# The Logical Analysis of Plurals and Mass Terms

## Godehard Link (1983)

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## 0 Overview

**Main contribution of the paper:** Extend the logical language presented in PTQ to cover plural and mass terms.

**The Plan:**

1. The ontology of plurals and mass terms.
2. The ingredients of LPM
3. Some applications to Montague Grammar
4. A brief final remark

## 1 The ontology of plurals and mass terms

**The Basic Question:** what do plural and mass nouns denote? Montague provided no account of plural and mass terms in his system, the domain of entities consisted only of a set of singular individuals.

1. Plural Terms

(1) a. *The kid met together
    b. The kids met together $\not\Rightarrow$ each one of the kids met together
    c. The kids are boys $\Rightarrow$ each one of the kids is a boy

How do we account for these different facts?

2. Mass Terms

Plurals and mass nouns behave similarly in some respects.

(2) a. If $a$ is water and $b$ is water, then the sum of $a$ and $b$ is water.
    b. If $a$ are horses and $b$ are horses, then the sum of $a$ and $b$ are horses.

Link dubbed this property the *cumulative reference property* of plurals and mass terms.

**However:** if two expressions $a$ and $b$ refer to entities co-occurring in space and time but have different sets of predicates that can be true of them, then $a \neq b$.

*Example:* take a ring recently made up from some old Egyptian gold. Then *the gold* the ring is made of is old, whereas *the ring* itself is new.

*Example:* take a committee constituted by all the faculty members under 30. Then, if the committee was constituted many years ago it is true that the committee is old, whereas the faculty members are young.

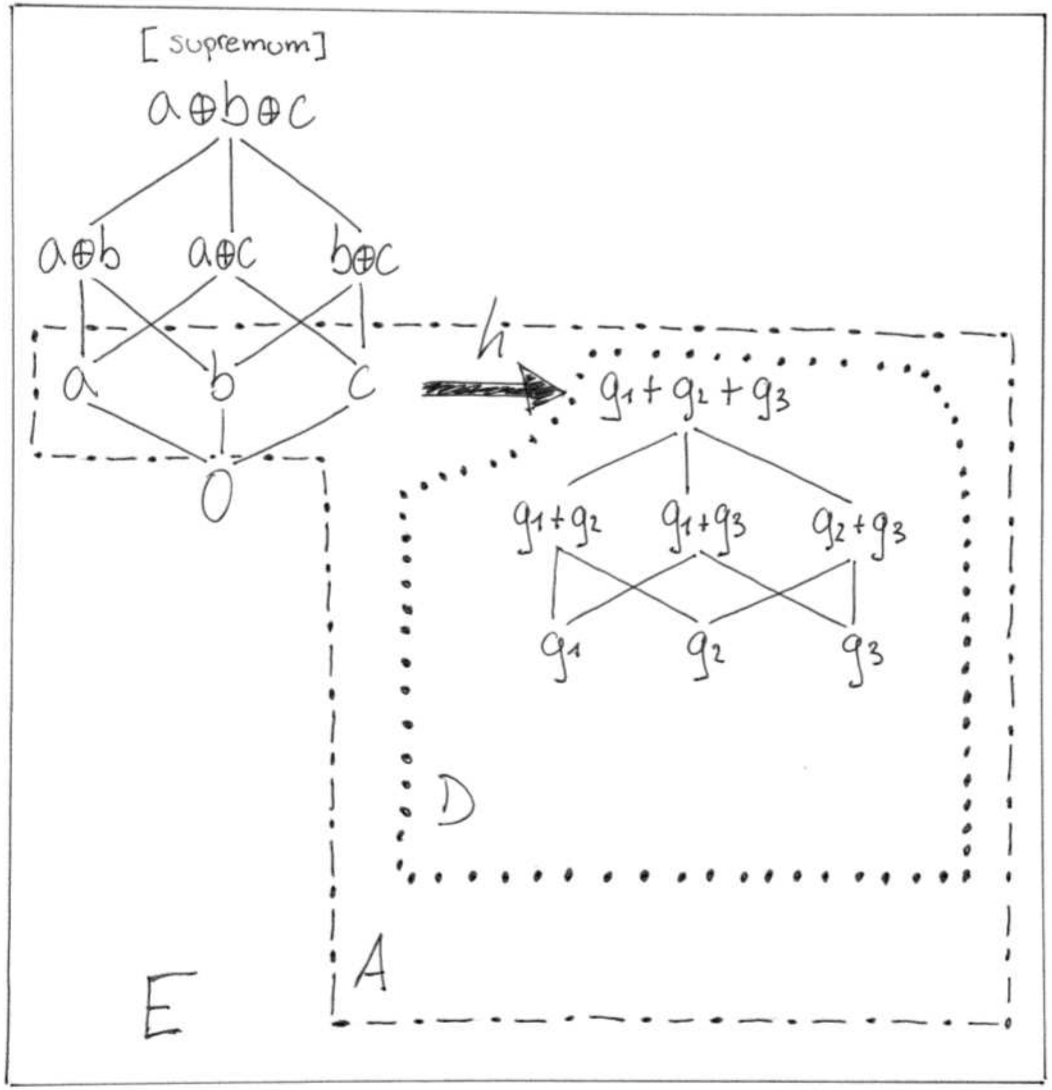
**Note:** the cumulative reference property behaves the same: if $a$ and $b$ are rings, the gold in $a$ and $b$ is still old, whereas the rings $a$ and $b$ are new.

The distinctiveness of these linguistic expressions points out that even if *the ring* and *the gold in the ring* share the portion of matter they are made of, they are not the same entity.

Link's proposal: enrich the -underlying- set theoretic metalanguage to account for properties like cumulative reference, but **without** appealing to sets.

**Idea 1:** within the domain of entities $E$ we can distinguish two different (sub)-domains: the domain $A$ of atomic individuals and the domain $D$ of "stuff" or portions of matter.

(3) $E$ as generated by the predicate $P$, where $P = \{a, b, c\}$:




$E$ can be defined as a (complete boolean) algebra closed under join: $\langle E, \oplus \rangle$. $A$, the set of atoms, and $D$, the set of individual portions of matter, are complete join-subsemilattices of $E$ generated by 1-place predicates $P$.

(4) <u>Ontology:</u>

a. Scattered Singular Entities:
$x, y \in D$ (*water, gold, air,* etc.)

b. Concrete Singular Entities:
$x, y \in A \setminus D$ (*the ring, the committee, the gold in this ring,* etc.)

c. Plural Entities:
$x, y \in E \setminus A$ (*the rings, the committees,* etc.)

**Note:**

1. Both mass terms and group nouns are *atoms*, they are different entities from the portions of matter or individuals that they are composed of (recall the example of *the ring* and *the committee above*).[1]
2. Plural entities are just *sums* of individuals (and not sets), as "concrete" as the individuals that serve to define them and of the same logical type.
3. Substances are abstract entities and cannot be defined in terms of their concrete manifestations (that's why *water* $\in D$ whereas *the water in this glass* $\in A$).
4. Joining together any two concrete individuals (in $E \setminus D$) returns a plural entity (in $E \setminus A$), whereas joining together two portions of matter (in $D$) returns another portion of matter, an atom in $D$.[2]
5. The null element 0 is assigned the role of dummy object to which no predicate applies and which can take care of denotation gaps in the theory (≈ our concept of garbage).

[1] In the case of group nouns, Link distinguishes to types of predicates: those that are permeable to the inner structure of group nouns and those that are not (*variant* vs. *invariant predicates*):

i. <u>Variant Predicate:</u>
The personnel committee is old
$\not\Rightarrow$ the professors is old

ii. <u>Invariant Predicate:</u>
The personnel committee met
$\Rightarrow$ the professors met

[2] Graphically this is represented by appealing to two different instantiations of the join operator '⊕' for concrete individuals and '+' for portions of matter.

6. Even though $D \subseteq A \subseteq E$, we cannot extend the algebra of the domain of individuals down to the algebra of portions of matter; they are different structures.

**Idea 2:** Elements in $E \setminus D$ and $D$ are connected by a *constitution relation*.

$h$ is the "materialization function" denoting the constitution relation. It maps any individual of $E$ into its corresponding portion of matter in $D$.

*Example:* the department of linguistics has two different committees: the personnel committee $c_1$ and the curriculum committee $c_2$. Both committees are formed by all the professors in the department ($p_1 \oplus p_n$), but they have different chairs, board members and secretaries; they are different "representative bodies", therefore, $c_1 \neq c_2 \neq p_1 \oplus p_n$.

(5) a. The personnel committee met yesterday
$\not\Rightarrow$ the curriculum committee met yesterday
b. The professors in the department are young
$\not\Rightarrow$ the curriculum committee is young

However, the materialization function maps every individual into its corresponding material parts. Thus, if follows that:

(6) $h(p_1 \oplus p_n) = h(c_1) = h(c_2)$

The same holds for the example with the rings:

*Example:* two rings, $r_1$ and $r_2$ are made of portions of old Egyptian gold $g_1$ and $g_2$ respectively. Then, the rings $r_1 \oplus r_2$ are made of the portions of matter in $g_1 + g_2$.

(7) a. $g_1 + g_2 = h(r_1 \oplus r_2)$
b. $g_1 + g_2 \neq r_1 \oplus r_2$

$g_1 + g_2$ is the "material fusion" of $g_1$ and $g_2$, it's still a scattered singular entity, whereas $r_1 \oplus r_2$ is the plural entity composed of both rings. Even though $g_1 + g_2$ and $r_1 \oplus r_2$ have the same "materialization", the theory is such that $g_1 + g_2$ constitutes but is not equal to $r_1 \oplus r_2$.

# 2 The Ingredients of LPM

## 2.1 Individuals

Plural morphology signals the presence of a pluralization operation '*' which generates all the individual sums of members of the extension of any 1-place predicate $P$.

That is, $E$ is closed under the join operation: $\langle E, \oplus \rangle$, where $a \oplus b$ is the "individual-sum" (i-sum) $a$ and $b$.

Sums are partially ordered through an ordering relation $\leq_i$ on $E$ expressed in the object language by the two place relation $\prod$: the *i(ndividual)-part* relation. $\prod$ is interpreted in the semantics as the Boolean relation $\sqcup_i$ (where $\sqcup_i$ denotes the join operation $\oplus$).

(8) a. $a \prod b \leftrightarrow a \oplus b = b$ iff $a \leq_i b$
b. $[\![a]\!] \leq_i [\![b]\!]$ iff $[\![a]\!] \sqcup_i [\![b]\!] = [\![b]\!]$
c. $[\![a \prod b]\!] = 1$ iff $[\![a]\!] \neq 0$ and iff $[\![a]\!] \leq_i [\![b]\!]$ (by (8b))
iff $[\![a]\!] \sqcup_i [\![b]\!] = [\![b]\!]$ (by def. of '$\sqcup_i$')
iff $[\![a \oplus b = b]\!]$

So, the semantic interpretation of plurals is as follows:

(9) a. $[\![{}^*P]\!] := [[\![P]\!]]$ *the complete join$_i$-lattice generated by P*
b. $[\![{}^*P]\!] := \{x \in E | \exists X \subseteq [\![P]\!] \ \& \ X \neq \emptyset \text{ st. } x = sup_i X\}$

If $P$ is a 1-place predicate and ${}^*P$ is the corresponding plural predicate, then we can define ${}^{\circledast}P$, the "proper plural predicate" of $P$.

(10) a. $[\![{}^{\circledast}P]\!] := [\![{}^*P]\!] \setminus A$
b. $\sigma x Px := \iota x({}^*Px \wedge \forall y({}^*Py \to y \prod x))$ *the sum of P's*
c. $\sigma^* x Px := \iota x({}^{\circledast}Px \wedge \forall y({}^*Py \to y \prod x))$ *the proper sum of P's*

$\sigma^* x Px$ carries the presupposition that there at least two $P$'s; in this case $\sigma^* x Px$ and $\sigma x Px$ coincide.

(11) a. $[\![\sigma x Px]\!] = \text{sup}_i [\![P]\!]$, where $\text{sup}_i \emptyset = 0$
b. $[\![\sigma^* x Px]\!] = [\![\sigma x Px]\!]$ if $[\![P]\!]$ has $>$ 2 elements, 0 otherwise

## 2.2 Stuff

$D$ is closed under join making $D$ a complete (but not-necessarily atomic) join-semilattice.

Like $E$, $D$ is closed under the join operation '$+$' on $D$ (denoted by $\sqcup_m$). Just like with individuals, there is a corresponding ordering relation called the "material part" (m-part) relation, denoted by "$\top$" (equivalent to $\prod$ in the count domain). It establishes a preorder (it's not anti-symmetric) on portions of matter $\leq_m$ (akin to $\leq_i$). If two objects are m-parts of each other then they are "materially equivalent".

(12) a. $a \prod b \vDash a \top b$
b. If $a \top b$ and $b \top a$, then $a$ is *materially equivalent* to $b$
c. $[\![x]\!] \leq_m [\![y]\!]$ iff $[\![x]\!] \sqcup_m [\![y]\!] = [\![y]\!]$

Semantically, $\top$ can be defined in terms of the materialization function $h$:

(13) $[\![a \top b]\!] = 1$ iff $[\![a]\!], [\![b]\!] \neq 0$ and $h([\![a]\!]) \leq_m h([\![b]\!])$
and $[\![a]\!] \leq_m [\![b]\!]$ if $a, b \in D$

Under the assumption that $a, b \in D$ semantic fact above follows trivially, given that $h$ denotes the identity function on $D$. Building on $h$ too, we can provide a semantic interpretation for the constitution relation, denoted by '$\triangleright$' in the object language and define the concept of material fusion:

(14) a. $[\![a \triangleright b]\!] = 1$ iff $[\![b]\!] \neq 0$ and $[\![a]\!] = h([\![b]\!])$ *constitution relation*
b. $[\![a + b]\!] = h([\![a]\!]) \sqcup_m h([\![b]\!])$ if $[\![a]\!] \neq 0$ and $[\![b]\!] \neq 0$ *material fusion*

Thus, if $P$ is a mass term, $[\![P]\!]$ is a number/quantity of portions of matter closed under join.

## 2.3 The relation between individuals and stuff

Interestingly, the material part-whole relation in (12c) can be used to order the individuals of $E$ materially: the semantic counterpart of the constitution relation '$\triangleright$' is precisely the semilattice homomorphism $h$ from $E \setminus 0$ to $D$.

(15) $h : E \setminus \{0\} \rightarrow D$ is a semilattice homomorphims st.
i. $h \upharpoonright D = id_D$ (for all $x \in D$, $h(x) = x$), and
ii. $h(\sup_B) = \sup h[B]$, for all $B \subseteq E \setminus \{0\}$

(16) The Homomorphic Relation:
a. $x \leq_i y \Rightarrow h(x) \leq h(y)$ $(\forall x, y \in E,$ where $x \neq 0)$
b. $x \leq_m y$ iff $h(x) \leq h(y)$ $(\forall x, y \in E \setminus \{0\})$

How is $h$ useful for linguistic analysis? Consider that every predicate $P$ has a *mass term correspondent* ${}^mP$.

(17) a. $[\![{}^mP]\!] = \sup h[[\![P]\!]]$ (compare with $[\![{}^*P]\!]$ in (9a))
$= \{x \in D | x \leq \sup h[[\![P]\!]]\}$
$= \{x \in D | \exists y \in [\![{}^*P]\!]$ st. $x \leq_m h(y)\}$
$= h(\sup_P)$

Note that if $P$ is already a mass term, it follows that $[\![P]\!] \subseteq [\![{}^mP]\!]$.

(18) a. There is apple in the salad
b. $\exists x({}^mPx \wedge Qx)$, where $P$ = *is an apple*, ${}^mP$ = *is apple* and $Q$ = *is in the salad.*

What is the relation between *the cards* and *the deck of cards*? We use the notion of "material fusion", denoted by the $\mu$-operator:

(19) $\mu x Px = \iota x(x \triangleright \sigma x Px)$

*Example:* Let $P$ = *is a card from one of the decks of cards* and $Q$ = *is a deck of cards*, then $\sigma x Px \neq \sigma x Qx$. And this is what we want, for if we have to decks of Bridge cards, then $\sigma x Px$ will contain 104 atoms, whereas $\sigma x Qx$ will contain only two. **However**, $\mu x Px = \mu x Qx$.

Note that the $\mu$-operator builds descriptions of concrete singular terms out of portions of matter:

(20) a. $[\![x \triangleright \sigma x Px]\!] = 1$ iff $[\![x \triangleright \sigma x Px]\!] \neq 0$ and $[\![x]\!] = h([\![x \triangleright \sigma x Px]\!])$
b. $\mu x Px = \iota x(h(\sigma x Px) \triangleright \sigma x Px)$

Similarly, recall the example of the rings: The constitution relation denoted by '$\triangleright$' relates *the gold or the ring* and *the ring*; if $r$ is a ring and $g$ is the gold in $r$, then $g \triangleright r$ (and same for plurals: $g_1 + g_2 \triangleright r_1 \oplus r_2$).

## 2.4 Some examples

(21) a. The child built the raft (*Px: x is a child*)
b. $\exists y(y = \iota x Px \wedge Qy)$ (*Qx: x built the raft*)

(22) a. The children built the raft
b. $\exists y(y = \sigma^* x Px \wedge Qy)$

(23) a. Tom and Jerry carried the piano (*t: Tom, j: Jerry*)
b. $P(t \oplus j)$ (*Px : x carried the piano*)

(24) a. John and Paul are pop stars and George is a pop star
b. ${}^{\circledast}P(j \oplus p) \wedge Pg \Rightarrow {}^{\circledast}P(j \oplus p \oplus g)$ (*Px : x is a popstar*)

(25) a. Water is wet
b. $\exists x(Px \rightarrow Qx)$ (*Px : x is water*)

(26) a. The water of the Rhine is dirty
b. $Q(\mu x Px)$ ($Px$ : *x is Rhine water*)

(27) a. The gold in the ring is old, but the ring is not old
b. $Q\iota x(Px \wedge x \rhd a) \wedge \neg Qa$ ($Px$ : *x is gold*, $Qx$ : *x is old*, *a: the ring*)

# 3 Application to PTQ

With these ingredients, Link builds a model $\mathcal{M}$ for LPM: an ordered pair $\langle \mathfrak{B}, [\![\cdot]\!] \rangle$ st.

1. $\mathfrak{B} = \langle E, A, D, h \rangle$ is a tuple where $E$ is the domain of individuals in $\mathcal{M}$, $A$ is the set of atoms in $\mathcal{M}$, $D$ is the set of portions of matter in $\mathcal{M}$ and $h$ is the materialization function in $\mathcal{M}$.

2. $[\![\cdot]\!]$ is a first order assignment of denotations to the primitive expressions of LPM.

The syntax and semantics are as in PTQ, with the following additions:

1. The category CN is split into MCN (mass NPs), SCN (singular count NPs) and PCN (plural count NPs).

2. There is plural rule, where $\zeta \in P_{SCN} \Rightarrow \zeta_{pl} \in P_{PCN}$.

So, he defines an Interpretation: $\langle E, A, D, h, I, J, [\![\cdot]\!] \rangle$, as defined above and in PTQ. Here are some translations for quantifiers: let **U** be the translation relation:

(28) a. *the* **U** $\lambda Q \lambda P \exists x[Q(x) \wedge P(x) \wedge \forall y[Q(y) \rightarrow y \prod x]]$
b. *some*, $\emptyset_{pl}$ **U** $\lambda Q \lambda P \exists x[Q(x) \wedge P(x) \wedge]$

Both *some* and *the* can apply to singular and plural phrases:

(29) a. *some child* **U** $\lambda P \exists x[child'(x) \wedge P(x)]$
b. *(some) children* **U** $\lambda P \exists x[{}^{\circledast}child'(x) \wedge P(x)]$
c. *(some) water* **U** $\lambda P \exists x[water'(x) \wedge P(x)]$

**Note:** it is only the CN phrase which differentiates between the appropriate singular and plural readings. As a consequence *some* can combine with conjoined CNs.

(30) a. ($\zeta$ and $\eta$) **U** $\lambda z \exists x \exists y[\zeta'(x) \wedge \eta'(y) \wedge z = x \oplus y]$
b. boy and girl who dated each other
c. $\lambda z \exists x \exists y[boy'(x) \wedge girl'(y) \wedge z = x \oplus y \wedge$ dated-each-other$(z)]$

This shows how pluralization has the force of group formation. As a consequence, we can combine phrases like ($\zeta$ and $\eta$) with quantifiers like *some* and get the correct interpretation.

(31) some ((boy and girl) such that they met) **U**
$\lambda P \exists z[\exists x \exists y[boy'(x) \wedge girl'(y) \wedge z = x \oplus y \wedge$ meet'$(z)] \wedge P(z)]$

# 4 Wrapping Up

## 4.1 One final remark

Link's view on plurals is exclusive, that is, plurals don't include atoms ($[\![{}^{\circledast}P]\!] = [\![{}^{*}P]\!] \setminus A$). However, from a linguistic pv., this is problematic:

i. Downward entailing cotexts:

(32) a. No professors are in class
b. Are there professors in class?

ii. Plural NPs with quantities less than 1:

(33) Put 0.5 grams of salt in the soup

## 4.2 Summary

1. The logic of plurals and mass nouns share a common structure, a lattice, the difference between both lattices being that the former is atomic, whereas the latter is not.

2. The star operator allows us to treat plural morphology compositionally is a way such that the parallels between plurals and mass terms are captured.

3. Plural entities and group nouns are equivalent in that they are interchangeable in some environment (with *invariant* predicates, see fn.1), but this not make them coreferential. This contrasts with reductionists approaches where both *the cards* and *the deck of cards* denote the same set.

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