

Contest Success Functions: Theory and Evidence

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Abstract

Two well-known forms of contest success functions – those attributed to Jack Hirschleifer and Gordon Tullock – predict the winner of a military, economic, political, or other conflict from the difference between the resources of each side and from the ratio of resources (respectively). The predictions of conflict models depend critically on which of these two formulations are used. In this paper we axiomatically and stochastically derive a new class of contest success functions which generalize the common properties of these different approaches, representing them as special cases. Using the new function we study a simple and widely-used conflict model and show that some properties of the existing form – for example the non-existence of an interior pure strategy Nash equilibrium in the model with the difference form – are due to the special assumption that the form presupposes. We also estimate the new contest success function by using data from 350 battles fought in seventeenth century Europe and 186 battles during World War II.

Keywords: Conflict; Contest Success Functions; Wars

1. Introduction

Gordon Tullock (1980) and Jack Hirshleifer (1989) provide an essential tool for the analysis of economic and other conflicts, now known as the contest success function, which shows how the probability of a group's victory in a conflict depends on the groups' resources devoted to winning, whether these be armed forces, advertising resources, or lobbying expenditure. Examples of conflict in social sciences include various appropriative activities, such as rent-seeking behaviors (Tullock, 1967, 1980; Hirshleifer, 1991; Grossman, 1994), as well as war and political demonstration (Fearon, 1995; Collier and Hoeffler, 2001; Sambanis, 2004; Kalyvas et al., 2008). Contest success functions have also played a critical role in modeling human biological evolution; early human lethal conflict is now recognized as a key factor in explaining human cooperation in evolutionary biology (Bowles 2008; Garcia and Bergh 2008; Aoki 1982; Boorman and Levitt 1973; for more comprehensive surveys see Garfinkel and Skaperdas (2007) and Konrad (2009)).

But the forms of the contest success functions proposed by Tullock and by Hirshleifer differ in essential ways. Here we provide a general class of the contest success functions which has the Tullock and Hirschleifer forms as special cases, and present some empirical estimates for the contest success functions using war data.

The Hirshleifer and Tullock functions predict contest outcomes (respectively) from the difference between the resources of each side (the difference form) and from the ratio of resources (the ratio form). The predictions of a given conflict model can be drastically changed simply by altering the form of the contest success function. In a widely-used conflict model, the ratio form does not admit the possibility of an equilibrium in which both competitors spend zero resources in conflict, since the party devoting no resource loses everything as long as its opponent spends a small amount of resources (Hirshleifer, 1989, 1991) In this model a unique interior Nash equilibrium exists when the contest success function is of the ratio form, while only the corner solution is possible in the case of the difference form. Because of this sensitivity of equilibrium prediction to the form of the contest success function, the choice of the specific form is crucial and the arbitrariness involved in the choice makes the model's predictions less convincing and compelling.

In addition, the relationship between the two forms has not been clear. Since Hirschleifer’s (1989) comparison, the two functional forms have been treated as distinctive alternatives, from which one has to choose one or the other. For example, in the existing axiomatic derivations the dependencies of winning probabilities on the ratio and on the difference in resources are treated as two exclusive axioms (Skaperdas, 1996; Clark and Riis, 1998; Muenster, 2009). The stochastic derivation of the difference form – the derivation of the contest success function from assumptions on the uncertainty of performance in contests – is known from McFadden (1974)’s derivation. However, a similar stochastic characterization did not exist until the derivation by Jia (2008). Thus, it is only recent that the *similarities* rather than the differences between these two forms have received the attention of researchers.

How can we choose the appropriate functional form given a conflict situation? How are the difference form and the ratio form related? To address these questions, we derive a generalized class of contest success functions having the properties of (1) retaining all the existing axioms that both forms satisfy – for example, monotonicity, anonymity, and independence from irrelevant alternatives – and (2) satisfying a new axiom replacing the additive and multiplicative properties which yield the difference and ratio forms respectively.

To find a new axiom, we first note that, analogously to production functions, a contest success function for a given contestant “produces” winning probabilities with one favorable input (his own resource) and with another adversarial input (his opponent’s resource), and the technology of conflict describes how these two inputs are offsetting and reducing each other in a conflict situation. Then one can examine the extent to which two inputs are “negatively” substitutable or counteract each other in producing the same amount of winning probability, as the standard production theory proceeds (Arrow et al., 1961). It turns out that for both of the difference and ratio forms the elasticity of “negative” substitution are constant at any combinations of two inputs. We call this elasticity the “elasticity of augmentation” since this measures the extent to which party 1 needs to *augment* its existing resources to keep its winning probability unchanged in response to an increase in party 2’s resources.

By using the equation that keeps this elasticity constant, we axiomatically

obtain a new class of contest success functions and, as a result, show that the difference form and the ratio form belong to a *single* family of contest success functions. In particular, choosing between two forms is equivalent to assuming the elasticity of augmentation to be one of two special values, one or zero. Analogously again, the difference and ratio forms correspond to the conflict versions of the linear (perfect substitutes) and Cobb-Douglas production functions, respectively. We name the new function the “constant elasticity of augmentation” function (CEA). In addition to the axiomatic derivation, we derive the CEA function using the stochastic approach and this, in turn, fills the gap between the two separate stochastic derivations of the difference and ratio forms.

Next we investigate the dependency of Nash equilibrium on the specification of contest success functions using the CEA function. By adopting a simple and widely used conflict model, we study the changes in Nash equilibrium as the elasticity of augmentation varies, and then show that some properties of a given contest success function – for example the non-existence of an interior pure strategy Nash equilibrium in the model with the difference form – are due to the special value of elasticity of augmentation; i.e., by changing the elasticity of augmentation slightly one can obtain a contest success function which is very close to the difference form, but admits an interior pure strategy Nash equilibrium.

If the existence of an interior pure strategy Nash equilibrium accounts for the popular use of the ratio form (Garfinkel and Skaperdas, 2007), then one can find numerous such contest success functions by varying the elasticity of augmentation. In other words, the elasticity of augmentation is an important determinant of whether “peaceful” outcome can be an equilibrium in a given conflict situation. An attractive feature of the CEA contest success function is that it can model cases in which mutual “disarmament” should be considered a possible outcome, even where on *a priori* grounds one thinks that the difference in forces should receive greater weight than the ratio in the determination of outcomes.

We do not believe that the elasticity of augmentation is uniform through out various conflict situations; the extent to which the amount of money spent by a firm, in the promotional activities to enlarge the market share, offsets

and counter-balances the money spent by its adversary would be different from the extent to which fighters in an army counter-attack and eliminate the fighters of its opponent. Thus we estimate the elasticity of augmentation for a specific conflict situation. Specifically using data from battles fought in seventeenth century Europe we provide a first estimate of the elasticity of augmentation. For comparison we also estimate the ratio form of the contest success function using data from battles of World war II. From these estimates we also conclude that the ratio form better describes the technology of conflict for wars in seventeenth century Europe than the difference form. Although we discuss the dyadic contest success functions in the paper, an extension to a more general case is straightforward.

The organization of the paper is as follows: Section 2 provides the axiomatic and stochastic derivations and the analysis of a simple conflict model using the CEA function. We present the empirical evidence in Section 3. In the Appendix we provide the proof of Theorem 2 and tables for empirical evidence.

2. Constant elasticity of augmentation contest success functions

2.1. Axiomatic and Stochastic Derivations

We recall that the difference form and the ratio form are specified as follows:

$$\text{Difference: } u_d(x_1, x_2) = \frac{\exp(\kappa x_1)}{\exp(\kappa x_1) + \exp(\kappa x_2)} \quad (1)$$

$$\text{Ratio: } u_r(x_1, x_2) = \begin{cases} \frac{(x_1)^\kappa}{(x_1)^\kappa + (x_2)^\kappa} & \text{if } x_1, x_2 > 0 \\ \frac{1}{2} & \text{if } x_1 = x_2 = 0 \end{cases} \quad (2)$$

where x_i denotes the resources or effort spent by party i and κ is a positive constant. Clearly the difference form gives the winning probabilities based on the difference between resources, $x_1 - x_2$, since $u_d(x_1, x_2) = 1/(1 + \exp(-\kappa(x_1 - x_2)))$, while the probability of winning in the ratio form depends only on the ratio, x_1/x_2 , because $u_r(x_1, x_2) = 1/(1 + (x_2/x_1)^\kappa)$. The parameter κ in equations (1) and (2) known as “mass effect parameter scaling the decisiveness of fighting effort disparities” measures the slope of the contest success function in an evenly balanced match (Hirshleifer, 1991).

First we introduce the marginal rate of augmentation (MRA) which measures the quantity of additional resources that party 1 needs to augment its existing resources in order to keep the probability of winning constant in response to an increase in the other player's resources. For a given iso-probability curve of party 1's contest success function $\bar{u} = u(x_1, x_2)$, MRA is the slope of the curve:

$$\text{MRA} := \frac{dx_2}{dx_1} = -\frac{u_{x_1}}{u_{x_2}}$$

where $u_{x_1} = \partial u / \partial x_1$, $u_{x_2} = \partial u / \partial x_2$. So when MRA is high more resources should be devoted to obtain the same probability of success. Using MRA we define the elasticity of augmentation as follows:

$$\begin{aligned} & \text{elasticity of augmentation } (\rho) \\ &= \frac{\text{percentage increase in MRA}}{\text{percentage increase in relative size of contestants' resources}} \\ &= \frac{d \ln(-u_{x_1}/u_{x_2})}{d \ln(x_2/x_1)} \end{aligned}$$

The elasticity of augmentation is a normalized percentage increase in MRA. If v is the winning probability of player 2, then from $u + v = 1$ we can also write

$$\rho = \frac{d \ln(u_{x_1}/v_{x_2})}{d \ln(x_2/x_1)},$$

so ρ is also interpreted to measure the extent to which the marginal winning probabilities for party 1 reduces the marginal winning probabilities for party 2. When ρ is low, we expect that party 1 would need to augment its resources by smaller amounts to maintain the same probability of success in response to an increase in its opponent's resources. By contrast, a high ρ implies that party 1 should increase its resources by greater amounts to gain the same probability of success. We present the CEA function:

$$u_\rho(x_1, x_2) = \frac{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right)}{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right) + \exp\left(\kappa \frac{1}{1-\rho} x_2^{1-\rho}\right)}. \quad (3)$$

Theorem 1 (Axiomatic Derivation). *The CEA function (3) is a unique function satisfying*

$$u(x_1, x_2) = \frac{f(x_1)}{f(x_1) + f(x_2)} \quad (4)$$

and

$$\rho = \frac{d \ln(-u_{x_1}/u_{x_2})}{d \ln(x_2/x_1)} \quad \text{for } x_1, x_2 \geq 0 \quad (5)$$

where $f(0) > 0$, f is increasing and differentiable for $x_1, x_2 \geq 0$ and $0 \leq \rho$, $\rho \neq 1$.

Proof. By rearranging (5) we obtain

$$(x_1)^\rho u_{x_1} + c(x_2)^\rho u_{x_2} = 0 \quad \text{for some } c \neq 0 \quad (6)$$

Using (4) and (6) we find

$$(x_1)^\rho f'(x_1)f(x_2) - c(x_2)^\rho f(x_1)f'(x_2) = 0 \quad \text{for } x_1, x_2 \geq 0 \quad (7)$$

By evaluating (7) at $x_1 = x_2 > 0$ we conclude $c = 1$. We set $x_2 = 1$ in (7) and find

$$\frac{f'(x_1)}{f(x_1)} = \frac{f'(1)}{f(1)} \frac{1}{x_1^\rho}. \quad (8)$$

Then by setting $f'(1)/f(1) = \kappa$ and solving (8) we obtain the CEA function (3). ■

By performing a simple calculation we verify that for the difference form the elasticity of augmentation is zero, whereas the elasticity is one in the case of the ratio form. In general, this follows from the fact that for the contest success function in the form of (4) the derivatives are given by the following formulas:

$$\frac{\partial u}{\partial x_1} = \frac{f'(x_1)}{f(x_1)} u(1 - u) \quad \text{and} \quad \frac{\partial u}{\partial x_2} = -\frac{f'(x_2)}{f(x_2)} u(1 - u). \quad (9)$$

When the elasticity of augmentation ρ equals zero, the function in (3) becomes the difference form (1), and it converges to the ratio form (2) as ρ approaches one by applying L'Hopital's rule. Skaperdas (1996) shows that a function of the form (4) satisfies the desirable axioms of contest success functions, which include monotonicity, anonymity, and independence from irrelevant alterna-

tives (See Skaperdas, 1996, pp.284-286). Hence we conclude that the CEA function is a unique function satisfying (5) and the desirable axioms of the contest success function.

Notice even when $\rho > 1$, the function (3) is still the legitimate contest success function. The parameter κ in the CEA function is the same mass effect parameter as in the ratio and difference forms, since by computing the marginal probabilities of winning for the CEA function in an evenly balanced match, we find

$$\frac{\partial u}{\partial x_1} = \frac{\kappa}{x_1^\rho} u(1-u) = \frac{\kappa}{4} \quad \text{for } x_1 = x_2 = 1. \quad (10)$$

Next we consider the stochastic derivation of the CEA function. We write $X \sim F(s)$ to indicate that the distribution of a random variable X is $F(s)$, and recall X follows a Gumbel type (type I) extreme value distribution if $X \sim \exp(-e^{-\kappa s})$ and a Fréchet type (type II) extreme value distribution if $X \sim \exp(-s^{-\kappa})$ for $s \geq 0$, where κ is a positive constant. We suppose the result of a contest depends on performance, h_i , and performance is in turn determined by x_i and a random factor ϵ_i : i.e., $h_i = h_i(x_i, \epsilon_i)$ (Garfinkel and Skaperdas, 2007; Jia, 2008).

If the specification of performance is in additive form, $h_i^d = x_i + \epsilon_i$, and ϵ_i follows a Gumbel type distribution, the difference form of the contest success function equals $\Pr\{h_1^d > h_2^d\}$ (McFadden, 1974). When ϵ_i follows a Fréchet type distribution and the specification of performance is in multiplicative form, $h_i^r = x_i \epsilon_i$, the ratio form of the contest success function can be derived from $\Pr\{h_1^r > h_2^r\}$ (Jia, 2008). We set

$$h_i(\rho) := \frac{x_i^{1-\rho} - 1}{1-\rho} + \frac{\epsilon_i^{1-\rho} - 1}{1-\rho} \quad (11)$$

and easily see that

$$h_i(0) = x_i + \epsilon_i \quad \text{and} \quad \lim_{\rho \rightarrow 1} h_i(\rho) = \ln(x_i \epsilon_i),$$

so $\Pr\{h_1(0) > h_2(0)\} = \Pr\{h_1^d > h_2^d\}$ and $\Pr\{\lim_{\rho \rightarrow 1} h_1(\rho) > \lim_{\rho \rightarrow 1} h_2(\rho)\} = \Pr\{h_1^r > h_2^r\}$. Thus equation (11) generalizes the additive and multiplicative

specifications of uncertainty in performance.

Since $\Pr\{x_1 + \epsilon_1 < x_2 + \epsilon_2\} = \Pr\{\epsilon_1 - \epsilon_2 < x_2 - x_1\}$ for given x_1 and x_2 , McFadden's result (1974) is obtained by showing that $\epsilon_1 - \epsilon_2 \sim 1/(1 + e^{-s})$ for $\epsilon_i \sim \exp(-e^{-\kappa s})$. Similarly because $\Pr\{x_1\epsilon_1 < x_2\epsilon_2\} = \Pr\{\log \epsilon_1 - \log \epsilon_2 < \log x_2 - \log x_1\}$ holds, Jia's derivation is equivalent to showing that $\log \epsilon_1 - \log \epsilon_2 \sim 1/(1 + e^{-s})$ for $\epsilon_i \sim \exp(-s^{-\kappa})$ for $s \geq 0$. Theorem 2 provides the generalization of these derivations. We put in the Appendix the definition of the rational power of real numbers $s^{\frac{n}{m}}$ used in Theorem 2.

Theorem 2 (Stochastic Derivation). *Suppose that $\rho = 1 - n/m$, m, n are natural numbers such that $m > n$ and $\epsilon_1, \epsilon_2 \sim F(s)$ i.i.d. and $F(s) = \exp\left(-e^{-\kappa \frac{1}{1-\rho} s^{1-\rho}}\right)$ for $-\infty < s < \infty$ and $h_i(\rho)$ is given by (11). Then*

$$\Pr\{h_1(\rho) > h_2(\rho)\} = \frac{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right)}{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right) + \exp\left(\kappa \frac{1}{1-\rho} x_2^{1-\rho}\right)}$$

Proof. See the Appendix. ■

Furthermore if $\epsilon_i \sim F_i(s)$ independently and $F_i(s) = \exp\left(-\gamma_i e^{-\kappa \frac{1}{1-\rho} s^{1-\rho}}\right)$, from lemma 1 in the appendix we have

$$\Pr\{h_1 > h_2\} = \frac{\gamma_1 \exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right)}{\gamma_1 \exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right) + \gamma_2 \exp\left(\kappa \frac{1}{1-\rho} x_2^{1-\rho}\right)} \quad (12)$$

As $\rho \rightarrow 1$, (12) approaches the generalized ratio form (Clark and Riis, 1998) and γ_1 represents the relative fighting effectiveness of party 1 against party 2 (see Dupuy, 1987; Kalyvas et al., 2008).

2.2. Nash Equilibrium in a conflict model with the CEA contest success function

In this section we study the condition for the elasticity of augmentation under which a symmetric pure strategy Nash equilibrium exists in a simple conflict model (Hirshleifer, 1989; Garfinkel and Skaperdas, 2007). Suppose that party 1 and 2 have resources x_1 and x_2 , where $x_1, x_2 \in [0, \bar{x}]$, and they compete for a prize of the value $2\bar{x}$, the sum of total available resources. The costs of

competing are the resources devoted to the contest, so we write expected payoffs for party 1 and 2 as follows:

$$\pi_1(x_1, x_2) = 2\bar{x} u_\rho(x_1, x_2) - x_1, \quad \pi_2(x_1, x_2) = 2\bar{x} v_\rho(x_1, x_2) - x_2, \quad (13)$$

where $v_\rho = 1 - u_\rho(x_1, x_2)$. We look for a symmetric pure-strategy Nash equilibrium. Denoting such an equilibrium by (x_1^*, x_2^*) , we find the first order condition for the best response of party 1:

$$2\bar{x} \frac{\kappa}{x_1^\rho} u_\rho(1 - u_\rho) - 1 = 0 \quad (14)$$

In the model with u_ρ being replaced by the ratio form u^r in (13), $(x_1, x_2) = (0, 0)$ cannot be a Nash equilibrium since an arbitrary small increase in resources from 0 will raise the probability of winning from 0.5 to 1 and hence the marginal probability of winning at 0 is infinity (Hirshleifer, 1989). This is one of the main reasons why Hirshleifer criticizes the ratio form: peace is never observed as an equilibrium outcome (Hirshleifer, 1991, p.132). This property also carries over to the class of CEA contest success functions with $\rho > 0$ since the derivative of u_ρ with respect to x_1 at $(0, 0)$ is also infinity. However, as we shall see shortly, when ρ is close to zero, the unique interior pure strategy Nash equilibrium is close to $(0, 0)$.

At a symmetric equilibrium, the winning probabilities for both parties are the same; i.e., $u_\rho = 1/2$. Had we used the difference form instead of the CEA function (or set $\rho = 0$), the left hand side of (14) would not have depended on x_1 . Because of this an interior symmetric equilibrium fails to exist in the difference form, and this accounts for the more popular use of the ratio form in conflict models. However an interior symmetric equilibrium exists in the CEA function with for all $\rho > 0$. Proposition 3 exhibits one sufficient condition for the existence of such an equilibrium. If a symmetric equilibrium (x_1^*, x_2^*) exists, from (14)

$$x_1^* = x_2^* = \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1}{\rho}} \quad (15)$$

Proposition 3. *Suppose that $(\frac{\kappa\bar{x}}{2})^{\frac{1-\rho}{\rho}} < \frac{1}{\kappa} \min\{\rho, 2\}$ and $0 < \rho < 1$. Then there exists an interior symmetric Nash Equilibrium.*

Proof. First from $(\frac{\kappa\bar{x}}{2})^{\frac{1-\rho}{\rho}} < \frac{2}{\kappa}$, we have $x_1^* = (\frac{\kappa\bar{x}}{2})^{\frac{1}{\rho}} < \bar{x}$. To show this is a

Nash equilibrium, we show that

$$\frac{\partial^2}{\partial x_1^2} \pi_1(x_1, x_2^*) \leq 0 \text{ for all } x_1.$$

We apply (9) twice to find

$$\frac{\partial^2}{\partial x_1^2} \pi_1(x_1, x_2) = 2\bar{x} u_\rho(1 - u_\rho) \frac{\kappa}{x_1^{\rho+1}} \left((1 - 2u_\rho)x_1^{1-\rho} - \frac{\rho}{\kappa} \right)$$

Now if $x_1 \geq x_2$, then $1 - 2u_\rho \leq 0$, so $\frac{\partial^2}{\partial x_1^2} \pi_1(x_1, x_2) \leq 0$. When $x_1 < x_2^* = \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1}{\rho}}$,

$$(1 - 2u_\rho)x_1^{1-\rho} - \frac{\rho}{\kappa} \leq x_1^{1-\rho} - \frac{\rho}{\kappa} \leq \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1-\rho}{\rho}} - \frac{\rho}{\kappa} \leq 0$$

from the hypothesis. ■

We can view the equation in (15) as a generalization of the solution in the case of the ratio form (For example, see equation (10) in Garfinkel and Skaperdas, 2007). Moreover we verify that $\lim_{\rho \rightarrow 0} x_1^* = 0$ if $\kappa\bar{x} < 2$; in the limiting case approaching the difference form, an interior pure strategy Nash equilibrium converges to 0. So when ρ is arbitrary close to 0, the contest success function behaves like the difference form, but admits an interior Nash equilibrium. Although the hypothesis of Proposition 3 requires $\rho < 1$, this condition is not necessary as the numerical example in Figure 1 shows.

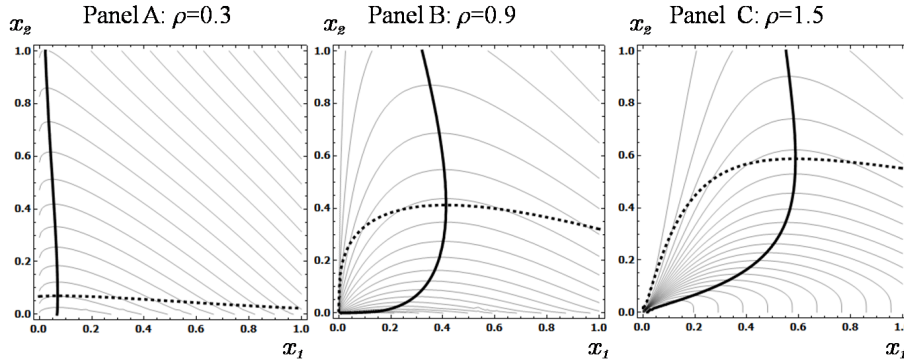


Figure 1: **Existence of an interior pure strategy Nash equilibrium.** We draw party 1's best response (thick line), party 2's best response (dashed line), party 1's indifference curves (thin line) in each panel. We use values, $\kappa = 0.9$, $\bar{x} = 1$.

3. Empirical Evidence

3.1. Estimation Method

To determine the appropriate contest success function for a specific conflict situation, we estimate the elasticity of augmentation using battle data from seventeenth century European wars provided in Bodart (1908, pp. 49-177) and from World War II in Dupuy (1987, pp. 293-295). We also estimate the difference and ratio forms for comparison. Military combat, as a violent, planned form of physical interaction between two hostile opponents, includes hierarchically war, campaign, battle, engagement, and duel (Dupuy, 1987). Among these we focus on two levels, war and battle. A war is an armed conflict or a state of belligerence usually lasting for months or years, while a battle involves combat between two armies with specific missions, normally lasting one or two days. The data set of wars in seventeenth century Europe contains data on 315 battles and each observation has a record of the winner, the loser, and the total numbers of personnel in winner and loser armies.

We observe that each battle gives two pieces of information: the winning probability of the winner and the losing probability of the loser. Because of this dual interpretation, we construct two data sets from the original data. In the first construction – the case presented in the text – we associate each battle with either a winning event or a losing event depending on a random draw. Alternatively, we expand the original battle data such that each battle represents both winning and losing events, hence obtaining a new data set with doubled observations. In this case, presented in the Appendix, we correct standard errors by clustering battles. We provide descriptive statistics for European battles in the Appendix.

Denoting the indicator of winning by y_i we use the following econometric models:

$$y_i \sim \text{Bernoulli}(\pi_i) \tag{16}$$

$$\begin{aligned} \text{(D)} \quad \pi_i &= F(\kappa(x_{1i} - x_{2i}) + \beta_1 + \beta_2 D_{\text{army } i}) \\ \text{(R)} \quad \pi_i &= F(\kappa(\ln x_{1i} - \ln x_{2i}) + \beta_1 + \beta_2 D_{\text{army } i}) \\ \text{(C)} \quad \pi_i &= F(\eta(x_{1i}^{1-\rho} - x_{2i}^{1-\rho}) + \beta_1 + \beta_2 D_{\text{army } i}), \end{aligned}$$

where $F(s) = 1/(1 + e^{-s})$, D_{army} is a dummy variable indicating which countries the armies in each observation belong to (see the Appendix).

The specifications of the models in equation (16) are the direct consequence of Theorem 2 and the dummy variable controls for combat effectiveness depending on country. Indeed using F and $\eta = \kappa(1 - \rho)$ we can write model **(C)** as

$$\pi_i = \frac{\exp^{\beta_1 + \beta_2 D_{army\ i}} \exp(\kappa \frac{1}{1-\rho} x_{1i}^{1-\rho})}{\exp^{\beta_1 + \beta_2 D_{army\ i}} \exp(\kappa \frac{1}{1-\rho} x_{1i}^{1-\rho}) + \exp(\frac{1}{1-\rho} x_{2i}^{1-\rho})}$$

so we can regard the term $\exp^{\beta_1 + \beta_2 D_{army\ i}}$ as a ratio of γ_i 's, γ_1/γ_2 , in (12). Note that model **(D)** is a standard logit regression and model **(R)** corresponds to the logit regression with the data log-transformed. Model **(D)**, **(R)**, and **(C)** estimate the difference form, the ratio form, and the CEA function, respectively. We estimate each parameter using the maximum likelihood method, which is the standard method in estimating logit models. We could not estimate the difference form and the elasticity of augmentation in the case of World War II, since the data provides only the ratios of combat powers.

3.2. Estimation Results

In Table 1 we note that all estimates of κ in the difference and ratio forms are positive, which shows that the estimated contest success functions satisfy the desirable property of contest success functions; i.e., a given party's winning probability is an increasing function of its own resources. In case of the CEA function we can recover the implied $\kappa = 3.03871$, using the relation $\kappa = (1 - \rho)\eta$. According to estimates of dummy variables, the English army fought most effectively in wars of seventeenth century Europe; the German army was more effective than the Allied army during World War II.

To compare κ 's in each model we compute one side's marginal probability of winning at an evenly balanced match when the number of combatants is half of the total available resources; i.e., if we denote the total available resources by \bar{x} , the marginal probability of winning is $0.25 \kappa / (\bar{x}/2)^\rho$ from equation (10). Since the mean number of combatants in the seventeenth century Europe wars is 21,035 (see the Appendix), we can use this number as a proxy for $\bar{x}/2$. Table 2 shows this computation.

We may interpret these numbers as follows: in response to an increase

	17C European War		CEA	World War II
	Difference	Ratio		Ratio
κ	$2.08 \times 10^{-5}^{**}$ (9.28×10^{-6})	0.750^{***} (0.123)		4.335^{***} (0.609)
η			-20.394* (12.095)	
ρ			1.149*** (0.214)	
French	1.193** (0.312)	1.199*** (0.334)	1.178*** (0.335)	
Imperial	0.871*** (0.323)	0.961*** (0.339)	0.954*** (0.338)	
Swedish	1.510*** (0.476)	1.812*** (0.499)	1.810*** (0.505)	
Spanish	-0.570* (0.326)	-0.638* (0.326)	-0.643* (0.354)	
English	1.842*** (0.556)	1.836*** (0.646)	1.826*** (0.643)	
Allied				0.614** (0.326)
constant	-0.657*** (0.242)	-0.669*** (0.245)	-0.652*** (0.245)	-1.227*** (0.461)
Number of Observations	315	315	315	186
Percentage of Correctly Predicted	65.40	69.84	69.52	84.95
Log-likelihood Value	-192.879	-179.090	-178.830	-59.890

Table 1: **Estimation of contest success functions.** The dummy variables in the European war estimation are “French”, “Imperial”, “Swedish”, “Spanish”, “English”. World war II data is from Dupuy (1986, p.293-5). We treat the 25th observation (Monte Camino II) as an outlier, so exclude it in the estimation. The dummy variable indicates whether the army is Allied army or not (so German army). Standard errors, in parentheses, are corrected for heteroskedasticity. *, **, and *** denote 90%, 95%, 99% significance level respectively.

	D	R	CEA
marginal winning probability $\times 10^4$	0.052	0.0891	0.0820

Table 2: **Estimates of the mass effect parameter**

of 10,000 in the number of combatants (from 21,035 original combatants), the difference form, the ratio form, and the CEA function predict increases in winning probabilities by 5.2%, 8.91%, 8.2%, respectively. In the case of World War II, using the fact that the average strength of armies in battles is around 14,000 (Dupuy, 1987, pp. 169) we obtain 0.774 (or 77.4%) which is much greater than those of the wars in seventeenth century Europe. This fact suggests that the contest success function for World War II is more “non-linear” than the one for the wars of seventeenth century Europe and “the tremendous advantage of being even just a little stronger than one’s opponent”, which Hirshleifer (1991, p 131) identifies as one of the stylized facts of warfare, seems to exist only during World War II.

What is the appropriate contest success function for the battles of seventeenth century Europe? As Table 1 as well as other alternative estimates in the Appendix show, the elasticity of augmentation was estimated to be slightly greater than unity. So a side in the battles of seventeenth century Europe would have needed to augment its resource more than proportionally in order to obtain the same winning probability in response to a given percentage increase in its opponent’s resources. In addition, between two existing forms, the estimations suggest that the ratio form describes better the technology of conflict in the wars of seventeenth century Europe than the difference form does.

In the paper we have axiomatically and stochastically derived a new class of contest success functions which has as special cases the two existing forms of the contest success functions: the difference and ratio forms attributed to Hirshleifer and Tullock. In addition we have shown that the CEA function has desirable analytical properties in a widely used conflict model, and hence provides flexibility in modeling conflict situations. We also estimate the CEA contest success function and the elasticity of augmentation by using data from battles fought in seventeenth century Europe and battles during World War II.

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Appendix: Lemma 4, Proof of Theorem 2, and Tables

In Theorem 2 we use the following definition of the rational power of real numbers, $s^{\frac{n}{m}}$: for m, n natural numbers

$$s^{\frac{n}{m}} := \begin{cases} (\sqrt[m]{s})^n & \text{if } s \geq 0 \\ -(\sqrt[m]{-s})^n & \text{if } s < 0 \end{cases} \quad (\text{A.1})$$

where $\sqrt[m]{s}$, for $s > 0$, denotes a unique positive real number y such that $y^m = s$.

Lemma 4. *Suppose that for $i = 1, 2$, $\epsilon_i \sim \exp(-\gamma_i e^{-\kappa s})$ independently where $\gamma_i > 0, \kappa > 0$ and $-\infty < s < \infty$. Then $\epsilon_1 - \epsilon_2 \sim \frac{\gamma_1}{\gamma_1 + \gamma_2 e^{-\kappa s}}$.*

Proof. It is easy to check $\Lambda(s) := \frac{\gamma_1}{\gamma_1 + \gamma_2 e^{-\kappa s}}$ is a distribution function. From the definition of ϵ_i , we have $\Pr\{\epsilon_2 \in ds\} = \exp(-\gamma_2 e^{-\kappa s}) \gamma_2 \exp(-\kappa s) \kappa ds$. Hence from the definition of conditional probability and the independence between ϵ_1 and ϵ_2 , we have

$$\begin{aligned} \Pr\{\epsilon_1 < \epsilon_2 + x\} &= \int_{-\infty}^{\infty} \Pr\{\epsilon_1 < s + x\} \Pr\{\epsilon_2 \in ds\} \\ &= \int_{-\infty}^{\infty} \exp(-\gamma_1 e^{-\kappa s - \kappa x}) \exp(-\gamma_2 e^{-\kappa s}) \gamma_2 \exp(-\kappa s) \kappa ds \\ &= \int_0^{\infty} \exp(-t(\gamma_1 e^{-\kappa x} + \gamma_2)) \gamma_2 dt \\ &= \frac{\gamma_2}{\gamma_1 e^{-\kappa x} + \gamma_2} \end{aligned}$$

We use the change of the variable, $t = \exp(-\kappa s)$, in the third line and the asserted claim follows. ■

Proof of Theorem 2. From (A.1) we see that $\frac{1}{1-\rho} s^{1-\rho}$ is continuous, increasing and $\frac{1}{1-\rho} s^{1-\rho} \rightarrow -\infty$ as $s \rightarrow -\infty$ and $\frac{1}{1-\rho} s^{1-\rho} \rightarrow \infty$ as $s \rightarrow \infty$, so $F(s)$ is indeed a distribution function. Since $\Pr\{h_1(\rho) > h_2(\rho)\} = \Pr\{\frac{1}{1-\rho} x_1^{1-\rho} - \frac{1}{1-\rho} x_2^{1-\rho} > \frac{1}{1-\rho} \epsilon_2^{1-\rho} - \frac{1}{1-\rho} \epsilon_1^{1-\rho}\}$, in the view of McFadden (1974) or Lemma 4, it is enough to show that $\frac{1}{1-\rho} \epsilon_1^{1-\rho} \sim \exp(-e^{-\kappa s})$. Again from (A.1) and the definition of $F(s)$ we have

$$\Pr\{\frac{1}{1-\rho} \epsilon_1^{1-\rho} < s\} = \Pr\{\epsilon_1 < \left(\frac{n}{m} s\right)^{\frac{m}{n}}\} = \exp(-e^{-\kappa s}).$$

■

War	Number of Battles
War of the Spanish Succession	108
Thirty Years' War	64
Austro-Turkish War	34
Great Northern War	29
Dutch War	19
War of the League of Augsburg	18
Other wars	43

Table A1: **17th century European wars.** Other wars include wars with less than ten battles. These are the Turkish War with Venice and Austria, English Civil War, Hungarian-Turkish War, Polish-Turkish War, Second English Civil War, The Fronde, War of the Quadruple Alliance, Polish-Swedish War, Spanish-Portuguese War, Swedish-Danish War, The First Northern War, War of Devolution, Chamber of Reunion, English Scottish War, Franco-Spanish War, Moldavian Campaign, Monmoth's Rebellion, Polish Insurgency, and Turkish-Venitian War. The classification of war is based on Dupuy and Dupuy (1986) and Palmer and Colton (1984).

Army	Number of engagements (Data set used in the text)	Number of engagements (Expanded data set)
French	84	145
Imperial	85	156
Swedish	26	61
Spanish	51	57
Turkish	23	49
English	26	47
Dutch	30	22
Russian	10	20
Others	116	116

Table A2: **Army by country.** The numbers count battles in which each army fought either as a winner or a loser. Note that the total sums of the numbers are greater than 315 and 630 respectively, since two or more armies often formed allies in battles. Others include Piemontese, Danish, Hungarian (Insurgents), Polish, Saon, Brandenburg-Prussia, Bavarian, Austrian, English (Parliament), German (Insurgents), Hungarian, English (Royal), French (Royalist), Portugues, Scottish, Siebenbuerger, Bohemian (Insurgents), German (States), Venetian, Bavarian (Insurgents), Cossack, English (Insurgents), French (Insurgents), Frondeur, German (Insurgents), Irish, Polish (Insurgents), Polish (Royal), Scottish (Insurgents), Spanish (Insurgents), Tatar. Each dummy variable assumes a value of 1 when the corresponding army belongs to the winning side. In the estimation presented in the text, we drop Turkish, Dutch, and Russian dummy variables since these variables turned out to be insignificant.

Statistics	Number of Personnel
Mean	21035
Maximum	260000
Minimum	1000
Standard Deviation	24047
Median	15000

Table A3: **Descriptive statistics for number of personnel involved in 17th century European war.**

Statistics	Combat Power Ratio
Number of observations	186
Mean	1.422
Maximum	7.54
Minimum	0.133
Standard Deviation	1.302
Median	1

Table A4: **Descriptive statistics for combat power ratio in World War II data.**

	17C Europe War		
	Difference	Ratio	CEA
κ	$2.08 \times 10^{-5**}$ (9.30×10^{-6})	$0.757***$ (0.122)	
η			-22.353 (17.783)
ρ			$1.177***$ (0.211)
French	$1.110***$ (0.215)	$1.200***$ (0.240)	$1.193***$ (0.240)
Imperial	$0.798***$ (0.221)	$0.927***$ (0.240)	$0.925***$ (0.239)
Swedish	$1.254***$ (0.332)	$1.591***$ (0.374)	$1.591***$ (0.379)
English	$1.563***$ (0.350)	$1.536***$ (0.399)	$1.536***$ (0.400)
constant	$-0.752***$ (0.122)	-0.827 (0.137)	-0.824 (0.137)
Number of Observations	630	630	630
Percentage of Correctly Predicted	66.35	69.84	73.96
Log-likelihood Value	-396.34723	-367.320	-354.55208

Table A5: **Alternative data set.** Each battle represents both a winning event and a losing event. The standard errors, in parentheses, are corrected for heteroskedasticity and clustering.

	Model I	Model II	Model III	Model IV
η	-193.696 (143.619)	-24.656 (21.522)	-63.534 (121.622)	-22.822 (16.134)
ρ	1.004*** (0.003)	1.198*** (0.203)	1.337*** (0.307)	1.168*** (0.203)
French	1.217*** (0.340)	1.255*** (0.347)	1.440*** (0.492)	1.450*** (0.431)
Imperial	0.964*** (0.344)	0.946*** (0.335)	0.942* (0.482)	1.108*** (0.404)
Swedish	1.854*** (0.503)	1.748*** (0.513)	1.746** (0.687)	2.095*** (0.573)
Spanish	-0.636* (0.359)	-0.642* (0.354)	-0.783 (0.505)	-0.496 (0.371)
English	1.858*** (0.656)	1.868*** (0.636)	2.512*** (0.875)	2.023*** (0.689)
Austro-Turkish War				0.267 (0.552)
Dutch War				-0.481 (0.679)
Great Northern War				-0.480 (0.599)
Thirty Years' War				-0.502 (0.504)
War of the League of Augsburg				0.311 (0.623)
War of the Spanish Succession				-0.677 (0.494)
Garrison A		-1.465 (1.515)		
Garrison B		-0.579 (0.246)		
constant	-0.687*** (0.249)	-0.579** (0.247)	-0.792** (0.404)	-0.469 (0.358)
Number of Observations	308	315	188	315
Log-likelihood Value	-173.294	-177.110	-95.492	-175.593

Table A6: **Alternative estimations.** Model 1: we exclude observations with armies of size greater than 100,000; Model 2: some observations indicate that the battle took place in a garrison. We use the dummy variable indicating this.; Model 3: we include only battles among eight major armies: French, Imperial, Swedish, Spanish, Turkish, English, Dutch, Russian.; Model 4: we include dummy variables for wars. The standard errors, in parentheses, are corrected for heteroskedasticity.

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