = 2$), not work and attend school ($j = 3$), and work and attend school simultaneously ($j = 4$). The CCCs the individual faces is the number of hours to work given that he chooses to participate in the labor market $(j = 2, 4)$. The shocks associated with these CCCs are the shocks to the wage-offer functions, and the unobserved state vector, $s_t$ is the returns to education. The distribution of returns to education is assumed to depend on the race of the individual, the individual's parental income, and the AFQT of the individual.

Define $y_{jt} = (d_{jt}, c_{jt})$ and let $u_{jt}(z_t, c_{jt}, e_{jt})$ be the individual's period-specific payoff. The individual chooses the vector $y_t = (y_{1t}, \cdots, y_{Jt})$ to sequentially maximize the expected discounted sum of payoffs:

$$E\left\{\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{t-1}d_{jt}[u_{jt}(z_t, c_{jt}, e_{jt})]\right\}, \tag{2.1}$$

where $\beta \in (0, 1)$ is the discount factor. In each period, $t$, the expectation is taken with respect to the joint distribution of $z_{t+1}, \cdots, z_T$ and $e_{t+1}, \cdots, e_T$. Let $f_{jt}(x_{t+1}, e_{t+1} | z_t, c_{jt}, e_t)$ be the probability function of $(x_{t+1}, e_{t+1})$ given $(z_t, e_t)$ and action $(j, c_{jt})$ taken in period $t$. The following restrictions are placed on the period-specific utility and probability functions.

**Assumption 2.1.** *For $j = 1, \cdots, J$,*
*1. the period-specific utility function, $u_{jt}(z_t, c_{jt}, e_{jt}) = u_{jt}(z_t, c_{jt}, r_{jt}) + \varepsilon_{jt}$, and*
*2. the transition function, $f_{jt}(z_{t+1}, e_{t+1} | z_t, c_{jt}, e_t) = f_{jt}(z_{t+1} | z_t, c_{jt}) g_\varepsilon(\varepsilon_{t+1}) g_r(r_{t+1})$.*
*3. The individual observes $(z_t, e_t)$ at the beginning of period $t$.*

Assumption 2.1.1 is the standard additive separability restriction of the period-specific utility function in the discrete-alternative-specific shocks. Note that, similar to Blevins (2014), the continuous-choice-specific shocks may enter the period-specific utility functions nonlinearly, which is necessary to avoid statistical degeneracy during estimation by maximum likelihood. Assumption 2.1.2 is the standard conditional independence assumption, with the additional restrictions that the discrete-alternative-specific shocks and the CCC-specific shocks are statistically independent. This assumption is stronger than the conditional independence assumption imposed in Blevins (2014), who assumes these two shocks are independent given the discrete choice and the observable state variables in period $t$. Assumption 2.1.2 can be extended to allow for the distribution of the CCC-specific shocks to be non-stationary and dependent on $x_t$, but at the cost of increased complexity in the exposition of the key results.

The key transition probability in the empirical application is the probability of completing a grade level given the individual enrolls in school $(j = 3,4)$. The probability of completing a grade level depends on the hours worked by the individual if he decides to both attend school and work $(j = 4)$.

Given Assumption 2.1.3, the individual computes the optimal continuous choice for each discrete alternative and then chooses the optimal discrete choice, given the optimal continuous choices. Assumption 2.1 implies the alternative-specific continuous choices are not functions of the discrete-choice-specific shocks, $\varepsilon_t$. Blevins (2014) considers the case in which $\varepsilon_t$ is first observed by the individual and $r_t$ is observed only after the discrete action is taken.

Let the optimal decision rule at period $t$ be given by $y_t^0 = \{(d_{jt}^0(z_t, e_t), c_{jt}^0(z_t, r_{jt})), j = 1, \cdots, J\}$, and define $u_{jt}^0(z_t, r_t) = u_{jt}(z_t, c_{jt}^0(z_t, r_{jt}), r_{jt})$. Let the ex-ante value function in period $t$, $V_t(z_t)$, be the discounted sum of expected future payoffs, prior to observing $e_t$, given the individual follows the optimal decision rule:

$$V_t(z_t) = E\left\{\sum_{\tau=t}^{T}\sum_{j=1}^{J}\beta^{\tau-t}d_{j\tau}^0(z_\tau, e_\tau)[u_{j\tau}^0(z_\tau, r_{j\tau}), r_{j\tau}) + \varepsilon_{j\tau}]\right\}.$$

The expected value function in period $t+1$, given $z_t$, the discrete choice, $j$, and corresponding CCC's, $c_{jt}$, is

$$\bar{V}_{jt+1}(z_t, c_{jt}) = \beta \int V_{t+1}(z_{t+1}) f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1}. \tag{2.2}$$

Let $\bar{V}_{jt+1}^0(z_t, r_{jt}) := \bar{V}_{jt+1}(z_t, c_{jt}^0(z_t, r_{jt}))$. The ex-ante value function can be then written recursively:

$$
\begin{aligned}
V_t(z_t) &= E\left\{\sum_{j=1}^{J} d^0_{jt}(z_t, e_t)\left[u^0_{jt}(z_t, r_{jt}) + \varepsilon_{jt} + \bar{V}^0_{jt+1}(z_t, r_{jt})\right]\right\} \\
&= \int\int \sum_{j=1}^{J} d^0_{jt}(z_t, e_t)\left[u^0_{jt}(z_t, r_{jt}) + \varepsilon_{jt} + \bar{V}^0_{jt+1}(z_t, r_{jt})\right] g_r(r_t) dr_t g_\varepsilon(\varepsilon_t) d\varepsilon_t, \\
&= \int\int \sum_{j=1}^{J} d^0_{jt}(z_t, e_t)\left[v^0_{jt}(z_t, r_{jt}) + \varepsilon_{jt}\right] g_r(r_t) dr_t g_\varepsilon(\varepsilon_t) d\varepsilon_t
\end{aligned}
$$

where $v^0_{jt}(z_t, r_{jt}) := u^0_{jt}(z_t, r_{jt}) + \bar{V}^0_{jt+1}(z_t, r_{jt})$. Define

$$
v_{jt}(z_t, c_{jt}, r_{jt}) = u_{jt}(z_t, c_{jt}, r_{jt}) + \bar{V}_{jt+1}(z_t, c_{jt}), \tag{2.3}
$$

to be the choice-specific conditional value function given $y_{jt}$, excluding $\varepsilon_{jt}$, so that $v^0_{jt}(z_t, r_{jt}) = v_{jt}(z_t, c^0_{jt}(z_t, r_{jt}), r_{jt})$. Then, under Assumption 2.1, the optimal CCCs associated with the discrete alternative $j$ in period $t$, satisfy

$$
\frac{\partial}{\partial c_{l_j t}} v_{jt}(z_t, c^0_{jt}(z_t, r_{jt}), r_{jt}) = 0, \tag{2.4}
$$

for $l_j = 1, \cdots, L_j$, and the optimal discrete choice of alternative $j$ is

$$
d^0_{jt}(z_t, e_t) = \begin{cases} 1 \quad \text{if} \quad v^0_{jt}(z_t, r_{jt}) + \varepsilon_{jt} > v^0_{kt}(z_t, r_{tk}) + \varepsilon_{kt} \quad \forall k \neq j \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.5}
$$

Finally, the optimal CCC, $c^*_{jt}(z_t, r_{jt})$, is given by

$$
c^*_{jt}(z_t, e_{jt}) = d^0_{jt}(z_t, e_t) c^0_{jt}(z_t, r_{jt}). \tag{2.6}
$$

## 2.2 CCP formulation

The probability of choosing alternative $j$ at time $t$, conditional on $z_t$, $r_t$, and the vector of choice-specific optimal CCCs, $c_t^0(z_t, r_t)$, is given by

$$p_{jt}^0(z_t, r_t) = E[d_{jt}^0(z_t, e_t)|z_t, r_t], \tag{2.7}$$

so that, for all $(z_t, r_t)$, $\sum_{j=1}^J p_{jt}^0(z_t, r_t) = 1$, and $p_{jt}^0(z_t, r_t) > 0$ for all $j$. Let $p_t^0(z_t, r_t) = (p_{1t}^0(z_t, r_t), \cdots, p_{Jt}^0(z_t, r_t))'$ be the vector of CCPs. Lemma 1 of Arcidiacono and Miller (2011) shows a function $\psi : [0,1]^J \mapsto \Re$ exists such that for $k = 1, \cdots, J$,

$$\psi_k(p_t^0(z_t, r_t)) \equiv V_t(z_t, r_t) - v_{tk}^0(z_t, r_{kt}). \tag{2.8}$$

Equation (2.8) is simply equation (3.5) of Arcidiacono and Miller (2011), modified so the choice probabilities and value functions are also conditional on the i.i.d. shocks associated with the CCCs.

The key insight is that if (2.8) holds for $k = 1, \cdots, J$, then for any $J$-dimensional vector of real numbers $a = (a_1, \cdots, a_J)$ with $\sum_{k=1}^J a_k = 1$,

$$V_t(z_t, r_t) = \sum_{k=1}^J a_k[v_{kt}^0(z_t, r_{kt}) + \psi_k(p_t^0(z_t, r_t))]. \tag{2.9}$$

Let $a_{jt+1}(z_{t+1}, r_{t+1}) = (a_{1jt+1}(z_{t+1}, r_{1t+1}), \cdots, a_{Jjt+1}(z_{t+1}, r_{Jt+1}))$, be the weights associated with the initial discrete choice, $j$, in period $t$. Substituting equation (2.9) into equation (2.3) gives

$$\begin{aligned} v_{jt}(z_t, c_{jt}, r_{jt}) &= u_{jt}(z_t, c_{jt}, r_{jt}) \\ &+ \beta \sum_{k=1}^J \int\int [v_{k,t+1}^0(z_{t+1}, r_{kt+1}) + \psi_k(p_{t+1}^0(z_{t+1}, r_{t+1}))] \\ &\times a_{kjt+1}(z_{t+1}, r_{kt+1}) g_r(r_{t+1}) dr_{t+1} f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1}. \end{aligned} \tag{2.10}$$

### 2.2.1 Clarifying example

To clarify the alternative representation, I provide a "stripped down" example of the model formation for which $J = 2$. In this example, I assume the individual-time–specific discrete-choice shock, $\varepsilon_{ijt}$, is distributed i.i.d., type 1 extreme value. The expected value function in equation (2.2) becomes.

$$\bar{V}_{jt+1}(z_t, c_{jt}) = \beta \int\int \ln \sum_{k=1}^{2} e^{v^0_{kt+1}(z_{t+1}, r_{kt+1})} g_r(r_{t+1}) dr_{t+1} f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1} + \beta\gamma, \quad (2.11)$$

and $\gamma$ is the Euler constant. Also, the period $t+1$ conditional choice probability of alternative $j = 1,2$ is given by

$$p^0_{jt+1}(z_{t+1}, r_{t+1}) = \frac{e^{v^0_{jt+1}(z_{t+1}, r_{jt+1})}}{\sum_{k=1}^{2} e^{v^0_{kt+1}(z_{t+1}, r_{kt+1})}}. \quad (2.12)$$

From equation (2.12), the following equality holds for $j = 1,2$:

$$\ln \sum_{k=1}^{2} e^{v^0_{kt+1}(z_{t+1}, r_{kt+1})} = v^0_{jt+1}(z_{t+1}, r_{jt+1}) - \ln p^0_{jt+1}(z_{t+1}, r_{t+1}). \quad (2.13)$$

Notice equation (2.13) is simply equation (2.8) under the assumptions of this example and evaluated at period $t+1$. Also, note the (LHS) of equation (2.13) is a term inside the integral on the (RHS) of equation (2.11). For alternative $j = 1,2$, let $a_{kjt+1}(z_{t+1}, r_{t+1})$ be weights associated with alternative $j$ in period $t$ and alternative $k$ in period $t+1$, with $a_{1jt+1}(z_{t+1}, r_{t+1}) + a_{2jt+1}(z_{t+1}, r_{t+1}) = 1$, $j = 1,2$. Then, from equation (2.13),

$$\begin{aligned} &\ln \sum_{k=1}^{2} e^{v^0_{kt+1}(z_{t+1}, r_{kt+1})} \\ &= \sum_{k=1}^{2} a_{kjt+1}(z_{t+1}, r_{t+1})[v^0_{kt+1}(z_{t+1}, r_{kt+1}) - \ln p^0_{kt+1}(z_{t+1}, r_{t+1})]. \end{aligned} \quad (2.14)$$

Substituting equation (2.14) into equation (2.11) obtains

$$\begin{aligned}
&\bar{V}_{jt+1}(z_t, c_{jt}) \\
&= \beta \int\int \sum_{k=1}^{2} a_{kjt+1}(z_{t+1}, r_{t+1})[v^0_{kt+1}(z_{t+1}, r_{kt+1}) - \ln p^0_{kt+1}(z_{t+1}, r_{t+1})] \\
&\times g_r(r_{t+1}) dr_{t+1} f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1} + \beta\gamma.
\end{aligned} \tag{2.15}$$

Now, substituting $\bar{V}_{jt+1}$ from equation (2.15) into equation (2.3), obtains

$$\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) = u_{jt}(z_t, c_{jt}, r_{jt}) \\
&+ \beta \int\int \sum_{k=1}^{2} a_{kjt+1}(z_{t+1}, r_{t+1})[v^0_{kt+1}(z_{t+1}, r_{kt+1}) - \ln p^0_{kt+1}(z_{t+1}, r_{t+1})] \\
&\times g_r(r_{t+1}) dr_{t+1} f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1} + \beta\gamma,
\end{aligned} \tag{2.16}$$

which is equation (2.10) under the assumptions of this clarifying example.

Equation (2.10) shows the value function conditional on $(z_t, r_t)$ can be written as the flow payoff of the choice plus any weighted sum of a function of the one-period-ahead CCPs plus the one-period-ahead conditional value functions, where the weights sum to 1. This extension of the results of Arcidiacono and Miller (2011) provides a powerful tool for obtaining finite dependence in a larger class of models than previously investigated.

# 3 Generalized finite dependence

This section shows that for any period $t < T - \rho$, the conditional value function can be expressed as a linear combination of $\rho$-periods-ahead period-specific utility functions and CCPs, and the $t + \rho + 1$ expected value functions. To that end, define $f^0_{jt}(z_{t+1}|z_t, r_{jt}) = f_{jt}(z_{t+1}|z_t, c^0_{jt}(z_t, r_{jt}))$. For any initial choice $(j, c_{jt})$, define the sequence

$$\{a_{kj\tau}(z_\tau, r_\tau|z_t, c_{jt}), \tau = t+1, \cdots, t+\rho, k = 1, \cdots, J\} \tag{3.1}$$

with $\sum_{k=1}^{J} a_{kj\tau}(z_\tau, r_\tau | z_t, c_{jt}) = 1$, and corresponding state transition functions,

$$\kappa_{j\tau}(z_{\tau+1}, | z_t, c_{jt}) =$$
$$\begin{cases} f_{jt}(z_{t+1}|z_t, c_{jt}) \quad \text{for } \tau = t \\ \sum_{k=1}^{J} \int\int f_{k\tau}^{0}(z_{\tau+1}|z_\tau, r_{k\tau}) a_{kj\tau}(z_\tau, r_\tau|z_t, c_{jt}) g_r(r_{k\tau}) dr_{k\tau} \kappa_{\tau-1,j}(z_\tau|z_t, c_{jt}) dz_\tau \\ \quad \text{for } \tau = t+1, \cdots, t+\rho. \end{cases} \tag{3.2}$$

**Theorem 3.1.** *Suppose Assumption 2.1 holds. For initial period $t < T - \rho$ and subsequent periods $\tau = \{t+1, \cdots, t+\rho\}$, the alternative-specific conditional value function can be expressed as a linear combination of $\rho$-periods-ahead period-specific utility functions and CCPs, with weights defined in (3.1), and the $\rho+1$-periods-ahead continuation value, where the utility functions and CCPs are evaluated at the optimal CCCs.*

The proof of Theorem 3.1 can be found in Appendix A.1.

**Definition 3.2.** For initial period $t < T - \rho$, a pair of initial choices, $(j, c_{jt})$ and $(j', c_{j't})$, exhibits generalized $\rho$-period dependence if the difference in their alternative-specific conditional value functions can be expressed a linear combination of $\rho$-periods-ahead period-specific utility functions and CCPs, where the utility functions and CCPs are evaluated at the optimal CCCs.

The following theorem provides sufficient conditions on the weights defined in equation (3.1) to achieve $\rho$-period finite dependence. Its proof can be found in Appendix A.2.

**Theorem 3.3.** *Suppose Assumption 2.1 holds. For initial period $t < T - \rho$, a pair of initial choices, $(j, c_{jt})$ and $(j', c_{j't})$, exhibits generalized $\rho$-period dependence if corresponding sequences,*
$\{(a_{kj\tau}(z_\tau, r_\tau|z_t, c_t), a_{kj'\tau}(z_\tau, r_\tau|z_t, c_t)), \tau = t+1, \cdots, t+\rho, k = 1, \cdots, J\}$ *exist for which*

$$\kappa_{jt+\rho}(z_{t+\rho+1}|z_t, c_{jt}) = \kappa_{j't+\rho}(z_{t+\rho+1}|z_t, c_{j't})$$

*almost everywhere with* $\sum_{k=1}^{J} a_{kk'\tau}(z_\tau, r_{k\tau}|z_t, c_t) = 1, \quad k' = j, j'$.

If the conditions imposed on the weights in Theorem 3.3 hold, then the difference in the conditional value functions is given by

$$
\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) - v_{j't}(z_t, c_{j't}, r_{j't}) = u_{jt}(z_t, c_{jt}, r_{jt}) - u_{j't}(z_t, c_{j't}, r_{j't}) \\
&+ \sum_{\tau=t+1}^{t+\rho} \sum_{k=1}^{J} \int \int \beta^{\tau-t} [u^0_{k\tau}(z_\tau, r_{k\tau}) + \psi_k[p^0_\tau(z_\tau, r_\tau)]] \\
&\times [a_{kj\tau}(z_\tau, r_{k\tau}|z_t, c_t)\kappa_{j\tau-1}(z_\tau|z_t, c_{jt}) - a_{kj'\tau}(z_\tau, r_{k\tau}|z_t, c_t)\kappa_{j'\tau-1}(z_\tau|z_t, c_{j't})] g_r(r_\tau) dr_\tau dz_\tau,
\end{aligned} \tag{3.3}
$$

which is the central result of this section. Notice the removal of the first subscript on CCC in the definition of the weights, because the discrete-choice–specific weights defined in Theorem 3.3 may depend on the CCCs associated with the competing discrete alternative.

## 3.1 One-period finite dependence

The advantage of CCP estimation of dynamic structural models relative to the full-solution method is its numerical advantage in terms of computation time, and this numerical advantage depends on the order of finite dependence, $\rho$. Smaller $\rho$ generally results in faster computation time, and more precise estimates of the structural parameters if the CCPs are estimated in a first stage as in Hotz, Miller, et al. (1994), Altug and Miller (1998), and Gayle and Miller (2003). In the context of this paper, the computational advantage of the CCP estimator also depends on whether the transition probabilities, and hence the weights that achieve finite dependence in Theorem 3.3, can be estimated outside the model. This section provides sufficient conditions for which one-period finite dependence in models where the transition functions are such that the weights can be computed outside the model. I consider two cases. In the first case, the transition function does not depend on the continuous choices, nor the unobserved effects, in which case, the CCCs and unobserved effects only enter the period-specific payoff functions. In the second case, the transition functions depend on the continuous choices, but not the unobserved effects, in which case, the unobserved effects enter the transition functions only through the optimal CCCs. In both cases, I assume the unobserved effect is permanent to the individual.

**Assumption 3.4.** *The unobserved effect, $s_t = s$ for all $t$.*

#### 3.1.1 Case 1

The first restriction on the transition functions considered is as follows.

**Assumption 3.5.** *The transition probability,* $f_{jt}(z_{t+1}|z_t,c_{jt}) = f_{jt}(x_{t+1}|x_t)$.

Assumption 3.5 restricts the transition probability of the observed states variables to not be a function of the continuous choices and the unobserved effect. This restriction holds in the empirical application for $j = 3$, the choice of enrolling in school and not work.

Theorem 3.3 implies one-period finite-dependence holds for initial choices $(j, j')$ if

$$\int \sum_{k=1}^{J} f_{kt+1}(x_{t+2}|x_{t+1}) \left[a_{kj}(x_{t+1}|x_t) f_{jt}(x_{t+1}|x_t) - a_{kj'}(x_{t+1}|x_t) f_{j't}(x_{t+1}|x_t)\right] dx_{t+1} = 0, \tag{3.4}$$

$$\sum_{k=1}^{J} a_{kk'}(x_{t+1}|x_t) = 1, \quad k' = j, j' \tag{3.5}$$

almost everywhere, where the time subscript on the weights are dropped for convenience, in which case, equation (3.3) becomes

$$\begin{aligned} v_{jt}(z_t,c_{jt},r_{jt}) - v_{j't}(z_t,c_{j't},r_{j't}) &= u_{jt}(z_t,c_{jt},r_{jt}) - u_{j't}(z_t,c_{j't},r_{j't}) \\ &+ \beta \sum_{k=1}^{J} \int \int [u^0_{kt+1}(z_{t+1},r_{kt+1}) + \psi_k(p^0_{t+1}(z_{t+1},r_{t+1}))] g_r(r_{t+1}) dr_{t+1} \\ &\times \left[a_{kj}(x_{t+1}|x_t) f_{jt}(x_{t+1}|x_t) - a_{kj'}(x_{t+1}|x_t) f_{j't}(x_{t+1}|x_t)\right] dx_{t+1}. \end{aligned} \tag{3.6}$$

In general, one would have to solve equations (3.4)-(3.5) to obtain weights that achieve finite dependence. However, special cases exist in the literature for which the weights that solve this system of equations are closed-form. Although these cases satisfy one-period finite dependence defined in Arcidiacono and Miller (2011), showing how they can be considered special cases of one-period finite stated defined in Definition 3.2 is instructive. It is sufficient to consider models where $J = 2$, because the weights that achieve one-period finite dependence can be computed pairwise.

Let $x_t = (x_{1t}, x_{2t})$, where $x_{1t}$ is a vector of strictly exogenous variables. I discuss three such cases in the following under the framework of the clarifying example where $J = 2$.

**Simple Transition** A simple transition function is defined by the restriction that the period $t+1$ conditional distribution of the endogenous state variables is independent of the period $t$ endogenous state variables given the joint distribution of the period $t$ and $t+1$ strictly exogenous state variables. In particular, the transition function takes the form

$$f_{jt}(x_{t+1}|x_t) = f_{jt}(x_{2t+1}|x_{1t+1}, x_{1t})\, f_t(x_{1t+1}|x_{1t}), \tag{3.7}$$

in which the LHS of equation (3.4) is given by

$$\begin{aligned}
&\int \sum_{k=1}^{2} f_{kt+1}(x_{2t+2}|x_{1t+2}, x_{1t+1})\, f_{t+1}(x_{1t+2}|x_{1t+1}) \\
&\times \int \left[a_{k2}(x_{t+1}|x_t)\, f_{2t}(x_{2t+1}|x_{1t+1}, x_{1t}) - a_{k1}(x_{t+1}|x_t)\, f_{1t}(x_{2t+1}|x_{1t+1}, x_{1t})\right] dx_{2t+1} \\
&\times f_t(x_{1t+1}|x_{1t}) dx_{1t+1}.
\end{aligned} \tag{3.8}$$

Therefore, setting $a_{11}(x_{t+1}|x_t) = a_{12}(x_{t+1}|x_t) = \gamma$ (and hence, $a_{21}(x_{t+1}|x_t) = a_{22}(x_{t+1}|x_t) = 1-\gamma$), for any $\gamma \in \Re$ satisfies equation (3.4).

**Renewal.** A model with the renewal property is one for which an action, say, alternative one, can be taken in period $t+1$ so the conditional distribution of the period $t+2$ endogenous state variables does not depend on the action taken in period $t$, given the joint distribution of the periods $(t, \cdots, t+2)$ exogenous state variables. In other words, equation (3.8) holds for only $j = 1$. Then, one-period finite dependence is obtained by setting $a_{11}(x_{t+1}|x_t) = a_{12}(x_{t+1}|x_t) = 1$ in equation (3.8). The bus-engine-replacement model of Rust (1987) is the central example of a model with the renewal property, where the state variable of interest is mileage of the bus, and the renewal action of replacing the bus engine (alternative 1 in our example) in period $t+1$ resets mileage to zero, thus making the distribution of mileage in period $t+2$ independent of the decision of whether to replace the bus engine in period $t$.

**Exchangeability.** A model with the exchangeability property is one for which taking "opposing" discrete actions in periods $t$ and $t+1$ results in the same distribution of the two-period-ahead state variables, that is,

$$\int f_{1t+1}(x_{t+2}|x_{t+1})f_{2t}(x_{t+1}|x_t)dx_{t+1} = \int f_{2t+1}(x_{t+2}|x_{t+1})f_{1t}(x_{t+1}|x_t)dx_{t+1} \tag{3.9}$$

Then, equation (3.4) is satisfied by setting $a_{12}(x_{t+1}|x_t) = a_{21}(x_{t+1}|x_t) = 1$. The exchangeability restriction holds in the typical labor-supply model in which, say, alternative 2 is the decision to work and the endogenous state variable is years of experience, $\sum_{\tau=1}^{t-1} d_{2\tau}$, which enters the classical Mincerian wage-offer function. Indeed, I implement these weights in the empirical application to obtain finite dependence of the difference between the conditional value of working and not attending school $(j = 2)$, and the conditional value of staying home $(j = 1)$.

Additional model frameworks may exist for which closed-form weights that satisfy the system of equations (3.4) - (3.5). However, this system of equations has to be solved numerically to obtain the weights that obtain finite dependence in more general frameworks. The following are two examples with no constants for which one-period finite dependence is satisfied, in which case, one-period finite dependence as defined in Arcidiacono and Miller (2011) is not achievable.

**Education attainment**. At any age, $t$, the individual chooses to enroll in school or stay home. The endogenous state variable is the completed grade level at age $t$, $x_t$. If an individual at age $t$ with grade level $x_t$ enrolls in school, she advances the grade level $(x_{t+1} = x_t + 1)$ with probability $\pi_t(x_t)$. It is straightforward to check that if $\pi_t(x_t) \neq \pi_{t+1}(x_t)$, no two sequences of choices are equivalent the resulting period $t+2$ distributions of education, in which case, equations (3.4) - (3.5) must be solved numerically. Indeed, $\rho$-period dependence is not achievable for any $\rho \geq 1$, due to the dependence of the probability of completing a grade level on age. With respect to the empirical application, these weights are computed using equation (3.11) below to achieve one-period finite dependence of the difference between the conditional value of enrolling in school and not work $(j = 3)$ and the conditional value of staying home $(j = 1)$.

**Fertility choice.** In each period, a household decides whether to try to have a child, and the endogenous variable is a live-birth outcome, whose probability depends on the household's history of birth outcomes. Let $b_t$ be equal to a if the household successfully gives birth, and 0 otherwise. Let $x_t = \sum_{\tau=1}^{t-1} \theta_\tau b_\tau$ be the household's history of live-birth outcomes.

The live-birth transition function is given by $f_{jt}(b_{t+1}|x_t)$. Because the probability of a live-birth depends on the pattern of previous live births, no two sequences of choices exist for which finite dependence as defined in Arcidiacono and Miller (2011) is achieved. One restriction to the model that achieves finite dependence is to assume the probability of a live birth depends on the history of live births only through the number of lives births (the number of kids), in which case the exchangeability condition holds. The results of Gayle and Miller (2003) show this restriction is substantial, especially in models of fertility choice and labor supply in which the spacing of births substantially affects the labor market success of females.

To provide sufficient conditions for one-period dependence in the general case, I assume the state variables are discrete.

**Assumption 3.6.** *The observed state variable, $x_t$, is discrete with cardinality $|x|$.*

Under these restrictions, the model exhibits one-period finite dependence if

$$\sum_{k=1}^{2}\sum_{i=1}^{|x|}\left[f_{kt+1}(x_{t+2}|x_i)a_{k2t+1}(x_i|x_t)f_{2t}(x_i|x_t) - f_{kt+1}(x_{t+2}|x_i)a_{k1t+1}(x_i|x_t)f_{1t}(x_i|x_t)\right] = 0 \tag{3.10}$$

for $x_{t+2} = x_1, \cdots, x_{|x|}$, with $a_{1jt+1}(x|x_t) + a_{2jt+1}(x|x_t) = 1,\ j = 1,2.$ Denote the probability of reaching state $x_{t+2}$ and $x_{t+1}$ from state $x_t$ and actions $j$ in period $t$ and $k$ in period $t+1$ by

$$f_{kj}(x_{t+2}, x_{t+1}|x_t) = f_{kt+1}(x_{t+2}|x_{t+1})f_{jt}(x_{t+1}|x_t).$$

Further define

$$f_{kj}(x|x_t) = \left(f_{kj}(x,x_1|x_t), \cdots, f_{kj}(x,x_{|x|}|x_t)\right), \quad f_{kj}(x_t) = \left(f_{kj}(x_1|x_t)', \cdots, f_{kj}(x_{|x|}|x_t)'\right)', \quad \text{and}$$
$$a_{kj}(x_t) = \left(a_{kjt+1}(x_1|x_t), \cdots, a_{kjt+1}(x_{|x|}|x_t)\right)'.$$

Equation (3.10) implies the $|x|$ system of equations $\sum_{k=1}^{2}[f_{k2}(x_t)a_{k2}(x_t) - f_{k1}(x_t)a_{k1}(x_t)] = 0.$

Imposing the condition $a_{1j}(x_t) = \iota_{|x|} - a_{2j}(x_t)$, $j = 1,2$ obtains

$$(f_{12}(x_t) - f_{22}(x_t))a_{12}(x_t) + (f_{21}(x_t) - f_{11}(x_t))a_{11}(x_t) = (f_{21}(x_t) - f_{22}(x_t))\iota_{|x|}, \Leftrightarrow \quad (3.11)$$
$$F_1(x_t)a(x_t) = F_2(x_t)\iota_{|x|}, \quad (3.12)$$

where $F_1(x_t) = [(f_{12}(x_t) - f_{22}(x_t)), (f_{21}(x_t) - f_{11}(x_t))]$, is a $2|x| \times |x|$ matrix, $F_2(x_t) = f_{21}(x_t) - f_{22}(x_t)$ is an $|x| \times |x|$ matrix, $a = (a_{12}(x_t)', a_{11}(x_t)')'$ is an $|x|$-dimensional vector, and $\iota_{|x|}$ is the $|x|$-dimensional vector of ones. In general, equation (3.11) constitutes $|x|$ equations with at most $2|x|$ unknowns, making it a consistent and underdetermined system with an infinitude of solutions. A column of $F_1(x_t)$ takes the form $(f_{kt+1}(x'|x) - f_{jt+1}(x'|x))f_{jt}(x|x_t)$, $x' = 1, \cdots, |x|$, which is nonzero if $f_{jt}(x|x_t) \neq 0$ and $f_{kt+1}(x'|x) \neq f_{jt+1}(x'|x)$ for at least one $x' \in \{x_1, \cdots, x_{|x|}\}$. In other words, this column is nonzero if state $x$ can be reached in period $t+1$ from $x_t$, given action $j$, and least one state in period $t+2$ exists for which the action taken in period $t+1$ is consequential for its occurrence. Therefore, a necessary condition for at least one solution to the system of equations (3.12) is that at least $|x|$ columns in $F_1$ satisfy these conditions. Theorem 3.7 states the corresponding sufficient conditions.

**Theorem 3.7.** *Suppose Assumptions 2.1,3.4, 3.5, and 3.6 hold. Then one-period finite dependence holds if $Pr(rank(F_1(x_t))) = |x|) = 1$.*

*Proof.* A solution to system (3.12) is given by

$$a = F_1^+(x_t)F_2(x_t)\iota_{|x|}, \quad (3.13)$$

where $^+$ denotes a generalized inverse. These weights in turn satisfy equations (3.4) - (3.5). ☐

Implementation of one-period finite dependence in practice requires estimation of $f_{jt}(x|x_t)$. This quantity can be obtained by nonparametric methods as follows:

$$\hat{f}_{jt}(x|x_t) = \frac{\sum_i^n 1\{x_{it+1} = x\}d_{jit}1\{x_{it} = x_t\}}{\sum_i^n d_{jit}1\{x_{it} = x_t\}}.$$

Notice these quantities can be computed once before estimation of the period-specific utility functions, which reduces the computational burden. To instead estimate $f_{jt}(x|x_t)$ by parametric methods, which is preferred when the dimension of $x$ is large is not uncommon.

### 3.1.2 Case 2

The second specification of the transition probabilities considered assumes the transition probability of the observed state variables depends on the continuous choices.

**Assumption 3.8.** *The transition probability,* $f_{jt}(z_{t+1}|z_t, c_{jt}) = f_{jt}(x_{t+1}|x_t, c_{jt})$.

An implication of Assumption 3.8 is the unobserved state variables affect the observed state variables only though the dependence of the optimal CCCs on the unobserved state variables. To simplify notation, $c_{jt}$ is treated as unidimensional in what follows.

Suppose Assumption 3.6 holds. Setting $\rho = 1$ in equation (3.2) obtains

$$\begin{aligned}
&\kappa_{2t+1}(x_{t+2}, |z_t, c_{2t}) - \kappa_{1t+1}(x_{t+2}, |z_t, c_{1t}) = \\
&\int \sum_{k=1}^{2} \sum_{i=1}^{|x|} \left[ f^0_{kt+1}(x_{t+2}|z_i, r_{kt+1}) a_{k2t+1}(x_i, r_{t+1}|z_t, c_t) f_{2t}(x_i|x_t, c_{2t}) \right. \\
&\left. - f^0_{kt+1}(x_{t+2}|z_i, r_{kt+1}) a_{k1t+1}(x_i, r_{t+1}|z_t, c_t) f_{1t}(x_i|x_t, c_{1t}) \right] g_r(r_{t+1}) dr_{t+1},
\end{aligned} \tag{3.14}$$

where $z_i = (x_i, s)$ and $f^0_{kt+1}(x_{t+2}|z_i, r_{t+1}) = f_{kt+1}(x_{t+2}|x_i, c^0_{kt+1}(z_i, r_{kt+1}))$. One-period finite dependence may be obtained pointwise over $r_{t+1}$. Specifically, define

$$\kappa_{jt+1}(x_{t+2}, |r_{t+1}, z_t, c_{jt}) = \sum_{k=1}^{2} \sum_{i=1}^{|x|} f^0_{kt+1}(x_{t+2}|z_i, r_{kt+1}) a_{kjt+1}(x_i, r_{t+1}|z_t, c_t) f_{jt}(x_i|x_t, c_{jt}), \tag{3.15}$$

so that $\kappa_{jt+1}(x_{t+2}, |z_t, c_{jt}) = \int \kappa_{jt+1}(x_{t+2}, |r_{t+1}, z_t, c_{jt}) g_r(r_{t+1}) dr_{t+1}$. As in the previous section, let

$$f_{kj}(x_{t+2}, x_{t+1}|r_{t+1}, z_t, c_{jt}) = f^0_{kt+1}(x_{t+2}|z_{t+1}, r_{kt+1}) f_{jt}(x_{t+1}|x_t, c_j)$$

be the probability of: (i) reaching state $x_{t+2}$ in period $t+2$, given states $(x_{t+1}, r_{t+1})$ are realized and action $(k, c^0_{kt+1}(z_{t+1}, r_{kt+1}))$ is taken in period $t+1$, and (ii) state $x_{t+1}$ is reached from state $x_t$, given action $(j, c_{jt})$ is taken in period $t$. Define

$$\begin{aligned}
f_{kj}(x|r,z_t,c_{jt}) &= \left(f_{kj}(x,x_1|r,z_t,c_{jt}),\cdots,f_{kj}(x,x_{|x|}|r,z_t,c_{jt})\right),\\
f_{kj}(r,z_t,c_{tj}) &= \left(f_{kj}(x_1|r,z_t,c_{jt})',\cdots,f_{kj}(x_{|x|}|r,z_t,c_{jt})'\right)',\\
a_{kj}(r,z_t,c_t) &= \left(a_{kjt+1}(x_1,|r,z_t,c_t),\cdots,a_{kjt+1}(x_{|x|}|r,z_t,c_t)\right)',\\
F_1(r,z_t,c_t) &= ((f_{12}(r,z_t,c_{2t})-f_{22}(r,z_t,c_{2t})),(f_{21}(r,z_t,c_{1t})-f_{11}(r,z_t,c_{1t}))),\\
F_2(r,z_t,c_t) &= f_{21}(r,z_t,c_{1t})-f_{22}(r,z_t,c_{2t}),\quad \text{and}\\
a(r,z_t,c_t) &= (a_{12}(r,z_t,c_t)',a_{11}(r,z_t,c_t)')'.
\end{aligned}$$

**Theorem 3.9.** *Suppose Assumptions 2.1,3.4, 3.8, and 3.6 hold. Then, one-period finite dependence holds if* $Pr(rank(F_1(r,z_t,c_t))) = |x|) = 1$ *for almost every* $r$*,* $z_t$*, and* $c_t$*.*

*Proof.* Under the conditions of the theorem and imposing the condition $a_{1j}(r,z_t,c_t) = \iota_{|x|} - a_{2j}(r,z_t,c_t)$, $j = 1,2$, the solution $a(r,z_t,c_t) = F_1^+(r,z_t,c_t)F_2(r,z_t,c_t)\iota_{|x|}$ satisfies

$$\kappa_{2t+1}(x,|r,z_t,c_{2t}) - \kappa_{1t+1}(x,|r,z_t,c_{1t}) = 0$$

for all $x \in \{x_1,\cdots,x_{|x|}\}$ with probability one and almost every $(r,z_t,c_t)$, implying

$$\kappa_{2t+1}(x,|z_t,c_{2t}) - \kappa_{1t+1}(x,|z_t,c_{1t}) = \int [\kappa_{2t+1}(x,|r,z_t,c_{2t}) - \kappa_{1t+1}(x,|r,z_t,c_{1t})]\, g_r(r)dr = 0 \tag{3.16}$$

with probability one. ☐

### 3.1.3 Approximating one-period finite dependence.

Implementation of the one-period finite dependence in Case 2 is more involved than Case 1. The first consideration is that direct implementation of Theorem 3.9 requires inversion of $F_1(r)$ for each trial values of the deep parameters of the model. One approach to circumventing this complication is to compute the weights $a_{t+1}$ on a fine grid of the continuous choice. To do so, let

$$f_{kj}(x_{t+2},x_{t+1}|c_{kt+1},x_t,c_{jt}) = f_{kt+1}(x_{t+2}|x_{t+1},c_{kt+1})\, f_{jt}(x_{t+1}|x_t,c_{jt})$$

be the probability of: (i) reaching state $x_{t+2}$ in period $t+2$, given states $x_{t+1}$ is realized and action $(k, c_{kt+1})$ is taken in period $t+1$, and (ii) state $x_{t+1}$ is reached from state $x_t$, given action $(j, c_{jt})$ are taken in period $t$. Assume $c_{jt}$ is bounded above with known bound in addition to the continuous choice being strictly positive; that is, $c_{jt} \in C_{tj} := (0, \bar{c}_{jt}]$, where $\bar{c}_{jt} < \infty$ is known. Consider a fine grid $\tilde{\mathcal{C}}_{jt} = \{0 < \tilde{c}_1 < \tilde{c}_2, \cdots, < \tilde{c}_m = \bar{c}_{jt}\}$ of $m$ points, where $m$ can potentially depend on $j$ and $t$, and define $\tilde{\mathcal{C}}_t = \tilde{\mathcal{C}}_{1t} \times \tilde{\mathcal{C}}_{2t}$. Define

$$\begin{aligned}
f_{kj}(x|x_t, c_{jt}) &= \left( f_{kj}(x, x_1|\tilde{c}_1, x_t, c_{jt}), f_{kj}(x, x_1|\tilde{c}_2, x_t, c_{jt}), \cdots, f_{kj}(x, x_{|x|}|\tilde{c}_m, x_t, c_{jt}) \right), \\
f_{kj}(x_t, c_{jt}) &= \left( f_{kj}(x_1|x_t, c_{jt}))', \cdots, f_{kj}(x_{|x|}|x_t, c_{jt}))' \right)', \\
a_{kj}(x_t, c_t) &= \left( a_{kjt+1}(x_1, \tilde{c}_1|x_t, c_t), a_{kjt+1}(x_1, \tilde{c}_2|x_t, c_t), \cdots, a_{kjt+1}(x_{|x|}, \tilde{c}_m|x_t, c_t) \right)', \\
F_1(x_t, c_t) &= \left( (f_{12}(x_t, c_{2t}) - f_{22}(x_t, c_{2t})), (f_{21}(x_t, c_{1t}) - f_{11}(x_t, c_{1t})) \right), \\
F_2(x_t, c_t) &= f_{21}(x_t, c_{1t}) - f_{22}(x_t, c_{2t}), \quad \text{and} \\
a(x_t, c_t) &= (a_{12}(x_t, c_t)', a_{11}(x_t, c_t)')'.
\end{aligned}$$

Then, $a(x_t, c_t) = F_1^+(x_t, c_t) F_2(x_t, c_t) \iota_{|x|}$ obtains

$$\begin{aligned}
&\tilde{\kappa}_{2t+1}(x_{t+2}, |z_t, c_{2t}) - \tilde{\kappa}_{1t+1}(x_{t+2}, |z_t, c_{1t}) = \\
&\int \sum_{k=1}^{2} \sum_{i=1}^{|x|} \big[ \tilde{f}_{kt+1}(x_{t+2}|z_i, r_{kt+1}) \tilde{a}_{k2t+1}(x_i, r_{t+1}|z_t, c_t) f_{2t}(x_i|x_t, c_{2t}) \\
&- \tilde{f}_{kt+1}(x_{t+2}|z_i, r_{kt+1}) \tilde{a}_{k1t+1}(x_i, r_{t+1}|z_t, c_t) f_{1t}(x_i|x_t, c_{1t}) \big] g_r(r_{t+1}) dr_{t+1} \approx 0,
\end{aligned} \tag{3.17}$$

where

$$\tilde{f}_{kt+1}(x_{t+2}|z_i, r_{kt+1}) = f_{kt+1}(x_{t+2}|x_i, \tilde{c}_{kt+1}(z_i, r_{kt+1})),$$

$\tilde{a}_{kjt+1}(x_i, r_{t+1}|z_t, c_t) = a_{kjt+1}(x_i, \tilde{c}_{kt+1}(z_i, r_{t+1})|z_t, c_t)$, and $\tilde{c}_{kt+1}(z_i, r_{kt+1})$ is the nearest $\tilde{c}$ to $c^0_{kt+1}(z_i, r_{t+1})$. One can then compute $a(x_t, c_t)$ for $c_t$ on the grid $\tilde{\mathcal{C}}_t$ to obtain weights outside of the main estimation. How well $\tilde{\kappa}_{jt+1}(x_{t+2}, |x_t, c_{jt})$ approximates $\kappa_{jt+1}(x_{t+2}, |x_t, c_{jt})$ depends on $m$, which I explored in section 8.

As in the previous section, estimates of $f_{jt}(x_{t+1}|x_t, c_{jt})$ are required to compute the weights that satisfy one-period dependence. This can be done by parametric or nonparametric methods, where in the latter case, a candidate estimator is given by

$$\hat{f}_{jt}(x|x_t, c_{jt}) = \frac{\sum_i^n 1\{x_{it+1} = x\} d_{jit} 1\{x_{it} = x_t\} K_\sigma(c_{jit} - c_{jt})}{\sum_i^n d_{jit} 1\{x_{it} = x_t\} K_\sigma(c_{jit} - c_{jt})},$$

where $K_\sigma$ is a kernel and $\sigma$ is a bandwidth.

The second concern is existence of measurement errors in observed continuous choice, which would lead to inconsistent estimates of $f_{jt}(x_{t+1}|x_t, c_{jt})$. However, consistent estimates of these transition probabilities can be obtained by the instrumental variables estimation methods such as in Newey and Powell (2003), where lagged values of the continuous choice may serve as the instruments under the conditions set out in Newey and Powell (2003).

### 3.1.4 Optimal continuous choice

This section presents sufficient conditions for uniqueness of the optimal CCCs, and discuss how they may be computed in the framework set out in the previous sections. The sufficient conditions for uniqueness of the optimal CCCs are outlined in the following assumption.

**Assumption 3.10.** *For $j = 1, \cdots, J$, $l_j = 1, \cdots, L_j$, and $(z_t, r_{jt})$,*
*(1) the period-specific utility function, $u_{jt}(z_t, c_{jt}, r_{jt})$, is strictly increasing, strictly concave, and twice continuously differentiable in $c_{l_j t}$ on int$(\mathcal{C}_{jt})$ with $\lim_{c_{l_j t} \to 0} \partial u_{jt}(z_t, c_{jt}, r_{jt}) / \partial c_{l_j t} = \infty$, and*
*(2) for all $(x_t, x_{t+1})$, $\sup_{c_{jt} \in \mathcal{C}_{jt}} \left| \partial^k f_{jt}(x_{t+1}|x_t, c_{jt}) / \partial c_{l_j t}^k \right| \leq \gamma < \infty$, $k = 0, 1$.*

Under Assumption 3.10, $c_{jt}^0(z_t, r_{jt})$ uniquely maximizes $v_{jt}(z_t, c_{jt}, r_{jt})$ over $\mathcal{C}_{jt}$, for $j = 1, \cdots, J$ and all $(z_t, r_{jt})$.

Under certain conditions, one-period finite dependence provides a simple and convenient representation of the condition for optimal CCCs. Specifically, let alternative $j = 1$ be the normalizing alternative and let $c_{1t}$ be either 0 or known. In the empirical application of this paper, the alternative $j = 1$ is to stay home, for which no continuous choice is associated.

Then, under Assumption 3.10, $c_{jt}^{0}(z_t, r_{jt})$ uniquely solves

$$
\begin{aligned}
0 &= \frac{\partial}{\partial c_{jt}}\left(v_{jt}(z_t, c_{jt}, r_{jt}) - v_{1t}(z_t, c_{1t}, r_{1t})\right) = \frac{\partial}{\partial c_{jt}} u_{jt}(z_t, c_{jt}, r_{jt}) \\
&+ \beta \sum_{k=1}^{J} \int \sum_{i=1}^{|x|} [u_k^0(z_i, r_{kt+1}) + \psi_k[p_{t+1}^0(z_i, r_{t+1})]] \\
&\times \frac{\partial}{\partial c_{jt}}[a_{kjt+1}(x_i, r_{t+1}|z_t, c_t) f_{jt}(x_i|x_t, c_{jt}) - a_{k1t+1}(x_i, r_{t+1}|z_t, c_t) f_{1t}(x_i|x_t, c_{1t})] g_r(r_{t+1}) dr_{t+1}
\end{aligned}
\tag{3.18}
$$

for $j = 2, \cdots, J$. This first-order condition becomes more complicated in the obvious way if the period-specific utilities are functions of lagged continuous choices.

# 4 Identification

This section discusses identification of the parameters of the model presented in section 2.1. The negative result of Rust (1994) and Magnac and Thesmar (2002) – that models of the form presented in 2.1 are generically non-identified – shows additional restrictions must be imposed on the structure of the model to identify the period-specific payoff functions. The class of models considered by Rust (1994) and Magnac and Thesmar (2002) are stationary and do not include continuous choices. Blevins (2014) provides sufficient conditions for identification of the period-specific payoff functions in stationary models with continuous choices and appropriate continuous state variables.

As discussed in Arcidiacono and Miller (2020), identification becomes more problematic in non-stationary environments. The model presented in section 2.1 is not only non-stationary, but also includes CCCs and permanent unobserved heterogeneity. Appendix A.3 provides sufficient conditions for nonparametric identification of the period-specific utility functions and the distribution of the permanent unobserved heterogeneity under the assumption the distribution takes a correlated finite-mixture form. The approach to identifying the parameters of interest taken in this paper imposes the restriction that level sets can be constructed from the difference between the transition function of a particular choice, $j^*$ and the others. In practice, such level sets can be achieved by index restrictions on the transition probabilities, or exclusion restrictions. The application in the paper imposes both restrictions where: $j^* = 3$ (enroll in school and not work), the probability of completing a grade level given enrollment is specified as a linear index probit, and this probability depends on labor market experience interacted with age and AFQT, which are excluded from period-specific utilities. Full development of the nonparemetric identification of the period-specific utility functions and the distribution of the permanent unobserved effects can be found in Appendix A.3.

# 5 Estimator

In this section, I propose a GMM estimator for the period-specific utility functions, as well as the distribution of the unobserved effects. I choose to propose a GMM estimator instead of the ML estimator for two reasons. First, the definition of the GMM estimator does not require specifying the distribution of measurement errors, which is of particular concern in discrete- and continuous-choice models, because observed continuous-choice variables are often measured with errors. Second, the GMM estimator is robust to the initial-conditions problem: consistent estimation of the parameters does not require observing the initialization of $x_t$ given $s$ or for it to be specified.

How to account for and estimate finite-mixture distributions in the GMM framework is unclear. Indeed, accounting for finite-mixture distributions typically requires the likelihood function to be fully specified, such as in Eckstein and Wolpin (1999), Keane and Wolpin (2000a), Keane and Wolpin (2001), and Arcidiacono and Miller (2011). The estimator in this section is designed to account for and estimate the finite-mixture distribution, and allow the distribution to be dependent on observed permanent characteristics of the individual. I assume consistent estimates of the state transition probabilities are obtainable in a first-stage estimation, as discussed in section 3.1.

The estimator proposed in this section assumes the period-specific utility functions, and the distribution of the shocks are known up to a finite-dimensional set of parameters. Specifically, $u_{jt}(z_t, s, c_{jt}, r_{jt}) = u_{jt}(z_t, s, c_{jt}, r_{jt}; B_1,), j = 2, \cdots, J$ are known up to $B_1 \in \Re^{D_{B_1}}$, $u_{jt}(z_t, s, c_{jt}, r_{jt})$ is known, and $g_r(r_t) = g_r(r_t; B_2)$ is known up to $B_2 \in \Re^{D_{B_2}}$. Define $B = (B_1, B_2) \in \mathcal{B} \subseteq \Re^{D_{B_1}+D_{B_2}}$. Identification of the model outlined in section A.3 of the appendix imposes a finite-mixture specification of the distribution of the permanent unobserved effects. This assumption is reproduced below for completeness.

**Assumption 5.1.** *1. For all $x_t \in \mathcal{X}$ and $w \in \mathcal{W}$, the conditional density $f_t(x_t|w) > 0$.*
*2. The permanent unobserved heterogeneity has finite support with probability mass function $\pi(s|x_t) = \pi(s|w)$, $s \in \mathcal{S}(w)$, and cardinality $Q(w)$, possibly depending on $w$.*

The estimator imposes the following additional restrictions on the conditional distribution of the mixing distribution and $w$.

**Assumption 5.2.** *1. $w$ is discrete valued with cardinality $|w|$.*
*2. The number of types and the support of the mixing distribution are not a function of $w$: $Q(w) = Q$; $s_q(w) = s_q,\ q = 1, \cdots, Q$.*
*3. The investigator has prior knowledge of the number of types, $Q$.*

Define $\pi(w_k) = (\pi_1(w_k), \cdots, \pi_Q(w_k))'$, $\pi = (\pi(w_k), k = 1, \cdots, |w|)$, $\theta_q = (B, s_q) \in \Theta_q := \mathcal{B} \times \mathcal{S}$, and $\theta = (B, \{s_1, \cdots, s_Q\}) \in \Theta := \mathcal{B} \times \mathcal{S}^Q$. For each $i$, and for $t = 1, \cdots, T$, define

$$
\begin{aligned}
\tilde{p}_t^0(x_{it};\theta_q) &= (p_{2t}^0(x_{it};\theta_q), \cdots, p_{Jt}^0(x_{it};\theta_q))', \\
\tilde{c}_t^0(x_{it};\theta_q) &= (d_{1it}c_{1t}^0(x_{it};\theta_q)', \cdots, d_{Jit}c_{Jt}^0(x_{it};\theta_q)')', \\
h_t(x_{it};\theta_q) &= (\tilde{p}_t^0(x_{it};\theta_q)', \tilde{c}_t^0(x_{it};\theta_q)')', \\
h(x_i;\theta_q) &= (h_1(x_{i1};\theta_q)', h_{T-1}(x_{iT-\rho};\theta_q)')', \\
h(x_i;\theta_q) &= (h(x_i;\theta_1), \cdots, h(x_i;\theta_Q)), \\
\rho(y_i, x_i;\theta,\pi) &= y_i - h(x_i;\theta)\pi(w_i),
\end{aligned}
\tag{5.1}
$$

where the vector $y_i$ is defined analogously.

The vector $\rho(y_i, x_i; \theta, \pi)$ is of $(L + J - 1)(T - \rho)$ dimension. Let $X_{it}$ be a vector of instruments with dimension $N_{X_t} \geq L + J - 1$, and define $X_i = diag\{X_{it}, t = 1, \cdots, T - 1\}$. Premultiplying equation (5.1) by $X_i$ obtains the following $N_X := \sum_{t=1}^{T-1} N_{X_t}$-dimensional vector:

$$
m_i(\theta,\pi) = X_i\rho(y_i, x_i;\theta,\pi). \tag{5.2}
$$

The GMM objective function is given by

$$\hat{S}(\theta;\pi) = \hat{m}(\theta;\pi)'\hat{\Omega}\hat{m}(\theta;\pi), \quad \text{where} \tag{5.3}$$

$$\hat{m}(\theta;\pi) = \frac{1}{n}\sum_{i=1}^{n} m_i(\theta;\pi), \tag{5.4}$$

and $\hat{\Omega}$ is consistent estimator for an $N_X \times N_X$ positive-definite weighting matrix, $\Omega$.

Although the objective function (5.3) is standard, additional restrictions need to be imposed to jointly estimate the type probabilities. For $w = w_k$, the likelihood of $d_{it}$ given $(x_{it}, \theta_q)$ implied by the model be given by

$$f(d_{it}|x_{it}, \theta_q) = \prod_{j=1}^{J} p_{jit}(x_{it}, \theta_q)^{d_{jit}}. \tag{5.5}$$

For $w_{it} = w_k$, for any value of $\theta$, and given type probabilities, $\pi(w_k)$, Bayes's rule implies the following vector of posterior type probabilities,

$$\pi_q(d_{it}, x_{it}; \theta, \pi(w_k)) = \frac{f(d_{it}|x_{it}, \theta_q)\pi_q(w_k)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'})\pi_{q'}(w_k)}, \quad q = 1, \cdots, Q. \tag{5.6}$$

The validity of the estimator is based on the following theorem, whose proof is provided in section A.7.

**Theorem 5.3.** *Suppose $(\theta, \pi)$ is identified; that is, the assumptions in Section A.3 hold. Then, $E[\pi_q(d_{it}, x_{it}; \theta, \pi(w_k))|x_{it}, w_k] = \pi_q(w_k)$ for $q = 1, \cdots, Q$, $k = 1, \cdots, |w|$, and almost every $x_{it} \in \mathcal{X}$ if, and only if $(\theta, \pi) = (\theta_0, \pi_0)$.*

For $k = 1, \cdots, |w|$, define

$$\boldsymbol{f}_{it}(\theta, \pi(w_k)) = diag\left\{\frac{f(d_{it}|x_{it}, \theta_q)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'})\pi_{q'}(w_k)}, q = 1, \cdots, Q\right\}, \tag{5.7}$$

$$m^{\pi}_{kit}(\theta, \pi(w_k)) = \pi(w_k) - \boldsymbol{f}_{it}(\theta, \pi(w_k))\pi(w_k),$$

$$\hat{m}^{\pi}_{k}(\theta, \pi(w_k)) = \frac{1}{n_k}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T} m^{\pi}_{kit}(\theta, \pi(w_k))I_i(w_k), \tag{5.8}$$

where $I_i(w)$ is the indicator variable equal to 1 if $w_i = w$, and 0 otherwise, and $n_k = \sum_{i=1}^n I_i(w_k)$. For fixed $\theta \in \Theta$, let $\hat{\pi}(w_k;\theta)$ be the solution to $\hat{m}_k^{\pi}(\theta, \hat{\pi}(w_k;\theta)) = 0$, and define $\hat{\pi}(\theta) = (\hat{\pi}(w_k;\theta), k = 1, \cdots, |w|)$. The estimator for $\theta_0$ is therefore defined as

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \hat{S}(\theta; \hat{\pi}(\theta)), \tag{5.9}$$

and the estimator for $\pi_0$ is given by $\hat{\pi}(\hat{\theta})$.

# 6 Computing the estimator

In this section, I present a method for computing the estimator proposed in the previous section. I describe updating the parameters of the model for individual $i$ in period $t$ at the $o+1$ iteration with $\left(\theta^{[o]}, \pi^{[o]}, \mathbf{c}^{[o]}, \mathbf{p}^{[o]}\right)$ in hand. In the development of the algorithm, I suppress dependence on $\left(\theta^{[o]}, \pi^{[o]}, \mathbf{c}^{[o]}, \mathbf{p}^{[o]}\right)$ and $(i,t)$, and I set the first alternative to be the normalizing alternative. Therefore, we have the notation

$$v_{jq}(x,c,r) = v_{jit}(x_{it}, c_{it}, r_{it}; \theta_q^{[o]}, \pi^{[o]}, \mathbf{c}^{[o]}, \mathbf{p}^{[o]}).$$

**Updating the CCCs**

The continuous choices, $c_{jq}^{[o+1]}(x, r_j)$ and $c_{jq}^{[o+1]}(x)$, are updated by solving equation (3.11) as follows:

$$\left.\frac{\partial}{\partial c_j}\right|_{c=c_q^{[o+1]}(x_t,r)} v_{j1q}(x,c,r) = 0, \tag{6.1}$$

$$c_{jq}^{[o+1]}(x) = \int c_{jq}^{[o+1]}(x, r_j) g_r(r_j) dr_j. \tag{6.2}$$

Conditions for uniqueness of the solution to equation (6.1) are given in Assumption 3.10, and the integration in equation (6.2) is taken numerically.

**Updating the CCPs**

Let

$$v(x, c_q^{[o+1]}(x,r), r) = \left(v_{1q}(x, c_{1q}^{[o+1]}(x,r_1), r), \cdots, v_{Jq}(x, c_{Jq}^{[o+1]}(x,r_J), r)\right)'.$$

For each $j \in \{1, \cdots, J\}$, the CCPs are updated as follows:

$$p_{jq}^{[o+1]}(x, c_q^{[o+1]}(x,r), r) = \Psi_j \left( v(x, c_q^{[o+1]}(x,r), r) \right), \tag{6.3}$$

$$p_{jq}^{[o+1]}(x) = \int p_{jq}^{[o+1]}(x, c_q^{[o+1]}(x,r), r) g_r(r) dr, \tag{6.4}$$

where the functional form of $\Psi_j$ is determined by the distribution of the alternative-specific shocks, $\varepsilon$. For example, if $\varepsilon$ is distributed i.i.d. type one extreme value,

$$p_{jq}^{[o+1]}(x, c_q^{[o+1]}(x,r), r) = \frac{e^{v_{jq}(x, c_{jq}^{[o+1]}(x, r_j), r)}}{\sum_{k=1}^{J} e^{v_{kq}(x, c_{kq}^{[o+1]}(x, r_k), r)}}.$$

**Updating $\pi$**

For $k = 1, \cdots, |w|$, equation (5.8) may be solved recursively by iterating

$$\pi_k^{[o'+1]} = \frac{1}{n_k} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} f_{it}^{[o+1]}(\theta^{[o]}, \pi_k^{[o']}) \pi_k^{[o']} I_i(w_k). \tag{6.5}$$

in $o'$ until convergence, where $\pi^{[o]}(w_k)$ is taken as the initial prior (see Richardson (1972) for an analysis of Bayesian-based iterative algorithms). The updated type probabilities $\pi^{[o+1]}(w_k)$ are these convergent values.

## 6.1 Updating $\theta$

Let $\hat{m}^{[o+1]}(\theta^{[o]}, \pi^{[o+1]}(\theta^{[o]}))$ be the moment function defined in equation (5.4) evaluated at $\left(\theta^{[o]}, \pi^{[o+1]}, \mathbf{c}^{[o+1]}, \mathbf{p}^{[o+1]}\right)$. Let $\hat{M}^{[o+1]}(\theta^{[o]}, \pi^{[o+1]}(\theta^{[o]}))$ and $\varphi^{[o+1]}(\theta^{[o]}, \pi^{[o+1]}(\theta^{[o]}))$ be

$$\hat{M}(\theta, \pi(\theta)) := \frac{\partial}{\partial \theta} \hat{m}(\theta, \pi(\theta)), \text{ and}$$

$$\varphi(\theta, \pi(\theta)) := -\left[\hat{M}(\theta, \pi(\theta))' \hat{\Omega} \hat{M}(\theta, \pi(\theta))\right]^{-1} \hat{M}(\theta, \pi(\theta))' \hat{\Omega} \hat{m}(\theta, \pi(\theta))$$

also evaluated at $\left(\theta^{[o]}, \pi^{[o+1]}, \mathbf{c}^{[o+1]}, \mathbf{p}^{[o+1]}\right)$. The updated values of $\theta$, $\theta^{[o+1]}$ are given by

$$\theta^{[o+1]} = \theta^{[o]} + \varphi^{[o+1]}(\theta^{[o]}, \pi^{[o+1]}(\theta^{[o]})). \tag{6.6}$$

The full algorithm for computing the estimates of the model is as follows.

*Algorithm*
1 - Initialize $\left(\theta^{[0]},\pi^{[0]},\mathbf{c}^{[0]},\mathbf{p}^{[0]}\right)$.
2 - For $o \geq 0$,
2.1 - Compute $\mathbf{c}^{[o+1]}$ by solving equation (6.2)
2.2 - Compute $\mathbf{p}^{[o+1]}$ using equation (6.4)
2.3 - Compute $\pi^{[o+1]}$ using equations (6.5)
2.4 - Compute $\theta^{[o+1]}$ using equations (6.6)
until convergence in $\theta$.

Convergence of the Gauss-Newton algorithm is not guaranteed for a variety of reasons, and if it does converge, it may be slow (see Dennis Jr. and Shanbel (1996) for discussion). Methods to improve the success and rate of convergence of the Gauss-Newton algorithm have been proposed in recent years (see Fan and Yuan (2005), Zhou and Chen (2010), and Ferreira, Goncalves, and Oliviera (2011)). Although I adopt components of these proposed algorithms in the simulation exercise and empirical application, a detailed discussion of these modifications is beyond the scope of the current paper.

What we do know, however, is that good initialization of the parameters of the model does improve the likelihood that the algorithm converges, and it reduces the number of iterations required to achieve convergence. Good initial values for the above algorithm may be obtained by first obtaining initial values of $\theta$, and type-invariant estimates of the CCCs and CCPs. These quantities can then be used to execute steps 2.1 - 2.4 of the algorithm to obtain initial type-specific CCCs and CCPs. With these initial values, the above algorithm iterates on the quantities in a way similar to Aguirregabiria and Mira (2002) and Arcidiacono and Miller (2011). Therefore, no loss of precision results from multistage estimation of the parameters of the model such as in Hotz and Miller (1993), Altug and Miller (1998), Bajari, Benkard, and Levin (2007) and other papers that apply such methods.

# 7 Limiting variance of the estimator

The estimator proposed in section 5 is $\sqrt{n}-$ consistent and asymptotically normal under standard conditions such as those in Newey and McFadden (1994). However, the form of the limiting variance is somewhat nonstandard because of the method for jointly estimating the type probabilities. This section provides the form of the limiting variance, whose derivation can be found in section A.8.

Define

$$m_{it}^{\pi}(\theta,\pi) = \pi - \boldsymbol{f}_{it}(\theta,\pi)\pi, \quad m_{i}^{\pi}(\theta,\pi) = \frac{1}{T}\sum_{t=1}^{T} m_{it}^{\pi}(\theta,\pi), \quad \text{and}$$
$$m_{0}^{\pi}(w;\theta,\pi) = E\left[m_{i}^{\pi}(\theta,\pi)|\, w\right],$$

where $\boldsymbol{f}_{it}(\theta,\pi)$ is defined in equation (5.7). Let $I$ and $\iota$ be the $q-$ dimensional identity matrix and iota vector, respectively. Define also $\boldsymbol{\pi} = diag\{\pi_1,\cdots,\pi_Q\}$, and let

$$M_{it}^{\pi}(\theta,\pi) = I - \boldsymbol{f}_{it}(\theta,\pi) + \boldsymbol{\pi}\boldsymbol{f}_{it}(\theta,\pi)\iota\iota'\boldsymbol{f}_{it}(\theta,\pi), \quad M_{i}^{\pi}(\theta,\pi) = \frac{1}{T}\sum_{t=1}^{T} M_{it}^{\pi}(\theta,\pi), \quad \text{and}$$
$$M_{0}^{\pi}(w;\theta,\pi) = E\left[M_{i}^{\pi}(\theta,\pi)|\, w\right].$$

Finally, let $m_i = m_i(\theta_0,\pi_0)$, $M_i(\theta,\pi) = \partial m_i(\theta,\pi)/\partial\theta$, and $M_0 = E[M_i(\theta_0,\pi_0)]$, $M_{\pi i}(\theta) = X_i h(x_i,\theta)$, $M_{\pi 0}(w) = E[M_{\pi i}(\theta_0)|w]$, $m_0^{\pi}(w) = m_0^{\pi}(w;\theta_0,\pi_0)$ and $M_0^{\pi}(w) = M_0^{\pi}(w;\theta_0,\pi_0)$. The limiting variance of the estimator, $\hat{\theta}$, is

$$\begin{aligned} V &= (M_0'\Omega M_0)^{-1}(M_0'\Omega\Sigma\Omega M_0)(M_0'\Omega M_0)^{-1}, \quad \text{where} \\ \Sigma &= E\left[\left(m_i + M_{\pi 0}(w_i)M_0^{\pi}(w_i)^{-1}m_i^{\pi}\right)'\left(m_i + M_{\pi 0}(w_i)M_0^{\pi}(w_i)^{-1}m_i^{\pi}\right)\right]. \end{aligned} \tag{7.1}$$

In practice, a consistent estimator for the asymptotic variance $V$ is required. One can be obtained via the plug-in approach, where the parameters in $V$ are replaced with their respective estimators, and the expectations are replaced with sample averages. The proof for consistency of this plug-in estimator is standard and can also be found in Newey and McFadden (1994).

# 8 Monte Carlo evidence

In this section, I present the results of two sets of Monte Carlo exercises to illustrate the finite-sample performance of the proposed estimator. Details of these exercises can be found in section A.11.

The first set of exercises investigate the performance of the approximation to finite dependence presented in section 3.1.3 under the restrictions on the transition probabilities imposed in Assumption 3.8. The performance of the estimator is compared with the estimator for which the CCPS and CCCs are obtained from the full solution and one for which the true CCPs and CCCs are known, in order to isolate the source of approximation error. The results show the proposed estimator performs well in finite samples with some loss of precision relative to the estimator based on solving the model. However, the results also show significant computational gains from the proposed estimator relative to the full-solution approach, with average computation time of the proposed estimator being four and a half times faster than the full-solution approach.

The second set of exercises investigate the performance of the estimator relative to the one based on the EM algorithm developed in Arcidiacono and Miller (2011) for models that satisfy the renewal property discussed in section 3.1. The proposed estimator performs significantly better in recovering the parameters of the model. However, the Arcidiacono Miller approach is significantly faster – being almost 19 times faster than the proposed method. These gains are largely due to the difference in the approaches to updating the type probabilities, and the proposed estimator is based on two-stage GMM.

# 9 Education and labor market choices and the heterogenous returns to education

This section implements the method developed in the previous sections to investigate life-cycle educational and labor market choices, using a sample of young men from the 1979 cohort of the NLSY. The primary objective is to estimate the long-run ex-ante distribution of returns to education. The model developed in this section is closely related to those developed in Keane and Wolpin (1997; 2000; 2001), and Eckstein and Wolpin (1999) in a variety of ways. Specifically, the model incorporates psychic costs of schooling and working, as well as direct and indirect benefits from working. The direct benefit from working is current income earned, and the indirect benefit is future potential income earned from the additional years of experience gained. To allow for simultaneity in these choices, I adopt the specification of Eckstein and Wolpin (1999) and Keane and Wolpin (2001), which includes the option to simultaneously work and attend school in the individual's choice set, for the number of hours worked while attending school to influence the chance the individual will advance the grade level, and incorporate psychic costs of working while in school. Additionally, similar to these papers, I allow for permanent unobserved heterogeneity and individual-time–specific shocks in the wage-offer function. A key point of departure is the permanent unobserved heterogeneity is specified as a random coefficient on the level of education, whose distribution is allowed to vary by race, parental income, and AFQT categories.

The inadequacy of the classical Mincer equation to obtain policy-relevant estimates of the returns to education has been well documented over the last two decades (see Heckman, Lochner, and Todd (2006) for a review of the relevant literature). Heckman, Lochner, and Todd (2008) cite notable extensions to the classical Mincer wage equation that are likely to reduce the biases in estimates of returns to education. These extensions include direct and psychic costs of schooling, nonseparability between experience and schooling, heterogeneity in returns to education, and disentangling marginal and average returns to schooling. Indeed, the authors show that if psychic costs of schooling are significant and ignored, the Mincer coefficient is expected to be larger than the internal rate of return. Other important factors that may affect estimates of the returns to education include the endogeneity of schooling and work-experience choices and uncertainty about the completed level of education. The structural model in this section is designed to incorporate these extensions.

The model allows for race, parental income, and AFQT to affect the grade transition probability, the utilities of leisure, the psychic costs of school enrollment and employment, and the wage-offer function. In investigating the determinants of education attainment, Cameron and Heckman (1998), Cameron and Heckman (2001), and Carneiro and Heckman (2003) argue the effect of parental income on educational attainment is primarily a result of it being a proxy for permanent income, which influences early childhood development, and less that it captures short-term credit constraints. Richer parents have more resources to invest in the environment that fosters early childhood development of cognitive abilities and social skill, which rewarded in terms of academic achievement. They argue a significant proportion of the racial disparity in academic achievement can largely be traced to differences in schooling abilities developed during early childhood. An implication of these arguments is that controlling for family income and AFQT should mitigate the effect of race in the structural model, and controlling for AFQT should mitigate the effect of parental income on the barriers to school enrollment, as well as the likelihood that a student will advance a grade level given enrollment. Interpretation of any residual effect of parental income on academic achievement, net of AFQT, becomes more convoluted. One source of any such an effect may be a result of permanent income on scholastic aptitude developed during early childhood in dimensions not captured in AFQT. Another channel could be that parental income, as a measure of permanent income, may be capturing access to better schools and academic assistance outside of normal school hours. These potential sources of the effect of parental income cannot be investigated with the data implemented in this paper but remain within the research agenda.

Performing the same analysis as done in Heckman, Lochner, and Todd (2008), where the psychic cost of working is included, shows this cost also drives a wedge between the Mincer coefficient and the internal rate of return to education. Although the resulting size of the Mincer coefficient relative to the internal rate of return is ambiguous, accounting for the existence of psychic costs of working is still necessary in order to obtain policy-relevant estimates of the returns to education using the Mincer specification of log-wages. Another important layer of complication comes from the psychic cost of working while attending school, because working while in school breaks the sequential framework of school to work assumed in the model analyzed by Heckman, Lochner, and Todd (2008).

The joint effect of race and AFQT on labor market success has been an extensively investigated phenomenon, with notable studies including Neal and Johnson (1996) and Keane and Wolpin (2000a). These authors find that accounting for AFQT substantially reduces, and in some cases eliminates, the racial gap in wages, a phenomenon that has become the general consensus in the literature. Models that find this result do not account for racial disparities in wage offers that may be education-specific. The model presented in this section incorporates both the direct and indirect effects of race on wages, where the direct effect is modelled as an indicator for whether the individual is black, and the indirect effect is modelled as black-white variation in the distribution of returns to education.

A variety of interpretations of the effects of parental income on labor market outcomes, if they exist, are plausible. One potential explanation of these effects is consistent with Heckman, Lochner, and Todd (2006), in that parental income is a proxy for permanent income, which influences early childhood development resulting in not only higher schooling abilities, but also higher market abilities. Another explanation is that individuals from richer families have greater access to the labor market in terms of employment opportunities and higher-paying job offers. The model developed in this section has the potential to shed some light on the more plausible interpretation of the two. Assuming AFQT also measures individual market ability, if the effect of parental income on the psychic costs labor force participation and wage offers diminishes substantially with the inclusion AFQT, parental income is more likely predominantly be a measure of market ability. On the other hand, if the size of this effect remains relatively unchanged with the inclusion of AFQT, parental income is more likely to capture the greater access to employment opportunities and high-paying jobs for individuals with rich parents.

## 9.1 The theoretical model

In each period, $t$, the individual is endowed with a fixed amount of time, which is normalized to 1. He faces four mutually exclusive and collectively exhaustive alternatives, $j$: to stay home ($j = 1$); to not attend school and work ($j = 2$); to not work and attend school ($j = 3$); and to both work and attend school ($j = 4$). Let $d_{jt}$ be equal to 1 if the individual chooses alternative $j$ in period $t$, and 0 otherwise. If the individual chooses to work in period $t$, he must decide how to allocate his time endowment between leisure, $l_t$, and labor supply, $h_t$, so that $l_t + h_t = 1$. Define $d_t^h$ to be equal to 1 if the individual chooses to work in period $t$, and 0 otherwise.

If the individual decides to work in period $t$, he gains an additional year of experience. If he decides to enroll in school, he advances the grade level with probability $F_t(h_t, x_t^a; \theta^a)$, where the variables composing $x_t^a$ include parental income (Par Inc) as at 1979 in \$10,000 1983 dollars, hours worked in the academic year (Hrs Worked), level of completed education (Edu), the indicator for whether the individual is a high school graduate (HS Grad), the indicator for whether he obtained some college education (Some Col); the indicator for whether he is a college graduate (Col Grad), and years of labor market experience (Exper). This specification is a key (though not only) source of uncertainty affecting completed level of education. The individual compares the benefits from working while in school, which include income generated and the level of labor market experience earned, to the costs, which include loss of leisure time and the potentially negative impact of working while in school on the likelihood that he will complete the grade level.

In each period, the individual receives a wage offer, $wage_t^o$, which is parameterized as follows:

$$\ln(wage_t^0) = x_t^w \theta^w + sEDU_t + r_t,$$

where $EDU_t$ is his level of education as $t$, $s$ is returns to his level of education, and $r_t$ is the period-specific shock to his wage offer, which is assumed to be distributed i.i.d. $N(0, \sigma_r)$. The explanatory variable, $x_t^w$, contains years of experience and its squared value, a dummy variable for the individual being black, parental income and its interaction with the black dummy variable, and AFQT.

I assume returns to education, $s$, are discretely distributed with $Q = 3$ support points, $s \in \{s_1, \cdots, s_Q\}$ and corresponding PMFs $\pi(w) = (\pi_1(w), \cdots, \pi_Q(w))$, where $w$ represents the eight categories of race (black and white), parental income (above and below median income), and AFQT (above and below median AFQT). I allow for the log of the wage offer to be measured with additive error, which is assumed to have zero mean with distribution independent of all the covariates in the model.

The contemporaneous utility function is given by

$$\begin{aligned} u_t(z_t) = {} & d_t^h \theta_1^u wage_t^o \ln(wage_t^o h_t) + \exp(x_t \theta_2^u) \ln(l_t) \\ & + d_t^h x_t \theta_3^u + d_t^E x_t \theta_4^u + d_t^h d_t^E x_t \theta_5^u + d_t' \varepsilon_t, \end{aligned} \tag{9.1}$$

where $d_t^E$ is equal to 1 if the individual enrolls in school in period $t$, and 0 otherwise, $x_t$ is a vector of observed demographic characteristics, $d_t = (d_{1t}, \cdots, d_{Jt})'$, and $\varepsilon_t = (\varepsilon_{1t}, \cdots, \varepsilon_{Jt})'$, where $\varepsilon_{jt}$ is the alternative-$j$-specific shock to utility, which is distributed i.i.d., type 1 extreme value. This specification of the contemporaneous utility imposes some desired restrictions on the preference for consumption, and leisure, as well as optimal hours worked.

Specifically, for $\theta_1^u > 0$, the contemporaneous utility is increasing and concave in both consumption and leisure, and optimal hours are restricted to lie strictly between 0 and 1. I impose these restrictions by setting $\theta_1^u = 0.1$.

The utility-of-leisure taste shifter includes the black dummy variable, parental income and its interaction with the black dummy variable, AFQT, age, and age squared. The schooling and employment taste shifters include the black dummy variable, parental income and its interaction with the black dummy variable, and AFQT, education, years of experience, and age. Unlike Keane and Wolpin (1997), Eckstein and Wolpin (1999), and Keane and Wolpin (2000a), I do no allow for permanent unobserved heterogeneity in the utility taste shifters, because how the monotonicity restriction required for identification in section A.3 would be maintained in the current framework is unclear. By allowing for permanent unobserved heterogeneity in the psychic cost of schooling and the constant in the log wage-offer function, Keane and Wolpin (2000a) remain agnostic about the source of heterogeneity in endowments, instead of prior commitment on the source of potential endowment effects as in the current model. The schooling and employment taste shifters also include lagged values of school enrollment and employment to capture intertemporal nonseparabilities in preferences.

## 9.2 Data

The data are taken from the NLSY79, a comprehensive panel data set that follows individuals who were 14 to 21 years of age as of January 1, 1979. The data set initially consisted of 12,686 individuals: a representative sample of 6,111 individuals, a supplemental sample of 5,295 Hispanics, non-Hispanic blacks, and economically disadvantaged, non-black, non-Hispanics, and a supplemental sample of 1,280 military youth. Interviews were conducted on an annual basis through 1994, after which the survey adopted a biennial interview schedule. This study makes use of the individuals observed for the first 28 years of interviews, from 1979 to 2006. The data are restricted to include non-Hispanic males and respondents with missing observations on the highest grade level completed that cannot be recovered with high confidence from other data information. The details on the sample restrictions are provided in Appendix A of Gayle (2006). I further restrict the data to individuals who were at most 16 years of age when they were first interviewed in 1979. With these restrictions, the data used in this application consist of 1,443 individuals in 1979.

## 9.3 Estimation

This section discusses calculation of the weights used to achieved one-period finite dependence for estimation of the model, which are outlined in section 3.1.

I set $a_{12}(x_{t+1}) = a_{21}(x_{t+1}) = 1$ when evaluating the difference of the values of working and not enrolling in school ($j = 2$) and staying home ($j = 1$), because the theoretical model satisfies the exchangeability condition. A key restriction imposed on the theoretical model for exchangeability to be satisfied is that the wage-offer function depends on years of experience, and not on accumulated hours worked as in Eckstein and Wolpin (1999). If the wage offer is specified to depend on accumulated hours, exchangeability can be achieved if it was assumed to depend discretely on accumulated hours.

The probability of advancing by a grade level given enrollment in school, $F_t(h_t, x_t^a; \theta^a)$, is specified to depend on the distribution of returns to education only through optimal hours worked, and I assume observed hours worked is not measured with error, so this probability can therefore be estimated by standard means. I assume the probability of advancing a grade level takes the logit form. I compute the weights that achieve one-period finite dependence when evaluating the difference in the values of enrolling in school and working ($j = 3$, $h_t = 0$), and staying home ($j = 1$, $h_t = 0$) using equation (3.11). To compute the weights that achieve one-period finite dependence when evaluating the difference in the values of working while attending school ($j = 4$, $h_t > 0$) and staying home ($j = 1$, $h_t = 0$), I implement the approximation approach developed in section 3.1. Specifically, I approximated the weights on a grid of 20 equally spaced grid points of hours worked ranging from 0 to 0.7. This range includes the maximum hours worked while enrolled in school by individuals in the sample of 0.53.

## 9.4 Results

### 9.4.1 Grade-promotion probability

Table 1 presents the results from estimation of the probability of completing a grade level given enrollment, which is assumed to take the logit form. Column (1) contains estimates of this transition probability without controlling for AFQT, and column (2) includes AFQT and its interaction with labor market experience. The results show that when AFQT is not accounted for, the coefficient on black is negative and significant, and the coefficient on parental income is positive and significant. Accounting for AFQT eliminates the effect of black on the grade-promotion probability and reduces the marginal effect of parental income by 58%, which becomes imprecisely estimated.

The results indicate that, given employment, additional hours worked reduces the probability of advancing to the next grade level, providing evidence for the crowding-out hypothesis, which is consistent with D'Amico (1984), Ehrenberg and Sherman (1987), and Eckstein and Wolpin (1999). Although the coefficient on labor force participation $d_t^h$, which captures the extensive margin of employment, is positive across specifications, it becomes imprecisely estimated after controlling for AFQT. This finding suggests a significant proportion of what is considered evidence for the congruence hypothesis can be explained by individuals with high scholastic ability taking advantage of their greater chance of advancing a grade level, regardless of whether they work while attending school, to earn income and gain labor market experience.

Table 1: Probability of Grade Promotion

| Specification | I | | II | |
|---|---|---|---|---|
| Variable | Estimate | Std. Err. | Estimate | Std. Err. |
| Constant | 4.1345 | 0.7981 | 4.7323 | 0.8220 |
| Black | -0.4012 | 0.1214 | 0.0271 | 0.1242 |
| Par Inc | 0.1224 | 0.0282 | 0.0516 | 0.0291 |
| Black × Par Inc | -0.0506 | 0.0551 | -0.0320 | 0.0555 |
| Hrs Worked | -3.4947 | 0.7517 | -3.1941 | 0.7612 |
| Employment | 0.2124 | 0.1035 | 0.1679 | 0.1049 |
| Edu | 0.9162 | 0.1418 | 0.4704 | 0.1466 |
| Edu Squared | -0.0273 | 0.0058 | -0.0144 | 0.0061 |
| HS Grad - Some Col | -0.3658 | 0.1481 | -0.7687 | 0.1539 |
| Col Grad | -0.6090 | 0.3155 | -0.9662 | 0.3192 |
| Exper | -0.4730 | 0.1198 | -0.3150 | 0.1205 |
| Age | -0.5038 | 0.0439 | -0.4099 | 0.0440 |
| Age × Exper | 0.0250 | 0.0048 | 0.0228 | 0.0049 |
| AFQT | —— | —— | 0.0326 | 0.0026 |
| AFQT × Exper | —— | —— | -0.0030 | 0.0005 |

### 9.4.2 Period-specific utility

**Utility of leisure.** The first panel of Table 2 presents the estimates of the parameters governing the period-specific utility of leisure. The results indicate the utility of leisure is increasing in both parental income and AFQT, and the effect of parental income on the utility of leisure is lower for blacks. I find no evidence of a direct effect being black on the utility of leisure. These results remain unchanged when AFQT is included in the utility-of-leisure taste shifter. The results also show that utility of leisure is decreasing and convex in age.

**Psychic value of labor force participation.** The results presented in the second panel of Table 2 imply significant psychic costs of working. The coefficient on black is statistically insignificant across both specifications, and the coefficient on the interaction of black and parental income becomes statistically insignificant when AFQT is included, suggesting racial variation in the barriers to employment can largely be explained by variation in market ability measured by AFQT. The coefficient on parental income is positive and significant across specifications, and the inclusion of AFQT does not reduce its magnitude, which supports the hypothesis that individuals from richer families have greater access to the labor market in terms of employment opportunities. The coefficient on AFQT is positive and significant, implying AFQT also measures market ability developed in early childhood. The results also imply significant nonpecuniary benefits to continuous employment.

**Psychic value of school enrollment.** The results in the third panel of Table 2 provide evidence of the preference for continuous schooling, and the psychic cost of school attendance is increasing in labor market experience and age. These results are consistent with those of Eckstein and Wolpin (1999), who provide a detailed discussion of the economic interpretation of these coefficients.

The results imply psychic costs of school attendance are decreasing in parental income and AFQT. I find no evidence from either specification that psychic costs of school enrollment vary by race. Furthermore, including AFQT reduces the magnitude of the (positive) effect of family income by 59%. The conclusion implied from these results and those from the estimated grade transition probability function, which is consistent with Cameron and Heckman (2001) and Carneiro and Heckman (2003) is that, given AFQT, educational attainment varies by parental income but not by race, and individuals with higher parental income are more likely to attain higher education. These conclusions come with an important caveat, which is discussed in the following.

**Psychic value of working while attending school.** The conclusions drawn so far from the results pertain only to individuals who exclusively enroll in school or exclusively participate in the labor market, and not for individuals who work while enrolled in school. The fourth panel of Table 2 provides the estimates of the psychic value of working while attending school. They show significant psychic costs of working while enrolled in school for blacks. The results also suggest white individuals with high parental income are less likely to work while enrolled in school, a result that is consistent with the existence of borrowing constraints found in Keane and Wolpin (2001). These results hold for both specifications and are not diminished by the inclusion of AFQT, which itself increases the psychic cost of working while in school. These results suggest significant differences in the demographic characteristics of individuals who work while in school relative to individuals who do not, and not accounting for these differences is likely to bias estimates of the distribution of returns to education.

Table 2: Period-specific Utility

| Specification | 1 | | 2 | |
|---|---|---|---|---|
| Variable | Estimate | Std. Err. | Estimate | Std. Err. |
| **Utility of leisure** $\exp(\theta_2^u x_t)\ln(1-h_t)$ | | | | |
| Constant | 1.3964 | 0.1070 | 2.7533 | 0.0934 |
| Black | -0.0476 | 0.0358 | 0.0609 | 0.0284 |
| Par Inc | 0.0525 | 0.0059 | 0.0424 | 0.0038 |
| Black × Par Inc | -0.0250 | 0.0161 | -0.0203 | 0.0135 |
| AFQT | —— | —— | 0.0038 | 0.0002 |
| Age | -0.0511 | 0.0076 | -0.1379 | 0.0067 |
| Age Squared | 0.0010 | 0.0001 | 0.0023 | 0.0001 |
| **Psychic value of labor force participation** $d_t^h x_t \theta_3^u$ | | | | |
| Constant | 0.1695 | 0.3980 | 0.3937 | 0.7350 |
| Black | -0.1141 | 0.3674 | 0.3608 | 0.5101 |
| Par Inc | 0.3936 | 0.0825 | 0.4348 | 0.0895 |
| Black × Par Inc | -0.3510 | 0.1681 | -0.2863 | 0.2253 |
| AFQT | —— | —— | 0.0581 | 0.0061 |
| Lagged Enrollment | 0.5903 | 0.0235 | 1.2012 | 0.0511 |
| Edu | 0.1720 | 0.0316 | 0.1324 | 0.0799 |
| Exper | 0.1664 | 0.0177 | 0.1816 | 0.0298 |
| Age | 0.0428 | 0.0176 | 0.0433 | 0.0308 |
| **Psychic value of school enrollment** $d_t^E x_t \theta_4^u$ | | | | |
| Constant | 0.4910 | 1.1258 | 1.6259 | 1.2196 |
| Black | -0.7636 | 0.5692 | -0.2700 | 0.6620 |
| Par Inc | 0.7629 | 0.0843 | 0.3094 | 0.0907 |
| Black × Par Inc | -0.1243 | 0.2262 | 0.0717 | 0.2563 |
| AFQT | —— | —— | 0.0951 | 0.0057 |
| Lagged Enrollment | 4.6888 | 0.2632 | 5.8061 | 0.3596 |
| Edu | 0.1603 | 0.0549 | 0.0981 | 0.0914 |
| Exper | -0.1216 | 0.0749 | -0.3431 | 0.0790 |
| Age | -0.1458 | 0.0685 | -0.2764 | 0.0728 |
| **Psychic value of working while attending school** $d_t^h d_t^E x_t \theta_5^u$ | | | | |
| Constant | -2.3564 | 0.2820 | -2.8900 | 0.3025 |
| Black | -1.5807 | 0.5531 | -1.9543 | 0.6532 |
| Par Inc | -0.5808 | 0.0986 | -0.5667 | 0.1014 |
| Black × Par Inc | 0.5440 | 0.2190 | 0.5082 | 0.2725 |
| AFQT | —— | —— | -0.0632 | 0.0060 |

## 9.5 Wage-offer equation

Table 3 presents the results from estimation of the wage-offer function. AFQT is included in both specifications in order to be consistent with the existing literature. I allow for the wage-offer function to be directly dependent on race and parental income, and for the distribution of returns to eduction to vary by racial and parental-income categories. In addition to racial and parental-income categories, Specification 2 allows for the distribution of returns to education to vary with AFQT categories.

The results from both specifications suggest the existence of a black-white gap in the level of log wages. The results also indicate wages are increasing in parental income, with higher increases for blacks. Therefore, the black-white gap in the level of log wages narrows as parental income increases, and is eliminated or reversed for parental income 37% above the median income in specification 1 and 65% above the median income in specification 2. Allowing for the distribution of returns to education to depend on AFQT results in a negative and statistically significant coefficient on AFQT in the wage-offer equation, indicating an overcorrection of the direct effect of AFQT on wage offers. These results suggest parental income, and not AFQT, explains the male black-white wage gap, which supports the hypothesis that individuals with richer parents have greater access to higher-paying jobs. These results are not immediately comparable to those of Neal and Johnson (1996) and Keane and Wolpin (2000a), who find the inclusion of AFQT in the wage-offer equation significantly reduces the black-white racial wage gap, because of the inclusion of parental income in the log-wage equation, as well as the correlated random-effects specification of the returns to education.

The ranges of the supports of returns to education are estimated to be from 0.062 to 0.10 when AFQT is not included and from 0.060 to 0.09 when AFQT is included, and the Wald test rejects the null of equality in support points at the 5% level of significance in both cases. These ranges of returns to education lie within the range of the estimates from other studies that implement OLS and IV methods and data from similar time periods (see Card, 1999, for a review of these studies). It includes the estimates of returns to education for white-collar workers in Keane and Wolpin (2000a), but lies at the lower end of estimates produced in Heckman, Lochner, and Todd (2008).

Table 3: Wage-Offer Equation

| Specification | 1 | | 2 | |
|---|---|---|---|---|
| Variable | Estimate | Std. Err. | Estimate | Std. Err. |
| Constant | 0.2888 | 0.0433 | 0.3143 | 0.0373 |
| $t/T$ | -0.9442 | 0.2677 | -0.8529 | 0.2209 |
| $(t/T)^2$ | -1.0522 | 0.3483 | -1.5486 | 0.3044 |
| $(t/T)^3$ | 1.6782 | 0.1947 | 2.0535 | 0.1822 |
| Exper | 0.1203 | 0.0078 | 0.1281 | 0.0022 |
| Exper Squared | -0.0031 | 0.0001 | -0.0031 | 0.0001 |
| Black | -0.1665 | 0.0280 | -0.1464 | 0.0277 |
| Par Inc | 0.0096 | 0.0040 | 0.0226 | 0.0037 |
| Black $\times$ Par Inc | 0.0678 | 0.0105 | 0.0325 | 0.0106 |
| AFQT | 0.0004 | 0.0003 | -0.0013 | 0.0003 |
| Wage Shock Std. Dev. | 0.6880 | 0.0192 | 0.9874 | 0.0238 |
| $s_1$ | 0.0622 | 0.0059 | 0.0602 | 0.0053 |
| $s_2$ | 0.0851 | 0.0061 | 0.0735 | 0.0054 |
| $s_3$ | 0.1000 | 0.0057 | 0.0906 | 0.0047 |

### 9.5.1 Returns to education

Table 4 reports the estimates of the distribution of the returns to education by racial, parental-income, and ability groups. The first panel reports the distribution when AFQT is not accounted for, and the second panel reports the distribution when it is allowed to vary with AFQT.

When AFQT is not accounted for, the distribution of returns to education for whites first-order stochastically dominates the distribution for blacks across income categories, and the distribution for high-parental-income individuals dominates the distribution for low-parental-income individuals across race. The probability that a white individual with high parental income receives high returns to education is 0.57, which is 21 percentage points higher than the probability that a white individual with low parental income receives high returns to education. The probability that a white individual with low parental income receives high returns to education is 5 percentage points higher than the probability that a black individual with high parental income receives high returns to education, which is in turn 7 percentage points higher than the probability that a black individual with low parental income receives high returns to education.

When AFQT is accounted for, the distribution of returns to education for individuals with high AFQT strongly first-order stochastically dominates the distribution for individuals with low AFQT within all race and parental-income categories. Also, when AFQT is accounted for, the racial difference in the distribution of returns to education reduces significantly within AFQT categories. Interestingly, this difference reverses in sign for individuals of low AFQT with high parental income; that is, the probability that a black individual with low AFQT and high parental income receives high returns to education is 4 percentage points higher than a white individual from the same AFQT and parental-income category. These results support the hypothesis that racial differences in returns to education are largely explained by differences in skill endowments developed during early childhood. The conclusion drawn from these and the above results is that, net of differences in skill endowments as measured by AFQT, no economically significant racial variation exists in the barriers to school enrollment, the likelihood of completing a grade level given enrollment, nor in the distribution of returns to education. However, a significant gap in wage offers exist, which narrows as parental income increases.

The results show parental income remains a significant determinant of the distribution of returns to education when AFQT is accounted for. The distribution of returns to education for individuals with high parental income stochastically dominates the distribution for individuals with low parental income for all AFQT and race categories, though marginally so for blacks with high AFQT. However, AFQT accounts for approximately 31% of the parental-income gap in the probability that a white male receives high returns to education, and the residual gap is approximately the same across AFQT categories. On the other hand, black males with low AFQT account for almost all of the parental-income–gap in the probability that a black male receives high returns to education, which supports the hypothesis that individuals with rich parents have a significant advantage in the labor market with respect to the returns they receive from an additional year of schooling. Combined with the above results, the conclusion drawn is that individuals with rich parents possess significant advantages in the labor market in terms of lower barriers to employment, higher wage offers independent of level of education, and higher returns to education. These labor market advantages of individuals from rich parents are independent of the skill endowments developed by during early childhood, except for the returns to education for blacks, for whom the gap in returns to education exists for the low-AFQT subgroup. An important caveat to these conclusions concerns the group of individuals who work while attending school, for whom the results indicate the psychic cost of working while in school is higher for blacks, individuals with high parental income, and individuals with high AFQT.

Table 4: Distribution of Returns to Education

| | | | | | |
|---|---|---|---|---|---|
| | | | Specification 1 | | |
| Support | 0.0622 | 0.0851 | 0.1000 | Mean | Std. Dev. |
| White | | | | | |
| Low Parental Income | 0.4555 | 0.1881 | 0.3564 | 0.0800 | 0.0171 |
| High Parental Income | 0.3071 | 0.1276 | 0.5653 | 0.0865 | 0.0169 |
| Black | | | | | |
| Low Parental Income | 0.5547 | 0.2117 | 0.2336 | 0.0759 | 0.0161 |
| High Parental Income | 0.5146 | 0.1797 | 0.3058 | 0.0778 | 0.0169 |
| | | | Specification 2 | | |
| Support | 0.0602 | 0.0735 | 0.0906 | Mean | Std. Dev. |
| | | | Low AFQT | | |
| White | | | | | |
| Low Parental Income | 0.4687 | 0.2808 | 0.2505 | 0.0716 | 0.0123 |
| High Parental Income | 0.4468 | 0.2325 | 0.3206 | 0.0731 | 0.0131 |
| Black | | | | | |
| Low Parental Income | 0.4925 | 0.2435 | 0.2639 | 0.0715 | 0.0127 |
| High Parental Income | 0.4319 | 0.2061 | 0.3620 | 0.0740 | 0.0135 |
| | | | High AFQT | | |
| White | | | | | |
| Low Parental Income | 0.1841 | 0.0865 | 0.7294 | 0.0836 | 0.0121 |
| High Parental Income | 0.1421 | 0.0662 | 0.7917 | 0.0852 | 0.0110 |
| Black | | | | | |
| Low Parental Income | 0.1815 | 0.0573 | 0.7612 | 0.0841 | 0.0119 |
| High Parental Income | 0.1705 | 0.0508 | 0.7787 | 0.0846 | 0.0117 |

# 10 Conclusion

CCP estimation of dynamic structural models has flourished over the last two decades, largely because of the potential for a quantifiable reduction in computational costs relative to the full-solution approach. Building on Hotz and Miller (1993), Arcidiacono and Miller (2011) show the expected value of future utilities from optimal decision-making can be expressed as functions of the flow payoffs and CCPs for any sequence of future choices. Any future choice sequence chosen for a given initial choice generates a corresponding sequence of distributions of states. The term $\rho$-period finite dependence is obtained if two distinct initial choices with two corresponding future choice sequences can be constructed so that their respective distributions of states are the same after $\rho$-periods in the future. The computational advantage of the CCP approach depends significantly on $\rho$, particulary if the CCPs are updated as recommended by Aguirregabiria and Mira (2002). Specifically, the computational cost reduces, and in many cases, the accuracy of the estimator improves with smaller $\rho$. Although Arcidiacono and Miller (2011) represents a significant advancement relative to Hotz and Miller (1993), application of the CCP method often still requires strong assumptions about the choice sequence or the state transition probabilities or both. The generalization of the concept of $\rho$-period finite dependence proposed is the paper overcomes these issues. Specifically, this extension obtains one-period dependence ($\rho = 1$) in a large class of dynamic structural models, which includes most models that have been estimated using the CCP method, and it imposes minimal restrictions on the state transition probabilities, which widens the scope of models for which the CCP method is computationally beneficial.

The class of models I consider includes continuous choices that are associated with the discrete choices, where the shocks associated with the continuous choices are observed by the agent simultaneously with or before the discrete-choice-specific shocks. This specification allows for selection on unobservables, which is in contrast to the results of Blevins (2014). With respect to identification of the parameters of the model, however, allowing for this form of selection on unobservables comes at the cost of restricting the distribution of the continuous-choices–specific shocks to being known up to a finite-dimensional set of parameters, a restriction that is unnecessary in Blevins (2014). The identification strategy of this model imposes three key restrictions. I assume the state variables can be varied in a way that changes the difference in the period-specific utility functions, but leaves the difference in the transition probabilities unchanged. This restriction can be attained by index restrictions on the transition probabilities or by the existence of state variables that drives the difference in the transition probabilities to 1, 0, or -1. The second restriction is that the differences in the period-specific utility functions is strictly monotonic in the permanent unobserved heterogeneity. The third is the support of the state variables is wide enough that any two distinct period-specific utility functions evaluated at any two distinct values of the unobserved effect cross each other. The Monte Carlo exercise performed in this paper shows the proposed estimator performs well in recovering the period-specific utility functions and the distribution of the permanent unobservables.

I apply the methods developed in this paper to estimate a model of educational attainment and labor supply to investigate properties of the distribution of the returns to education, using data from the NLSY79. The main lesson learnt from this application is AFQT accounts for all of the racial disparity and significant portion of the parental-income disparity in academic achievement: the likelihood of enrolling in school and the likelihood of completing a grade level given enrollment. This result is consistent with Cameron and Heckman (2001) and Carneiro and Heckman (2003), who argue most of the racial and parental income gap in academic achievement can be explained by differences in early childhood development of scholastic ability as measured by AFQT. An important caveat pertains to individuals who work while attending school, for whom I find the nonpecuniary costs of working while enrolled in school are higher for blacks and individuals with high parental income, and these costs are not diminished with the inclusion of AFQT.

On the other hand, although AFQT accounts for the racial disparities in the barriers to labor market participation and the distribution of returns to education, it does not account for the parental-income disparities in these economic quantities. Furthermore, I find family income, and not AFQT, has a mitigating effect on racial disparity in wage offers. The primary conclusion of Carneiro and Heckman (2003) is that, because education is a key determinant of earnings, public investment designed to eliminate the racial and parental-income gap in earnings should be more targeted toward enhancing the scholastic ability of the young, and less on policy interventions such as tuition subsidies designed to enhance the education level of older individuals with low skills. The results of this paper suggest public investment should also target enhancing employment opportunities and earnings, independent of education, of older individuals from poorer families. If the ultimate goal is to address the socio-economic disparity in labor market success, the results of this paper provide a refinement of how a dollar of public spending should be allocated between enhancing skills of the young on one hand, and improving employment opportunities and wages of older individuals entering the labor market on the other. Further analysis of the allocation of public spending on the young versus the old would involve counterfactual policy analysis similar to Keane and Wolpin (2000b) and Cameron and Heckman (2001), which is beyond the scope of the current paper but belongs to the research agenda.

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# A LEMMA AND THEOREMS

## A.1 Proof of Theorem 3.1

*Proof.* For any initial choice $(j, c_{jt})$, evaluating equation (2.3) at period $t+1$ and $c_{jt+1} = c^0_{jt+1}(z_t, r_{jt})$, and substituting into equation (2.10) obtains

$$\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) = u_{jt}(z_t, c_{jt}, r_{jt}) \\
&+ \beta \sum_{k=1}^{J} \int\int [u^0_{kt+1}(z_{t+1}, r_{kt+1}) + \psi_k(p^0_{t+1}(z_{t+1}, r_{t+1})) \\
&+ \beta \int\int V_{t+2}(z_{t+2}, r_{t+2}) g_r(r_{t+2}) dr_{t+2} f^0_{kt}(z_{t+2}|z_{t+1}, r_{kt+1}) dz_{t+2}] \\
&\times a_{kjt+1}(z_{t+1}, r_{kt+1}) g_r(r_{t+1}) dr_{t+1} f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1},
\end{aligned} \tag{A.1}$$

so that

$$\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) = u_{jt}(z_t, c_{jt}, r_{jt}) \\
&+ \beta \sum_{k=1}^{J} \int\int [u^0_{kt+1}(z_{t+1}, r_{kt+1}) + \psi_k(p^0_{t+1}(z_{t+1}, r_{t+1}))] \\
&\times a_{kjt+1}(z_{t+1}, r_{kt+1}) g_r(r_{t+1}) dr_{t+1} f_{jt}(z_{t+1}|z_t, c_{jt}) dz_{t+1} \\
&+ \beta^2 \int\int V_{t+2}(z_{t+2}, r_{t+2}) g_r(r_{t+2}) dr_{t+2} \kappa_{jt+1}(z_{t+2}|z_t, c_{jt}) dz_{t+2}.
\end{aligned} \tag{A.2}$$

By forward substitution, equations (2.10) and (A.1) obtain

$$\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) = u_{jt}(z_t, c_{jt}, r_{jt}) \\
&+ \sum_{\tau=t+1}^{t+\rho} \sum_{k=1}^{J} \int\int \beta^{\tau-t} [u^0_{k\tau}(z_\tau, r_{k\tau}) + \psi_k[p^0_\tau(z_\tau, r_\tau)]] \\
&\times a_{kj\tau}(z_\tau, r_{k\tau}) g_r(r_\tau) \kappa_{\tau-1,j}(z_\tau|z_t, c_{jt}) dr_\tau dz_\tau \\
&+ \beta^{t+\rho+1} \int\int V_{t+\rho+1}(z_{t+\rho+1}, r_{t+\rho+1}) \\
&\times g_r(r_{t+\rho+1}) dr_{t+\rho+1} \kappa_{jt+\rho+1}(z_{t+\rho+1}|z_t, c_{jt}) dz_{t+\rho+1}.
\end{aligned} \tag{A.3}$$

□

## A.2 Proof of Theorem 3.3

*Proof.* Using equation (A.3), the difference in the conditional value functions associated with two alternative initial choices, $(j, c_{jt})$ and $(j', c_{j't})$, becomes

$$\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) - v_{j't}(z_t, c_{j't}, r_{j't}) = u_{jt}(z_t, c_{jt}, r_{jt}) - u_{j't}(z_t, c_{j't}, r_{j't}) \\
&+ \sum_{\tau=t+1}^{t+\rho} \sum_{k=1}^{J} \int \int \beta^{\tau-t} [u^0_{k\tau}(z_\tau, r_{k\tau}) + \psi_k[p^0_\tau(z_\tau, r_\tau)]] \\
&\times [a_{kj\tau}(z_\tau, r_{k\tau}|z_t, c_{jt})\kappa_{j\tau-1}(z_\tau|z_t, c_{jt}) - a_{kj'\tau}(z_\tau, r_{k\tau}|z_t, c_{j't})\kappa_{j'\tau-1}(z_\tau|z_t, c_{j't})]g_r(r_\tau)dr_\tau dz_\tau \\
&+ \beta^{t+\rho+1} \int \int V_{t+\rho+1}(z_{t+\rho+1}, r_{t+\rho+1})g_r(r_{t+\rho+1})dr_{t+\rho+1} \\
&\times [\kappa_{jt+\rho}(z_{t+\rho+1}|z_t, c_{jt}) - \kappa_{j't+\rho}(z_{t+\rho+1}|z_t, c_{j't})]dz_{t+\rho+1}.
\end{aligned} \tag{A.4}$$

If

$$\kappa_{jt+\rho}(z_{t+\rho+1}|z_t, c_{jt}) = \kappa_{j't+\rho}(z_{t+\rho+1}|z_t, c_{j't})$$

almost everywhere with $\sum_{k=1}^{J} a_{kk'\tau}(z_\tau, r_{k\tau}) = 1, \quad k' = j, j'$, the last term on the RHS of equation (A.4) is eliminated to obtain

$$\begin{aligned}
&v_{jt}(z_t, c_{jt}, r_{jt}) - v_{j't}(z_t, c_{j't}, r_{j't}) = u_{jt}(z_t, c_{jt}, r_{jt}) - u_{j't}(z_t, c_{j't}, r_{j't}) \\
&+ \sum_{\tau=t+1}^{t+\rho} \sum_{k=1}^{J} \int \int \beta^{\tau-t} [u^0_{k\tau}(z_\tau, r_{k\tau}) + \psi_k[p^0_\tau(z_\tau, r_\tau)]] \\
&\times [a_{kj\tau}(z_\tau, r_{k\tau}|z_t, c_t)\kappa_{j\tau-1}(z_\tau|z_t, c_{jt}) - a_{kj'\tau}(z_\tau, r_{k\tau}|z_t, c_t)\kappa_{j'\tau-1}(z_\tau|z_t, c_{j't})]g_r(r_\tau)dr_\tau dz_\tau.
\end{aligned}$$

□

## A.3 Identification

Define $u_{jkt}(x_t, s, c_t, r_t) = u_{jt}(x_t, s, c_{jt}, r_{jt}) - u_{kt}(x_t, s, c_{kt}, r_{kt})$. Define $v_{jkt}(x_t, s, c_t, r_t)$ and $f_{jkt}(x_{t+1}|x_t, c_t)$ analogously. In what follows, I assume $f_{jt}(x_{t+1}|x_t, c_{jt}), j = 1, \cdots, J$ are identified in the population from the observables $(x_{t+1}, x_t, c_{jt})$. For a given alternative $j^*$ and any $c_t \in \mathcal{C}_t$, for any measurable function $\Gamma : \mathcal{X} \to \mathcal{X}$ and constants, $\gamma = \{\gamma_j, j \in \mathcal{J} \backslash j^*\}$, define the set

$$\mathcal{X}_t(\Gamma, \gamma, c_t) = \left\{ x_t \in \mathcal{X} : \int \Gamma(x_{t+1}) f_{jj^*t}(x_{t+1}|x_t, c_t) = \gamma_j, j \in \mathcal{J} \backslash j^* \right\}. \tag{A.5}$$

The following assumption provides sufficient conditions for the uniqueness of the mapping between the differences between the conditional value functions and the period-specific payoff functions.

**Assumption A.1.** *1. The discount factor $\beta \in (0,1]$ is known.*
*2. The distribution of the alternative-specific shocks, $g_\varepsilon$, is known and twice continuously differentiable and log-concave with support $\Re^J$.*
*3. The distribution of the shocks associated with the continuous choices, $g_r$, is known and continuous.*
*4. $u_{1t}(x_t, s, c_{1t}, r_{1t})$ is known for all $(x_t, s, c_{1t}, r_{1t})$.*
*5. For each $j \in \mathcal{J}$, $\bar{x_t}$ exists for which $u_{jt}(\bar{x_t}, s, c_{jt}, r_{jt})$ is known for all $(s, c_{1t}, r_{1t})$.*
*6. For $\Gamma = \beta V_{t+1}(\cdot, s)$, the set $\mathcal{X}_{1t} := \left\{\mathcal{X}_t(\Gamma, \gamma, c_t) : \gamma \in \Re^{J-1}, c_t \in \mathcal{C}_t\right\}$ is such that $P(\mathcal{X}_{1t}) > 0$.*

Parts 1 and 2 of Assumption A.1 are standard in these models. Magnac and Thesmar (2002) show the discount factor can be identified from exclusion restrictions. I conjecture such exclusion restrictions can deliver identification of the discount factor in the model presented in section 2.1, but such explorations are beyond the scope of this paper. Blevins (2014) proposes sufficient conditions for identification of the shocks associated with the continuous choices under the assumption that these shocks are revealed before the shocks associated with the discrete choices. Part 4 of Assumption A.1 states the period-specific payoff associated with one alternative is fully known. This assumption is weaker than assuming the utility of the outside alternative is identically zero, and is an emerging standard in the literature of identification of these models (see Blevins, 2014, Arcidiacono and Miller, 2020, for examples). Part 5 of Assumption A.1 is a location normalization, without which the period-specific payoff functions would be identified only up to a function of the continuous choices. Location restrictions of this type are developed and discussed extensively in Matzkin (2003), Chesher (2003) and Matzkin (2007), for examples. Blevins (2014) implement similar location restrictions, the difference being the author imposes the restriction based on a value of the continuous choice. Location restrictions are replaced with traditional rank restrictions if the period-specific payoff functions are parametrically specified.

Define $U = \left\{u_{jt}(x_t, s, c_{jt}, r_{jt}),\ \text{for all } (x_t, s, c_{jt}, r_{jt}),\ j = 1, \cdots, J,\ t = 1, \cdots, T\right\}$. The following lemma is proved in section A.4

**Lemma A.2.** *Suppose Assumptions 2.1, 3.10, A.1 hold. Let $\tilde{U}$ and $U$ generate $\tilde{v}_{jt}(x_t, s, c_t, r_t)$ and $v_{jt}(x_t, s, c_t, r_t)$ according to the Bellman equation (2.3). Then $\tilde{v}_{jj^*t}(x_t, s, c_t, r_t) = v_{jj^*t}(x_t, s, c_t, r_t)$, for all $j \in J \backslash j^*$ implies $\tilde{u}_{jt}(x_t, s, c_{jt}, r_{jt}) = u_{jt}(x_t, s, c_{jt}, r_{jt})$, $j = 1, \cdots, J$ on $\mathcal{X}_{1t}$.*

The next assumption imposes restrictions on the period-specific functions that are sufficient for the difference in the conditional value of an alternative and the others, $v_{jkt}$ to be strictly monotonic in the permanent unobserved heterogeneity, which is assumed to be unidimensional. In the empirical application, this alternative is $j = 2$, the choice of working and not enrolling in school.

**Assumption A.3.** *For some $j \in \mathcal{J}$, for all $k \in \mathcal{J} \backslash j$ and all $(c_t, r_t)$, $t = 1, \cdots, T$, the following hold.*
*1. For $\Gamma = \beta\left[V_{t+1}(\cdot, \tilde{s}) - V_{t+1}(\cdot, s)\right]$, $x_t$ and corresponding constants $\{\gamma_k, k = 1, \cdots, J\}$ exist for which $x_t \in \mathcal{X}_t(\Gamma, \gamma, c_t)$ and $u_{jkt}(x_t, \tilde{s}, c_t, r_t) - u_{jkt}(x_t, s, c_t, r_t) > \gamma_k - \gamma_j$ for $\tilde{s} > s$. Define $\mathcal{X}_{2t} = \left\{\mathcal{X}_t(\Gamma, \gamma) : \gamma \in \Re^{J-1}, c_t \in \mathcal{C}_t\right\}$.*
*2. For any $\tilde{U} \neq U$ and $\tilde{s} \neq s$, and for $\Gamma = \beta\left[\tilde{V}_{t+1}(\cdot, \tilde{s}) - V_{t+1}(\cdot, s)\right]$, $x_t$ and corresponding constants $\{\gamma_k, k = 1, \cdots, J\}$ exist for which $x_t \in \mathcal{X}_t(\Gamma, \gamma, c_t)$ and $\tilde{u}_{jkt}(x_t, \tilde{s}, c_t, r_t) - u_{jkt}(x_t, s, c_t, r_t) > \gamma_k - \gamma_j$. Define $\mathcal{X}_{3t} = \left\{\mathcal{X}_t(\Gamma, \gamma) : \gamma \in \Re^{J-1}, c_t \in \mathcal{C}_t\right\}$.*
*3. $P\left(\tilde{\mathcal{X}}_t\right) > 0$, where $\tilde{\mathcal{X}}_t := \{\mathcal{X}_{1t} \bigcap \mathcal{X}_{2t} \bigcap \mathcal{X}_{3t}\}$.*

Part 1 of Assumption A.3 provides conditions for which the difference in conditional value functions are strictly increasing in the unobserved state variable on a set of the observed state vectors for all values of the CCCs and their associated shocks. The identification results to follow also hold under sufficient conditions that restricts the difference in conditional value functions to be decreasing in the unobserved state variable. However, one of these monotonicity restrictions must be imposed, and we assume the investigator has prior knowledge of the direction. For example, the unobserved heterogeneity in the empirical application is the returns to education, in which case, the wage-offer function is strictly increasing in returns to education, and the period-specific utility of working is strictly increasing in the wage offer. Part 2 of Assumption A.3 is a crossing restriction, which states that for two distinct differences in period-specific payoff functions and values of the permanent heterogeneity, a value of the observed state variable exists for which the two payoffs are sufficiently separated.

**Lemma A.4.** *Suppose the conditions of Lemma A.1 and Assumption A.3 hold. Then, for some $j \in \mathcal{J}$, for all $k \in \mathcal{J} \backslash j$, and $r_t$, the following hold:*

1. $v^0_{jkt}(x_t, s, r_t)$ *is strictly increasing in s for all* $x_t \in \tilde{X}_t$.
2. *For any* $\tilde{s} \neq s$, $x_t \in \tilde{X}_t$ *exists for which* $\tilde{v}^0_{jkt}(x_t, \tilde{s}, r_t) > v^0_{jkt}(x_t, s, r_t)$.

The following assumption imposes restrictions on the distribution of the permanent unobserved heterogeneity. In particular, I assume the distribution is finitely supported and dependent on a time-invariant subset of $x_t$, $w \in \mathcal{W} \subseteq \Re^{D_w}$. Magnac and Thesmar (2002) presents sufficient conditions for identification of the period-specific utility function in two-period binary-choice models, where they impose restrictions directly on the CCPs conditioned on the permanent heterogeneity relative to their corresponding unconditional CCPs.

**Assumption A.5.** *1. For all $x_t \in \mathcal{X}$ and $w \in \mathcal{W}$, the conditional density $f_t(x_t|w) > 0$.*
*2. The permanent unobserved heterogeneity has finite support with probability mass function $\pi(s|x_t) = \pi(s|w)$, $s \in \mathcal{S}(w)$ and cardinality $Q(w)$, possibly depending on w.*

Let $\pi(s_q(w)|w) = \pi_q(w)$, $\pi(w) = (\pi_1(w), \cdots, \pi_{Q(w)}(w))'$, $\Pi(w) = (\{s_1(w), \cdots, s_{Q(w)}(w)\}, \pi(w)', Q(w))$, and $\Pi = \{\Pi(w), w \in \mathcal{W}\}$. Equation (2.7) obtains

$$p^0_{jt}(x_t, s; U) = \int p^0_{jt}(x_t, s, r; U) g_r(r) dr, \quad \text{and}$$
$$p^0_{jt}(x_t; U, \Pi(w)) = \sum_{q=1}^{Q(w)} p^0_{jt}(x_t, s_q(w); U) \pi_q(w). \tag{A.6}$$

Define $\mathbf{P}(x; U, \Pi) = (p^0_{jt}(x_t; U, \Pi(w)), j = 1, \cdots, J, t = 1, \cdots, T)$ and, analogously define $\mathbf{P}_0(x) = (p_{0jt}(x_t), j = 1, \cdots, J, t = 1, \cdots, T,)$, where $p_{0jt}(x_t)$ is the true CCP, which is observable in the population. Let $(U_0, \Pi_0)$ be the true parameter vector; that is, the probabilities generated from the model at $(U_0, \Pi_0)$ coincide with the population probabilities: $\mathbf{P}(x; U_0, \Pi_0) = \mathbf{P}_0(x)$. Define $\tilde{\mathcal{X}} = \bigcup_{t=1}^{T} \tilde{\mathcal{X}}_t$. Note that, although the identification theorem is stated for all $t$, it also holds for cases in which neither the initial nor final period of the agent's decision process is observed, in which case, the period-specific payoffs are identified for only the observable periods. The identification theorem is stated as follows.

**Theorem A.6.** *Suppose the conditions of Lemma A.2 and Assumption A.5 hold. Then* $(U_0, \Pi_0)$*, is identified on* $\tilde{X}$ *in the sense that any* $(\tilde{U}, \tilde{\Pi})$ *satisfying* $\mathbf{P}(x; \tilde{U}, \tilde{\Pi}) = \mathbf{P}_0(x)$ *implies* $(\tilde{U}, \tilde{\Pi}) = (U_0, \Pi_0)$ *on* $\tilde{X}$ *with probability one.*

The proof of Theorem A.6 is provided in Appendix A.6.

*Remark* A.7. The estimator developed in the paper assumes the period-specific utility functions are known up to a finite-dimensional set of parameters, which are not functions of the state variables. I assume this restriction extends the set upon which $U_0$ is identified in Theorem A.6 from $\tilde{X}$ to $X$. I also assume the distribution of the shocks associated with the continuous choices, $r$, can also be parametrically specified, given the parametric restrictions on the period-specific utility function.

## A.4 Proof of Lemma A.2

*Proof.* Recall that for each $j = 1, \cdots, J$, and under Assumption 2.1, the conditional value function in equation (2.3) is given by

$$v_{jt}(x_t, s, c_{jt}, r_{jt}) = u_{jt}(x_t, s, c_{jt}, r_{jt}) + \beta \int V_{t+1}(x_t, s) f_{jt}(x_{t+1}|x_t, c) dx_{t+1}.$$

This representation of the conditional value function is unique for given $U$ under Assumption 3.10 and parts 1-3 of Assumption A.1, by the uniqueness of the back induction, and so is

$$v_{jj^*t}(x_t, s, c_{jt}, r_{jt}) = u_{jj^*t}(x_t, s, c_{jt}, r_{jt}) + \beta \int V_{t+1}(x_t, s) f_{jj^*t}(x_{t+1}|x_t, c_{jt}) dx_{t+1}. \qquad \text{(A.7)}$$

Likewise, the alternative set of utilities, $\tilde{U}$, uniquely generate

$$\tilde{v}_{jj^*t}(x_t, s, c_{jt}, r_{jt}) = \tilde{u}_{jj^*t}(x_t, s, c_{jt}, r_{jt}) + \beta \int \tilde{V}_{t+1}(x_t, s) f_{jj^*t}(x_{t+1}|x_t, c_{jt}) dx_{t+1}. \qquad \text{(A.8)}$$

Then, $\tilde{v}_{jj^*t}(x_t, s, c_{jt}, r_{jt}) = v_{jj^*t}(x_t, s, c_{jt}, r_{jt})$ obtains

$$\tilde{u}_{jj^*t}(x_t, s, c_{jt}, r_{jt}) - u_{jj^*t}(x_t, s, c_{jt}, r_{jt}) = \beta \int \left[V_{t+1}(x_t, s) - \tilde{V}_{t+1}(x_t, s)\right] f_{jj^*t}(x_{t+1}|x_t, c) dx_{t+1}. \tag{A.9}$$

Under part 6 of Assumption A.1, for any $(s, r_t)$ and fixed $c_t$, by varying $x_t$ on $\mathcal{X}_t(\Gamma, \gamma, c_t)$ where $\Gamma = \beta V_{t+1}(\cdot, s)$, equation (A.9) obtains

$$\tilde{u}_{jj^*t}(x_t, s, c_{jt}, r_{jt}) - u_{jj^*t}(x_t, s, c_{jt}, r_{jt}) = \gamma_j \tag{A.10}$$

on $\Gamma = \beta V_{t+1}(\cdot, s)$, which holds for all $j \neq j^*$, including $j = 1$. By part 4 of Assumption A.1 and equation (A.10),

$$\tilde{u}_{j^*t}(x_t, s, c_{jt}, r_{jt}) - u_{j^*t}(x_t, s, c_{jt}, r_{jt}) = \gamma_1. \tag{A.11}$$

Part 5 of Assumption A.1 and equation (A.11) imply $\gamma_1 = 0$, so that

$$\tilde{u}_{j^*t}(x_t, s, c_{jt}, r_{jt}) = u_{j^*t}(x_t, s, c_{jt}, r_{jt}),$$

which case, part 5 of Assumption A.1 and equation (A.10) imply

$$\tilde{u}_{jt}(x_t, s, c_{jt}, r_{jt}) = u_{jt}(x_t, s, c_{jt}, r_{jt}), \; j = 1, \cdots, J$$

on $\mathcal{X}_t(\Gamma, \gamma, c_t)$. Noting these results hold for all $\gamma$ and $c_t$ for which $\mathcal{X}_t(\Gamma, \gamma, c_t) \subset \mathcal{X}_{1t}$ completes the proof. $\square$

## A.5 Proof of Lemma A.4

For $j$ defined in Assumption A.3, and $x_t \in \mathcal{X}_{2t}$ defined Assumption A.3.1, $\tilde{s} > s$ implies

$$\begin{aligned}
& v_{jkt}(x_t,\tilde{s},c_t,r_t) - v_{jkt}(x_t,s,c_t,r_t) = u_{jkt}(x_t,\tilde{s},c_t,r_t) - u_{jkt}(x_t,s,c_t,r_t) \\
& + \beta \int [V_{t+1}(x_{t+1},\tilde{s}) - V_{t+1}(x_{t+1},s)] \, f_{jkt}(x_{t+1}|x_t,c_t) \\
& = u_{jkt}(x_t,\tilde{s},c_t,r_t) - u_{jkt}(x_t,s,c_t,r_t) \\
& + \beta \int [V_{t+1}(x_{t+1},\tilde{s}) - V_{t+1}(x_{t+1},s)] \, f_{jj^*t}(x_{t+1}|x_t,c_t) \\
& - \beta \int [V_{t+1}(x_{t+1},\tilde{s}) - V_{t+1}(x_{t+1},s)] \, f_{kj^*t}(x_{t+1}|x_t,c_t) \\
& = u_{jkt}(x_t,\tilde{s},c_t,r_t) - u_{jkt}(x_t,s,c_t,r_t) + \gamma_j - \gamma_k > 0
\end{aligned} \tag{A.12}$$

for any $r_t$ and any $c_t$, including $c_t^0(x_t,s,r_t)$. This completes part 1 of the theorem.

For any $(x_t,c_t,r_t)$ and $\tilde{s} \neq s$,

$$\begin{aligned}
& \tilde{v}_{jkt}(x_t,\tilde{s},c_t,r_t) - v_{jkt}(x_t,s,c_t,r_t) = \tilde{u}_{jkt}(x_t,\tilde{s},c_t,r_t) - u_{jkt}(x_t,s,c_t,r_t) \\
& + \beta \int \left[\tilde{V}_{t+1}(x_{t+1},\tilde{s}) - V_{t+1}(x_{t+1},s)\right] f_{jj^*t}(x_{t+1}|x_t,c_t) dx_{t+1} \\
& - \beta \int \left[\tilde{V}_{t+1}(x_{t+1},\tilde{s}) - V_{t+1}(x_{t+1},s)\right] f_{kj^*t}(x_{t+1}|x_t,c_t) dx_{t+1}.
\end{aligned} \tag{A.13}$$

Therefore, for $x_t \in \mathcal{X}_{3t}$ defined in Assumption A.3.2, equation (A.13) obtains

$$\tilde{v}_{jkt}(x_t,\tilde{s},c_t,r_t) - v_{jkt}(x_t,s,c_t,r_t) = \tilde{u}_{jkt}(x_t,\tilde{s},c_t,r_t) - u_{jkt}(x_t,s,c_t,r_t) + \gamma_j - \gamma_k > 0. \tag{A.14}$$

Now, for any $r_t$ and $\tilde{s} \neq s$,

$$\begin{aligned}
& \tilde{v}^0_{jkt}(x_t,\tilde{s},r_t) - v^0_{jkt}(x_t,s,r_t) \\
& = \tilde{v}_{jkt}(x_t,\tilde{s},\tilde{c}^0_t(x_t,\tilde{s},r_t),r_t) - v_{jkt}(x_t,s,c^0_t(x_t,s,r_t),r_t) \\
& = \left[\tilde{v}_{jkt}(x_t,\tilde{s},\tilde{c}^0_t(x_t,\tilde{s},r_t),r_t) - \tilde{v}_{jkt}(x_t,\tilde{s},c^0_t(x_t,s,r_t),r_t)\right] \\
& + \left[\tilde{v}_{jkt}(x_t,\tilde{s},c^0_t(x_t,s,r_t),r_t) - v_{jkt}(x_t,s,c^0_t(x_t,s,r_t),r_t)\right].
\end{aligned} \tag{A.15}$$

By uniqueness of the optimal CCC, the first term in brackets on the RHS of equation (A.15) is positive, and equation (A.14) implies the second is positive on $\mathcal{X}_{3t}$. This completes part 2 of the theorem.

## A.6 Proof of Theorem A.6

*Proof.* Consider choice $j$ satisfying Assumption A.3. Under the conditions of the theorem, Lemmas A.2 and A.4 imply the following are true:

1. For all $s$, $\tilde{U} \neq U$ on $\tilde{\mathcal{X}}$ implies $p^0_{jt}(x_t, s; \tilde{U}) \neq p^0_{jt}(x_t, s; U)$.
2. $p^0_{jt}(x_t, s; U)$ is strictly increasing in $s$ for all $x_t \in \tilde{\mathcal{X}}_t$.
3. For any $\tilde{U} \neq U$ and $\tilde{s} \neq s$, $x_t \in \tilde{\mathcal{X}}_t$ exist for which $p^0_{jt}(x_t, \tilde{s}; \tilde{U}) > p^0_{jt}(x_t, s; U)$.

Define

$$\mathcal{P}_1(U) = \left\{ p^0_{jt}(x_t, s_q; U) : x_t \in \mathcal{X}_t, s_q \in \mathcal{S}, q \geq 1, 2, \cdots \right\}, \tag{A.16}$$

$$\mathcal{P}_2(w, U) = \Bigg\{ p^0_{jt}(x_t; U, \Pi(w)) : p^0_{jt}(x_t; U, \Pi(w)) = \sum_{q=1}^{Q(w)} p^0_{jt}(x_t, s_q(w); U)\pi_q(w), \pi_q(w) > 0,$$
$$\sum_{q=1}^{Q(w)} \pi_q(w) = 1, p^0_{jt}(x_t, s_q(w); U) \in \mathcal{P}_1(w, U), Q(w) \geq 1, 2, \cdots \Bigg\}. \tag{A.17}$$

By parts 2 and 3 of Assumption A.1, $\mathcal{P}_1(U)$ is a linearly independent set for any fixed $U$ with probability 1. Therefore, result 1 above implies $p^0_{jt}(x_t; U, \Pi(w)) \in \mathcal{P}_2(w, U)$ has a unique representation as a linear combination of finitely many elements of $\mathcal{P}_1(U)$ (see Kreyszig, 1989); that is, any $U$ induces a unique corresponding finite-mixing distribution $\Pi(w; U) = (\{s_1(w; U), \cdots, s_Q(w; U)\}, \pi(w; U), Q(w; U))$. By hypothesis,

$$p_{0jt}(x_t) = p^0_{jt}(x_t; U_0, \Pi(w; U_0)) = \sum_{q=1}^{Q(w; U_0)} p^0_{jt}(x_t, s_q(w; U_0); U_0)\pi_q(w; U_0).$$

Suppose $\tilde{U} \neq U_0$ exist for which $p_{0jt}(x_t) = p^0_{jt}(x_t; \tilde{U}, \Pi(w; \tilde{U}))$ so that

$$\sum_{q=1}^{Q(w;\tilde{U})} p^0_{jt}(x_t, s_q(w;\tilde{U}); \tilde{U})\pi_q(w;\tilde{U}) = \sum_{q=1}^{Q(w;U_0)} p^0_{jt}(x_t, s_q(w;U_0); U_0)\pi_q(w;U_0). \tag{A.18}$$

Because the weights sum to 1, equation (A.18) can be written as follows:

$$\begin{aligned} & p^0_{jt}(x_t, s_1(w;\tilde{U}); \tilde{U}) - p^0_{jt}(x_t, s_1(w;U_0); U_0) = \\ & \sum_{q=2}^{Q(w;U_0)} (p^0_{jt}(x_t, s_q(w;U_0); U_0) - p^0_{jt}(x_t, s_1(w;U_0)); U_0)\pi_q(w;U_0) \\ & - \sum_{q=2}^{Q(w;\tilde{U})} (p^0_{jt}(x_t, s_q(w;\tilde{U}); \tilde{U}) - p^0_{jt}(x_t, s_1(w;\tilde{U})); \tilde{U})\pi_q(w;\tilde{U}). \end{aligned} \tag{A.19}$$

Relabel the abscissa if necessary so that $s_1(w;U_0) > s_q(w;U_0), q = 2, \cdots, Q(w;U_0)$ and $s_1(w;\tilde{U}) < s_q(w;\tilde{U}), q = 2, \cdots, Q(w;\tilde{U})$. By result 2 above, the RHS of equation (A.19) is negative for all $x_t \in \tilde{\mathcal{X}}(w)$. However, by result 3 above, $x_t \in \tilde{\mathcal{X}}(w)$ exists for which the LHS of equation (A.19) is positive, contradicting the claim in equation (A.18) and therefore implying $\tilde{U} = U_0$ on $\tilde{\mathcal{X}}(w)$, which in turn implies $\Pi(w;\tilde{U}) = \Pi(w;U_0) = \Pi_0(w)$. Noting these results hold for any $w \in \mathcal{W}$ completes the proof.

□

## A.7 Proof of Theorem 5.3

*Proof.* For $q = 1, \cdots, Q$, $k = 1, \cdots, |w|$, equation (5.6) obtains

$$\begin{aligned} & E[\pi_q(d_{it}, x_{it}; \theta, \pi(w_k))|x_{it}, w_k] = E\left[ \left. \frac{f(d_{it}|x_{it}, \theta_q)\pi_q(w_k)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'})\pi_{q'}(w_k)} \right| x_{it}, w_k \right] \\ & = \int \frac{f(d_{it}|x_{it}, \theta_q)\pi_q(w_k)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'})\pi_{q'}(w_k)} f_0(d_{it}|x_{it}, w_k)\mathrm{d}d_{it} \\ & = \pi_q(w_k) \int \frac{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{0q'})\pi_{0q'}(w_k)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'})\pi_{q'}(w_k)} f(d_{it}|x_{it}, \theta_q)\mathrm{d}d_{it} \end{aligned} \tag{A.20}$$

for almost every $x_{it} \in \mathcal{X}$, where $f_0(d_{it}|x_{it}, w_k)$ and is the population-conditional density of $d_{it}$ given $(x_{it}, w_k)$. The last equality is a result of identification of the parameters of the model, which implies the unique representation: $f_0(d_{it}|x_{it}, w_k) = \sum_{q=1}^{Q} f(d_{it}|x_{it}, \theta_{0q})\pi_{0q}(w_k)$. Suppose $(\theta, \pi) = (\theta_0, \pi_0)$. Then equation (A.20) obtains $E[\pi_q(d_{it}, x_{it}; \theta, \pi_0(w_k))|x_{it}, w_k] = \pi_{0q}(w_k)$. Conversely, suppose $E[\pi_q(d_{it}, x_{it}; \theta, \pi_(w_k))|x_{it}, w_k] = \pi_q(w_k)$ for almost every $x_{it} \in \mathcal{X}$. Then, equation (A.20) implies

$$\int \frac{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{0q'})\pi_{0q'}(w_k)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'})\pi_{q'}(w_k)} f(d_{it}|x_{it}, \theta_q)\mathrm{d}d_{it} = 1, \tag{A.21}$$

because the RHS of equation (A.20) varies with $x_{it}$ otherwise. By identification of the model, equation (A.21) implies $(\theta, \pi) = (\theta_0, \pi_0)$. □

## A.8 Asymptotic properties of the estimator

I use the following notations in all assumptions, theorems, and proofs: $\sup_\theta = \sup_{\theta\in\Theta}$, $\sup_\pi = \sup_{\pi\in\Delta^{Q-1}}$, $\sup_{\theta,\pi} = \sup_{\theta\in\Theta}\sup_{\pi\in\Delta^{Q-1}}$, and $\sum_k = \sum_{k=1}^{|w|}$. The first assumption imposes the typical random-sampling restriction of the sampling process.

**Assumption A.8.** *As sample of n independent realizations is drawn from $F(d, c, x)$. For each $i = 1, \cdots, n$, $(d_{it}, c_{it}, x_{it}, t = 1, \cdots, T)$ is observed.*

The next assumption imposes restrictions on the parameter space and the admissible functional forms of the period-specific utility functions.

**Assumption A.9.** *1. The sets $\mathcal{X}$ and $\Theta$ are compact; 2. $\theta_0 \in int(\Theta)$ and for j=0,1,2, and $k = 1, \cdots, |w|$, $\|\partial^j \pi_0(w_k; \theta)/\partial\theta^j\|$ exists at each $\theta \in int(\Theta)$; 3. $E[\|c\|^2] < \infty$ and $E[\|h(x_i, \theta_0)\|^2] < \infty$ ; 4. $h(x; \theta)$ is twice continuously differentiable at each $\theta \in int(\Theta)$, with $\|\partial^j h(x; \theta)/\partial\theta^j\| \leq \tilde{h}_j(x)$, $j = 0, 1, 2$ for all $\theta \in \Theta$ and some $\tilde{h}_j(x)$ satisfying $E[\tilde{h}_j(x)] < \infty$ for $j = 0, 1, 2$; and 5. For $k = 1, \cdots, |w|$, $n_k/n \to c_k > 0$.*

**Assumption A.10.** *$\hat{\Omega}$ is symmetric and positive definite with $\|\hat{\Omega} - \Omega\| = o_p(1)$.*

Some additional definitions and notations are required to proceed. Define

$$m_{it}^{\pi}(\theta,\pi) = \pi - \boldsymbol{f}_{it}(\theta,\pi)\pi, \quad m_{i}^{\pi}(\theta,\pi) = \frac{1}{T}\sum_{t=1}^{T} m_{it}^{\pi}(\theta,\pi),$$
$$m_{0}^{\pi}(w;\theta,\pi) = E\left[m_{i}^{\pi}(\theta,\pi)|\, w\right], \quad \hat{m}^{\pi}(w_k;\theta,\pi) = \frac{1}{n_k}\sum_{i=1}^{n} m_{i}^{\pi}(\theta,\pi)I_i(w_k).$$

Let $I$ and $\iota$ be the $q-$ dimensional identity matrix and iota vector, respectively. Define also $\boldsymbol{\pi} = diag\{\pi_1,\cdots,\pi_Q\}$, and let

$$M_{it}^{\pi}(\theta,\pi) = I - \boldsymbol{f}_{it}(\theta,\pi) + \boldsymbol{\pi}\boldsymbol{f}_{it}(\theta,\pi)\iota\iota'\boldsymbol{f}_{it}(\theta,\pi), \quad M_{i}^{\pi}(\theta,\pi) = \frac{1}{T}\sum_{t=1}^{T} M_{it}^{\pi}(\theta,\pi),$$
$$M_{0}^{\pi}(w;\theta,\pi) = E\left[M_{i}^{\pi}(\theta,\pi)|\, w\right], \quad \hat{M}^{\pi}(w_k;\theta,\pi) = \frac{1}{n_k}\sum_{i=1}^{n} M_{i}^{\pi}(\theta,\pi)I_i(w_k).$$

The proof of the following consistency theorem is in Appendix A.9.

**Theorem A.11.** *Suppose (i) Assumption A.3 holds, (ii) Assumption 5.2 holds, and (iii) Assumptions A.8, A.9, and A.10 hold. Then,* $\hat{\theta} \xrightarrow{p} \theta_0$, *and for* $k \in \{1,\cdots,|w|\}$,

$$\left\|\frac{\partial^j}{\partial\theta^j}\hat{\pi}(w_k;\hat{\theta}) - \frac{\partial^j}{\partial\theta^j}\pi_0(w_k;\theta_0)\right\| \xrightarrow{p} 0, \quad j = 0,1.$$

Let $m_i = m_i(\theta_0,\pi_0)$, $M_i(\theta,\pi) = \partial m_i(\theta,\pi)/\partial\theta$, and $M_0 = E[M_i(\theta_0,\pi_0)]$, $M_{\pi i}(\theta) = X_i h(x_i,\theta)$, $M_{\pi 0}(w) = E[M_{\pi i}(\theta_0)|w]$, $m_0^{\pi}(w) = m_0^{\pi}(w;\theta_0,\pi_0)$ and $M_0^{\pi}(w) = M_0^{\pi}(w;\theta_0,\pi_0)$.

**Theorem A.12.** *Suppose the conditions of theorem A.11 hold, and* $\theta_0$ *is in the interior of* $\Theta$. *Then,*

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{p} N(0,V),$$

*where* $V = (M_0'\Omega M_0)^{-1}(M_0'\Omega\Sigma\Omega M_0)(M_0'\Omega M_0)^{-1}$, *and*

$$\Sigma = E\left[\left(m_i + M_{\pi 0}(w_i)M_0^{\pi}(w_i)^{-1}m_i^{\pi}\right)'\left(m_i + M_{\pi 0}(w_i)M_0^{\pi}(w_i)^{-1}m_i^{\pi}\right)\right].$$

The proof of Theorem A.12 is in Appendix A.10.

## A.9 Proof of Theorem A.11

*Proof.* For $k = 1, \cdots, |w|$, define $\hat{h}(w_k, \theta) = \sum_{i=1}^{n} h(x_i, \theta) I_i(w_k)/n_k$ and $h_0(w_k \theta) = E[h(x_i, \theta)|w_i = w_k]$. Under Assumptions A.8, A.9.1, A.9.3, and A.9.5, Lemma 2.4 of Newey and McFadden (1994) implies

$$\sup_{\theta} \left\| \frac{\partial^j}{\partial \theta^j} \hat{h}(w_k, \theta) - \frac{\partial^j}{\partial \theta^j} h_0(w_k, \theta) \right\| \xrightarrow{p} 0, \quad j = 0, 1, 2. \tag{A.22}$$

I suppress the dependence of the type probabilities and empirical and theoretical expectations and on $w$. Note the denominator term in $\boldsymbol{f}_{it}(\theta, \pi)$ satisfies $0 < \sum_q f(d_{it}|x_{it}, \theta_q)\pi_q < 1$ uniformly over $\theta \in \Theta$ and $\pi \in \triangle^{Q-1}$. Then, by defining $\boldsymbol{f}_i(\theta, \pi) = \sum_{i=1}^{T} \boldsymbol{f}_{it}(\theta, \pi)/T$, $\hat{\boldsymbol{f}}(\theta, \pi) = \sum_{i=1}^{n} \boldsymbol{f}_i(\theta, \pi)/n$ and $\boldsymbol{f}_0(\theta, \pi) = E[\boldsymbol{f}_i(\theta, \pi)]$, equation (A.22) implies

$$\sup_{\theta, \pi} \left\| \frac{\partial^{j+l}}{\partial \theta^j \pi^l} \hat{\boldsymbol{f}}(\theta, \pi) - \frac{\partial^{j+l}}{\partial \theta^j \pi^l} \boldsymbol{f}_0(\theta, \pi) \right\| \xrightarrow{p} 0, \quad j, l = 0, 1, 2. \tag{A.23}$$

Consistent with equation (A.20), $\hat{m}_q^{\pi}(\theta, \hat{\pi}(\theta)) = 0$ for any $\theta \in \Theta$. The mean-value expansion around $\pi_0(\theta)$ obtains

$$\begin{aligned}
0 = {} & \hat{m}_q^{\pi}(\theta, \pi_0(\theta)) \\
& + \frac{1}{nT} \sum_{i,t} \left[ \left( 1 - \frac{f(d_{it}|x_{it}, \theta_q)}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'}) \bar{\pi}_{q'}(\theta)} \right) \left( \hat{\pi}_q(\theta) - \pi_{0q}(\theta) \right) \right. \\
& + \frac{f(d_{it}|x_{it}, \theta_q) \bar{\pi}_q}{\sum_{q'=1}^{Q} f(d_{it}|x_{it}, \theta_{q'}(\theta)) \bar{\pi}_{q'}(\theta)} \\
& \left. \times \sum_{q' \neq q} \frac{f(d_{it}|x_{it}, \theta_{q'})}{\sum_{q''=1}^{Q} f(d_{it}|x_{it}, \theta_{q''}) \bar{\pi}_{q''}(\theta)} \left( \hat{\pi}_{q'}(\theta) - \pi_{0q'}(\theta) \right) \right]
\end{aligned} \tag{A.24}$$

identically over $\theta$ over $\Theta$, where $\bar{\pi}(\theta)$ are mean values. Stacking equation (A.24) in $q$ obtains

$$0 = \hat{m}^{\pi}(\theta, \pi_0(\theta)) + \hat{M}^{\pi}(\theta, \bar{\pi}(\theta)) \left( \hat{\pi}(\theta) - \pi_0(\theta) \right). \tag{A.25}$$

Note each $\bar{\pi}_q(\theta)$ is strictly positive, so $M_{it}^{\pi}(\theta,\bar{\pi}(\theta))$ can be written as follows:

$$M_{it}^{\pi}(\theta,\bar{\pi}(\theta)) = \bar{\boldsymbol{\pi}}(\theta)\left[\bar{\boldsymbol{\pi}}(\theta)^{-1}(I - \boldsymbol{f}_{it}(\theta,\bar{\pi})) + \boldsymbol{f}_{it}(\theta.\bar{\pi})\iota\iota'\boldsymbol{f}_{it}(\theta,\bar{\pi})\right].$$

Note also that $\bar{\boldsymbol{\pi}}(\theta)^{-1}(I - \boldsymbol{f}_{it}(\theta,\bar{\pi})) + \boldsymbol{f}_{it}(\theta.\bar{\pi})\iota\iota'\boldsymbol{f}_{it}(\theta,\bar{\pi})$ is symmetric with strictly positive elements, and

$$\begin{aligned}
&det\left[\bar{\boldsymbol{\pi}}(\theta)^{-1}(I - \boldsymbol{f}_{it}(\theta,\bar{\pi})) + \boldsymbol{f}_{it}(\theta.\bar{\pi})\iota\iota'\boldsymbol{f}_{it}(\theta,\bar{\pi})\right] \\
&= \left[1 + \iota'\boldsymbol{f}_{it}(\theta,\bar{\pi})(I - \boldsymbol{f}_{it}(\theta,\bar{\pi}))^{-1}\bar{\boldsymbol{\pi}}(\theta)\boldsymbol{f}_{it}(\theta.\bar{\pi})\iota\right] det\left[\bar{\boldsymbol{\pi}}(\theta)^{-1}(I - \boldsymbol{f}_{it}(\theta,\bar{\pi}))\right] \\
&> 0,
\end{aligned} \tag{A.26}$$

so that $\bar{\boldsymbol{\pi}}(\theta)^{-1}(I - \boldsymbol{f}_{it}(\theta,\bar{\pi})) + \boldsymbol{f}_{it}(\theta.\bar{\pi})\iota\iota'\boldsymbol{f}_{it}(\theta,\bar{\pi})$ is symmetric and positive definite, which in turn implies $\frac{1}{nT}\sum_{i,t}\bar{\boldsymbol{\pi}}(\theta)^{-1}(I - \boldsymbol{f}_{it}(\theta,\bar{\pi})) + \boldsymbol{f}_{it}(\theta.\bar{\pi})\iota\iota'\boldsymbol{f}_{it}(\theta,\bar{\pi})$ is symmetric and positive definite. The diagonal matrix $\boldsymbol{\pi}(\theta)$ is also symmetric and positive definite. From these results, conclude that $\hat{M}^{\pi}(\theta,\bar{\pi}(\theta))$ is invertible so that equation (A.25) obtains

$$\hat{\pi}(\theta) - \pi_0(\theta) = -\hat{M}^{\pi}(\theta,\bar{\pi}(\theta))^{-1}\hat{m}^{\pi}(\theta,\pi_0(\theta)) \tag{A.27}$$

identically in $\theta$ over $\Theta$. Dy definition,

$$\hat{\pi}(\theta) = \hat{\boldsymbol{f}}(\theta,\hat{\pi}(\theta))\hat{\pi}(\theta) \text{ and} \tag{A.28}$$

$$\pi_0(\theta) = \boldsymbol{f}_0(\theta,\pi_0(\theta))\pi_0(\theta) \tag{A.29}$$

hold identically in $\theta$ over $\Theta$, and equation (A.29) implies $\hat{m}_{it}^{\pi}(\theta,\pi_0(\theta))$ can be written as follows:

$$m_i^{\pi}(\theta,\pi_0(\theta)) = \boldsymbol{f}_0(\theta,\pi_0(\theta))\pi_0(\theta) - \boldsymbol{f}_i(\theta,\pi_0(\theta))\pi_0(\theta). \tag{A.30}$$

Equations (A.23) and (A.30), along with Assumption A.9.3, imply

$$\sup_{\theta}\left\|\frac{\partial^j}{\partial\theta^j}\hat{m}^{\pi}(\theta,\pi_0(\theta))\right\| \xrightarrow{p} 0 \quad j = 0,1,2. \tag{A.31}$$

Because $\boldsymbol{f}_{it}(\theta,\pi)$ is a diagonal matrix with probabilities that sum to 1, $\sup_{\theta,\pi}\|\boldsymbol{f}_{it}(\theta,\pi)\| = 1$. Therefore,

$$
\begin{aligned}
\|M_{it}^{\pi}(\theta,\pi)\| &\le \|Q+1+\|\boldsymbol{\pi}\|\|\|\boldsymbol{f}_{it}(\theta,\pi)\iota'\boldsymbol{f}_{it}(\theta,\pi)\| \le \mathcal{T}_1 < \infty \\
\left\|\frac{\partial}{\partial\pi}M_{it}^{\pi}(\theta,\pi)\right\| &= \left\|(2\boldsymbol{\pi}\boldsymbol{f}_{it}(\theta,\pi)\iota' - I)\frac{\partial}{\partial\pi}\boldsymbol{f}_{it}(\theta,\pi) + \boldsymbol{f}_{it}(\theta,\pi)\iota'\boldsymbol{f}_{it}(\theta,\pi)\right\| \\
&\le \mathcal{T}_2 + \mathcal{T}_3\left\|\frac{\partial}{\partial\pi}\boldsymbol{f}_{it}(\theta,\pi)\right\|,
\end{aligned}
$$

where $\mathcal{T}_2$ and $\mathcal{T}_3$ are positive and finite constants. These results and equation (A.23) imply

$$
\sup_{\theta,\pi}\left\|\frac{\partial^j}{\partial\pi^j}\hat{M}^{\pi}(\theta,\pi) - \frac{\partial^j}{\partial\pi^j}M_0^{\pi}(\theta,\pi)\right\| \xrightarrow{p} 0, \quad j = 0,1. \tag{A.32}
$$

Equations (A.27), (A.31), and (A.32) obtain

$$
\sup_{\theta}\|\hat{\pi}(\theta) - \pi_0(\theta)\| \xrightarrow{p} 0. \tag{A.33}
$$

The mean-value theorem, equations (A.23), (A.32) and (A.33), and Assumption A.9.2 imply

$$
\begin{aligned}
&\sup_{\theta}\left\|\hat{M}^{\pi}(\theta,\hat{\pi}(\theta)) - M_0^{\pi}(\theta,\pi_0(\theta))\right\| \\
&= \sup_{\theta}\left\|\hat{M}^{\pi}(\theta,\hat{\pi}(\theta)) - M_0^{\pi}(\theta,\hat{\pi}(\theta)) + \frac{\partial}{\partial\pi}M_0^{\pi}(\theta,\bar{\pi}(\theta))(\hat{\pi}(\theta) - \pi_0(\theta))\right\| \\
&\le \sup_{\theta,\pi}\left\|\hat{M}^{\pi}(\theta,\pi) - M_0^{\pi}(\theta,\pi)\right\| + \sup_{\theta,\pi}\left\|\frac{\partial}{\partial\pi}M_0^{\pi}(\theta,\pi)\right\|\sup_{\theta}\|\hat{\pi}(\theta) - \pi_0(\theta)\| \\
&= o_p(1).
\end{aligned} \tag{A.34}
$$

$$
\begin{aligned}
&\sup_{\theta}\left\|\frac{\partial}{\partial\theta}\hat{\boldsymbol{f}}(\theta,\hat{\pi}(\theta)) - \frac{\partial}{\partial\theta}\boldsymbol{f}_0(\theta,\pi_0(\theta))\right\| \\
&= \sup_{\theta}\left\|\frac{\partial}{\partial\theta}\hat{\boldsymbol{f}}(\theta,\hat{\pi}(\theta)) - \frac{\partial}{\partial\theta}\boldsymbol{f}_0(\theta,\hat{\pi}(\theta)) + \frac{\partial^2}{\partial\theta\partial\pi}\boldsymbol{f}_0(\theta,\bar{\pi}(\theta))(\hat{\pi}(\theta) - \pi_0(\theta))\right\| \\
&\le \sup_{\theta,\pi}\left\|\frac{\partial}{\partial\theta}\hat{\boldsymbol{f}}(\theta,\pi) - \frac{\partial}{\partial\theta}\boldsymbol{f}_0(\theta,\pi)\right\| + \sup_{\theta,\pi}\left\|\frac{\partial^2}{\partial\theta\partial\pi}\boldsymbol{f}_0(\theta,\pi)\right\|\sup_{\theta,\pi}\|\hat{\pi}(\theta) - \pi_0(\theta)\| \\
&= o_p(1).
\end{aligned} \tag{A.35}
$$

Because equations (A.28) and (A.29) hold identically in $\theta$ over $\Theta$, the envelope condition implies

$$\hat{M}^{\pi}(\theta,\hat{\pi}(\theta))\frac{\partial}{\partial\theta}\hat{\pi}(\theta)+\hat{\pi}(\theta)'\otimes I\frac{\partial}{\partial\theta}vec\left(\hat{\boldsymbol{f}}(\theta,\hat{\pi}(\theta))\right)=0, \text{ and} \tag{A.36}$$

$$M_0^{\pi}(\theta,\pi_0(\theta))\frac{\partial}{\partial\theta}\pi_0(\theta)+\pi_0(\theta)'\otimes I\frac{\partial}{\partial\theta}vec\left(\boldsymbol{f}_0(\theta,\pi_0(\theta))\right)=0 \tag{A.37}$$

hold identically in $\theta$ over $int(\Theta)$, where $\otimes$ is the Kronecker product operator and $vec$ is the vectorization operator. By noting the equality $\hat{a}\hat{b}-ab=(\hat{a}-a)(\hat{b}-b)+a(\hat{b}-b)+(\hat{a}-a)b$, the difference between equations (A.37) and (A.36) gives

$$\begin{aligned}
&\hat{M}^{\pi}(\theta,\hat{\pi}(\theta))\left(\frac{\partial}{\partial\theta}\hat{\pi}(\theta)-\frac{\partial}{\partial\theta}\pi_0(\theta)\right)=-\left(\hat{M}^{\pi}(\theta,\hat{\pi}(\theta))-M_0^{\pi}(\theta,\pi_0(\theta))\right)\frac{\partial}{\partial\theta}\pi_0(\theta)\\
&+(\hat{\pi}(\theta)-\pi_0(\theta))'\otimes I\left(\frac{\partial}{\partial\theta}vec\left(\hat{\boldsymbol{f}}(\theta,\hat{\pi}(\theta))\right)-\frac{\partial}{\partial\theta}vec\left(\boldsymbol{f}_0(\theta,\pi_0(\theta))\right)\right)\\
&+\pi_0(\theta)'\otimes I\left(\frac{\partial}{\partial\theta}vec\left(\hat{\boldsymbol{f}}(\theta,\hat{\pi}(\theta))\right)-\frac{\partial}{\partial\theta}vec\left(\boldsymbol{f}_0(\theta,\pi_0(\theta))\right)\right)\\
&+(\hat{\pi}(\theta)-\pi_0(\theta))'\otimes I\frac{\partial}{\partial\theta}vec\left(\boldsymbol{f}_0(\theta,\pi_0(\theta))\right).
\end{aligned} \tag{A.38}$$

Equations (A.33) - (A.38) and Assumption A.9.2 imply

$$\sup_{\theta}\left\|\frac{\partial}{\partial\theta}\hat{\pi}(\theta)-\frac{\partial}{\partial\theta}\pi_0(\theta)\right\|\xrightarrow{p}0. \tag{A.39}$$

Now, under Assumptions A.8, A.9.1, and A.9.4,

$$\|m_i(\theta,\pi_0(\theta))\|\leq\|X_i\|(\|y_i\|+\|h(x_i,\theta)\|\|\pi_0(\theta)\|)\leq\mathcal{T}_1\|y_i\|+\mathcal{T}_2\tilde{h}_0(x_i)$$

for positive and finite constants $\mathcal{T}_1$ and $\mathcal{T}_2$, so that Assumption A.9.3 and Lemma 2.4 of Newey and McFadden (1994) imply

$$\sup_{\theta}\|\hat{m}(\theta,\pi_0(\theta))-m_0(\theta,\pi_0(\theta))\|\xrightarrow{p}0. \tag{A.40}$$

Therefore, equations (A.46), (A.33), and (A.40) imply

$$
\begin{aligned}
&\|\hat{m}(\theta,\hat{\pi}(\theta)) - m_0(\theta,\pi_0(\theta))\| \leq \|\hat{m}(\theta,\hat{\pi}(\theta)) - m_0(\theta,\hat{\pi}(\theta))\| \\
&+ \|E[X_i h(x_i,\theta)]\| \, \|\hat{\pi}(\theta) - \pi_0(\theta)\| \\
&+ \left\|\hat{M}_\pi(\theta) - M_{\pi 0}(\theta)\right\| \|\hat{\pi}(\theta) - \pi_0(\theta)\| \xrightarrow{p} 0,
\end{aligned} \tag{A.41}
$$

which, along with Assumption A.10, obtains $\hat{\theta} \xrightarrow{p} \theta_0$. Finally, this result, Assumption A.9.2, and equations (A.33), (A.39), and (A.40) imply

$$
\begin{aligned}
\|\hat{\pi}(\hat{\theta}) - \pi_0(\theta_0)\| &= \left\|\hat{\pi}(\hat{\theta}) - \pi_0(\hat{\theta}) + \frac{\partial \pi_0(\overline{\theta})}{\partial \theta}(\hat{\theta} - \theta_0)\right\| \\
&\leq \sup_\theta \|\hat{\pi}(\theta) - \pi_0(\theta)\| + \sup_\theta \left\|\frac{\partial \pi_0(\theta)}{\partial \theta}\right\| \|\hat{\theta} - \theta_0\| \xrightarrow{p} 0,
\end{aligned} \tag{A.42}
$$

and

$$
\begin{aligned}
\left\|\frac{\partial}{\partial \theta}\hat{\pi}(\hat{\theta}) - \frac{\partial}{\partial \theta}\pi_0(\theta_0)\right\| &= \left\|\frac{\partial}{\partial \theta}\hat{\pi}(\hat{\theta}) - \frac{\partial}{\partial \theta}\pi_0(\hat{\theta}) + \frac{\partial^2 \pi_0(\overline{\theta})}{\partial \theta^2}(\hat{\theta} - \theta_0)\right\| \\
&\leq \sup_\theta \|\hat{\pi}(\theta) - \pi_0(\theta)\| + \sup_\theta \left\|\frac{\partial^2 \pi_0(\theta)}{\partial \theta^2}\right\| \|\hat{\theta} - \theta_0\| \xrightarrow{p} 0.
\end{aligned} \tag{A.43}
$$

The reintroduction of the type probabilities depending on $w$ only involves performing the above analysis with all theoretical and empirical expectations replaced with their corresponding conditional expectations, conditioned on the event $\{w_i = w_k\}, k = 1, \cdots, |w|$ and noting that $n/n_k \to c_k < \infty$ for $k = 1, \cdots, |w|$. With these modifications, conclude that

$$
\max_k \left\|\frac{\partial^j}{\partial \theta^j}\hat{\pi}(w_k;\hat{\theta}) - \frac{\partial^j}{\partial \theta^j}\pi_0(w_k;\theta_0)\right\| \xrightarrow{p} 0, \quad j = 0, 1.
$$

$\square$

## A.10 Proof of Theorem A.12

*Proof.* By recalling that $\boldsymbol{f}_i(\theta,\pi)$ is a diagonal matrix with probabilities that sum to 1,

$$E[\|\pi_0(w_k;\theta)'\boldsymbol{f}_i(\theta,\pi_0(w_k;\theta))'\boldsymbol{f}_i(\theta,\pi_0(w_k;\theta))\pi_0(w_k;\theta)\||w_k] < \infty$$

for $k=1,\cdots,|w|$. This result implies that for any $\theta\in\Theta$, $m_i^{\pi}(\theta,\pi_0(w_k;\theta))$ defined in equation (A.30) satisfies $E\left[\left|m_i^{\pi}(\theta,\pi_0(w_k;\theta))'m_i^{\pi}(\theta,\pi_0(w_k;\theta))\right||w_k\right] < \infty$. Therefore, under Assumptions A.8 and A.9, and by noting $n_k/n \to c_k > 0, k=1,\cdots,|w|$ and equation (A.34), application of the Lindeberg-Levy CLT to equation (A.27) gives $\|\hat{m}^{\pi}(w_k;\theta,\pi_0(w_k;\theta))\| = O_p(1/\sqrt{n}),\ k=1,\cdots,|w|$. This convergence result, and equations (A.27) and (A.32) imply

$$\hat{\pi}(w_k;\theta)-\pi_0(w_k;\theta) = -M_0^{\pi}(w_k;\theta,\pi_0(w_k;\theta))^{-1}\frac{1}{n_k}\sum_{i=1}^{n} m_i^{\pi}(w_k;\theta,\pi_0(w_k;\theta))I_i(w_k)+o_p(1/\sqrt{n})$$

for any $\theta\in\Theta$ and for $k=1,\cdots,|w|$. Also, under Assumptions A.8 and A.9, $E[\|m_i'm_i\|] < \infty$ so that, by the Lindegerg-Levi CLT,

$$\|\hat{m}\| = O_p(1/\sqrt{n}). \tag{A.44}$$

Define $\hat{M}_\pi(w_k;\theta) = \sum_{i=1}^n M_{\pi i}(\theta) I_i(w_k)/n_k$, for $k = 1, \cdots, |w|$ and let $\hat{M}(\theta,\pi) = \sum_{i=1}^n M_i(\theta,\pi)/n$. The mean-value expansion obtains

$$
\begin{aligned}
&\hat{m}(\hat{\theta},\hat{\pi}(\hat{\theta})) = \\
&\hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) - \frac{1}{n}\sum_{i=1}^n X_i h(x_i,\theta_0)(\hat{\pi}(w_i;\theta_0) - \pi_0(w_i;\theta_0)) \\
&= \hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) - \frac{1}{n}\sum_{i=1}^n X_i h(x_i,\theta_0)\sum_k I_i(w_k)(\hat{\pi}(w_k;\theta_0) - \pi_0(w_k;\theta_0)) \\
&= \hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) - \sum_k \left[\frac{1}{n_k}\sum_{i=1}^n X_i h(x_i,\theta_0) I_i(w_k)\right]\frac{n_k}{n}(\hat{\pi}(w_k;\theta_0) - \pi_0(w_k;\theta_0)) \\
&= \hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) - \sum_k \hat{M}_\pi(w_k;\theta_0)\frac{n_k}{n}(\hat{\pi}(w_k;\theta_0) - \pi_0(w_k;\theta_0)) \\
&= \hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) + \sum_k M_{\pi 0}(w_k) M_0^\pi(w_k)^{-1}\frac{1}{n}\sum_{i=1}^n m_i^\pi(w_k;\theta_0,\pi_0(\theta_0)) I_i(w_k) + o_p(1/\sqrt{n}) \\
&= \hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) + \frac{1}{n}\sum_{i=1}^n\sum_k M_{\pi 0}(w_k) M_0^\pi(w_k)^{-1} m_i^\pi(w_k) I_i(w_k) + o_p(1/\sqrt{n}) \\
&= \hat{m} + \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) + \frac{1}{n}\sum_{i=1}^n M_{\pi 0}(w_i) M_0^\pi(w_i)^{-1} m_i^\pi + o_p(1/\sqrt{n}) \\
&= \left[\hat{M}(\bar{\theta},\hat{\pi}(\bar{\theta}))\right](\hat{\theta}-\theta_0) + \frac{1}{n}\sum_{i=1}^n \left[m_i + M_{\pi 0}(w_i) M_0^\pi(w_i)^{-1} m_i^\pi\right] + o_p(1/\sqrt{n}), \qquad \text{(A.45)}
\end{aligned}
$$

where $\bar{\theta}$ and $\bar{\pi}$ are mean values. Next, under the same conditions that obtains equation (A.22),

$$
\sup_\theta \left\| \frac{\partial^j}{\partial\theta^j}\hat{M}_\pi(w_k,\theta) - \frac{\partial^j}{\partial\theta^j} M_{\pi 0}(w_k,\theta)\right\| \xrightarrow{p} 0, \quad j = 0,1,2. \qquad \text{(A.46)}
$$

$$
\hat{M}(\theta,\pi(\theta)) = \sum_k \hat{M}_\pi(w_k;\theta)\frac{\partial}{\partial\theta}\pi(w_k;\theta) + \sum_k \pi(w_k;\theta)' \otimes I_{N_X}\frac{\partial}{\partial\theta} vec\left(\hat{M}_\pi(w_k;\theta)\right),
$$

so that

$$
\begin{aligned}
&\left\|\hat{M}(\hat{\theta},\hat{\pi}(\hat{\theta}))-M_0(\theta_0,\pi_0(\theta_0))\right\| \\
&\leq \sum_k \left\|\hat{M}_\pi(w_k;\hat{\theta})\frac{\partial}{\partial\theta}\hat{\pi}(w_k;\hat{\theta})-M_{\pi 0}(w_k;\theta_0)\frac{\partial}{\partial}\pi_0(\theta_0)\right\| \\
&+\sum_k \left\|\hat{\pi}(w_k;\hat{\theta})'\otimes I_{N_X}\frac{\partial}{\partial\theta}vec\left(\hat{M}_\pi(w_k;\hat{\theta})\right)-\pi_0(w_k;\theta_0)'\otimes I_{N_X}\frac{\partial}{\partial\theta}vec\left(M_{\pi 0}(w_k;\theta_0)\right)\right\| \\
&= o_p(1), \qquad \text{(A.47)}
\end{aligned}
$$

where the last equality is obtained using Assumptions A.8 and A.9, the consistency results of Theorem A.11, equation (A.46), and calculations similar to equation (A.38). This result also holds when $\hat{\theta}$ is replaced with the mean values $\bar{\theta}$.

The first-order condition $\hat{M}(\hat{\theta},\hat{\pi}(\hat{\theta}))'\hat{\Omega}\hat{m}(\hat{\theta},\hat{\pi}(\hat{\theta}))=0$, equations (A.45) and (A.47), and Assumption A.10 obtain

$$
\sqrt{n}(\hat{\theta}-\theta_0)=-(M_0'\Omega M_0)^{-1}M_0'\Omega\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[m_i+M_{\pi 0}(w_i)M_0^{\pi}(w_i)^{-1}m_i^{\pi}\right]+o_p(1).
$$

Application of the Lindeberg-Levi CLT completes the proof.

□

## A.11 Monte-Carlo Exercise

This appendix presents the results of two sets of Monte Carlo exercises to illustrate the finite-sample performance of the proposed estimator. The first set of exercises investigate the performance of the approximation to finite dependence presented in section 3.1 under the restrictions on the transition probabilities imposed in Assumption 3.8. The second set of exercises investigate the performance of the estimator relative to the one based on the EM algorithm developed in Arcidiacono and Miller (2011) for models that satisfy the renewal property discussed in section 3.1. The discount factor is set to be 0.95 in all cases.

For the first exercise, the number of discrete alternatives is $J=2$, and the period-specific utility function is specified as follows:

$$u_t(z_t, c_t, r_t, \varepsilon_t) = d_{2t}\left[\ln(1 + c_t\phi_1(z_t, r_t; \theta)) + \phi_2(x_t)\ln(1 - c_t) + \varepsilon_{2t}\right] + d_{1t}\varepsilon_{1t}, \tag{A.48}$$
$$\phi_1(z_t, r_t; \theta) = \exp(\theta_1 + \theta_2 x_{1t} + s x_{2t} + r_t),$$
$$\phi_2(x_t) = \exp(0.05 x_3).$$

The observed state variable, $x_{1t} = \sum_{\tau=1}^{t-1} d_{2\tau}/10$ and $f_{jt}(x_{2t+1}|x_t, c_t)$, is distributed discretized normal $N(\mu_{jt}(x_t, c_{jt}), 1)$ with $|x_2| = 7$ support points, where

$$\mu_{jt}(x_t, c_{jt}) = 2.5 + d_{2t}(0.4 - 0.8c_t) + 0.05\phi_3(t) + 0.01 d_{2t-1} - 0.005 x_{1t}, \text{ and} \tag{A.49}$$
$$\phi_3(t) = 0.067t - 0.001t^2. \tag{A.50}$$

The time-invariant observed stated variable, $x_3$, is distributed discrete-uniform on the interval $[0, 10]$ with $|x_3| = 100$ support points, and the shock associated with the CCC, $r_t$ is distributed discretized normal $N(0, 0.1)$ with $|r| = 3$ support points. I set the number of types in the population to two with $s \in \{0.1, 0.5\}$ and corresponding probabilities $(0.6, 0.4)$, and the shocks, $\varepsilon_{jt}, j = 1, 2$, to be distributed i.i.d., type 1 extreme value. I estimate $\theta_2, \theta_4$, and the distribution of $s$ in each design.

The preference parameters are $\theta_1 = 1$, $\theta_2 = 0.2$. The simulated data are generated by solving the dynamic programming for 40 periods and simulating 100 replications of 1,000 individuals. Estimation is based on the last 20 periods, and I contaminate the log CCCs with additive measurement errors, which are distributed i.i.d. normal with zero mean and variance equal to 10% of the variance in the simulated CCCs.

The parameters are estimated by two-stage GMM, and the counterfactual CCPs and CCCs are updated by regressing the type-specific log-odds ratio and the type-specific CCSs implied from the model on a third-order polynomial of the state variables in each period. The weights that achieve approximate one-period finite dependence are computed on 15 equidistant grid points between 0.1 and 0.5. Table 1 presents the results of simulation exercise for this first model design.

Table A1: Finite-Sample Properties of the Estimator for Design 1

| | $\theta_1$ | $\theta_2$ | $s_1$ | $s_2$ | $\pi_1$ |
|---|---|---|---|---|---|
| True Value | 1.00 | 0.20 | 0.10 | 0.50 | 0.60 |
| | Full Solution. Time:[a] 12.42 (7.30) mins. | | | | |
| MB | 0.0249 | -0.0046 | -0.0031 | -0.0134 | 0.0105 |
| MAB | 0.0253 | 0.0046 | 0.0039 | 0.0192 | 0.0224 |
| RMSE | 0.0305 | 0.0061 | 0.0046 | 0.0235 | 0.0301 |
| | Known CCP and CCC. Time: 2.39 (1.60) mins. | | | | |
| MB | 0.0313 | -0.0084 | -0.0044 | -0.0147 | -0.0154 |
| MAB | 0.0314 | 0.0084 | 0.0050 | 0.0214 | 0.0240 |
| RMSE | 0.0367 | 0.0108 | 0.0063 | 0.0241 | 0.0313 |
| | Proposed Estimator. Time: 2.73 (2.01) mins. | | | | |
| MB | 0.0291 | -0.0073 | -0.0043 | -0.0149 | -0.0162 |
| MAB | 0.0294 | 0.0074 | 0.0049 | 0.0212 | 0.0245 |
| RMSE | 0.0350 | 0.0099 | 0.0063 | 0.0241 | 0.0319 |

[a] Standard deviations in parentheses.

The second simulation exercise investigates the performance of the proposed estimator in an environment where the model satisfies the renewal property, in which case, the weights that achieve one-period finite dependence are closed form. The estimator is compared with EM algorithm approach of Arcidiacono and Miller (2011). Again, the model is one of two discrete alternatives, and the period-specific utility function is given by

$$u_t(z_t, \varepsilon_t) = d_{2t}\left[s + \theta_1 x_1 + \theta_2 \ln(1 + x_{2t}/10) + \varepsilon_{1t}\right] + d_{1t}\varepsilon_{1t}. \qquad \text{(A.51)}$$

The time-invariant observed state variable, $x_2$, is distributed discrete-uniform on $[0,1]$ with $|x_1| = 200$ support points. The endogenous state variable $x_{2t}$ evolves as follows: $x_{2t+1} = 0$ if

$d_{1t} = 1$, and $x_{2t+1} = x_{2t} + r_{t+1}$ if $d_{2t} = 1$, where $r_t \in \{1, 2, 3\}$ and

$$Pr(r_{t+1} = r|x_t) = \frac{e^{r\phi_t(x_t)}}{\sum_{k=1}^{3} e^{r_k \phi_t(x_t)}}, \quad \text{where} \tag{A.52}$$
$$\phi_t(x_t) = 0.5 - 0.01t + 0.5x_1 - 0.001x_{2t}.$$

The number of unobserved types is set to two with $s \in 1, 3$ with corresponding probabilites $(0.6, 0.4)$. In the context of the optimal replace-of-bus-engines framework of Rust (1987), $t$ is the age of the bus, $x_1$ is the brand of the bus, and $x_{2t}$ is accumulated mileage. Action $j = 1$ is replacing the bus, in which case, the accumulated mileage of the bus is zero in the next period. If Harold Zurcher chooses not to replace the bus engine, accumulated mileage increases by $r_{t+1}$. Equation (A.52) implies the probability of high usage of the bus decreases with age and accumulated mileage of the bus.

I set $\theta_1 = 1$, $\theta_2 = -0.5$. As in the previous Monte-Carlo exercise, the simulated data are generated by solving the dynamic programming problem for 60 periods and simulating 100 replications of 1,000 individuals with the initial condition $x_{21} = 0$. Estimation is based on the last 20 periods. To construct the likelihood for the EM algorithm, the initial distribution of the endogenous state variable conditioned on the unobserved state variable is computed under the assumption that the unobserved state variable is observed. Table A2 presents the results of a simulation exercise for the second model design.

Table A2: Finite-Sample Properties of the Estimator for Design 2

| | $\theta_1$ | $\theta_2$ | $s_1$ | $s_2$ | $\pi_1$ |
|---|---|---|---|---|---|
| True Value | 1.0 | -0.5 | 1.0 | 3.0 | 0.6 |
| | Proposed Estimator. Time:[a] 0.28 (0.13) mins. | | | | |
| MB | 0.0260 | 0.0015 | 0.0012 | -0.0006 | -0.0044 |
| MAB | 0.0278 | 0.0083 | 0.0086 | 0.0078 | 0.0172 |
| RMSE | 0.0457 | 0.0131 | 0.0134 | 0.0124 | 0.0216 |
| | EM Algorithm. Time: 0.015 (0.003) mins. | | | | |
| MB | -0.0197 | -0.0151 | -0.0240 | -0.0108 | 0.0105 |
| MAB | 0.0312 | 0.0323 | 0.0391 | 0.0216 | 0.0224 |
| RMSE | 0.0397 | 0.0403 | 0.0522 | 0.0276 | 0.0301 |

[a] Standard deviations in parentheses.