

First Order Logic: Formal Semantics and Models ¹

(1) Goal for These Notes

Just as we did for PL, we need a mathematically rigorous explication of what it is for a sentence of FOL to be ‘true’.

- This will give us an appropriately precise definition of ‘entailment’, which will ultimately allow us to prove that FOL is sound and complete...

(2) Towards a Formalization of ‘Truth’ for FOL

Let’s begin by considering sentences of FOL *without quantifiers*, and how an informal semantics (‘key’) determines a truth-value for them.

Example Sentence: (Fa & ~Pbc)

Determining Its Truth (Informally)

a. *Step One: Consult the Key*

A ‘key’ for an FOL encoding maps individual constants of FOL to proper names, and predicate letters of FOL to properties/relations.

Fx: x is French a: Angelika Kratzer
Pxy: x is older than y b: Seth Cable c: Rajesh Bhatt

b. *Step Two: Consult the Facts*

Next, we consider whether the properties/relations denoted by the English expressions actually hold of the entities named by the English names.

- Angelika Kratzer is not French.
- Seth Cable is not older than Rajesh Bhatt

This determines the truth-value of the atomic sentences of FOL.

c. *Step Three:*

Finally, once we have the truth-values of the atomic sentences, the meanings of the (English) logical connectives determine a truth-value for the whole formula.

~Pbc = ‘It is not the case that Seth is older than Rajesh’ = *True*
Fa = ‘Angelika Kratzer is French’ = *False*
(Fa & ~Pbc) = *false*

¹ These notes are based upon material in the following required readings: Gamut (1991), Chapter 3 pp. 87-101; Partee *et al.* (1993), Chapter 7 pp. 140-152, Chapter 13 pp. 321-331.

It seems, then, that an informal interpretation of FOL determines a truth-value for an FOL sentence (without quantification) by

- *Mapping the individual constants to entities (indirectly, through English translations)*
- *Mapping the predicate letter to some set (property) or set of n -tuples (relation)*
- *For atomic sentences, the sentence is true if the individual(s) named by the constant are 'in' the set or relation denoted by the predicate letter.*
- *For complex sentences, the main connective determines the truth-value of the whole formula, based on the truth-values of the component formulae*

Thus, it seems we can develop a more abstract, mathematically precise conception of 'FOL interpretation' in the following way...

2. A Review of First Order Logic (FOL): Formal Semantics and Models ²

(7) Definition of a 'Model' for First Order Logic

A model \mathcal{M} is a pair $\langle D, I \rangle$ consisting of:

- a. A *non-empty* set D , called the 'domain of \mathcal{M} '
- b. A function I , whose domain is the individual constants and predicate letters, and whose range satisfies the following conditions:
 - (i) If α is an individual constant, then $I(\alpha) \in D$
 - (ii) If Φ is an n -ary predicate letter, then $I(\Phi) \subseteq D^n$

Note:

- If Φ is a unary predicate letter, then $I(\Phi) \subseteq D^1 = D$ (a subset of D)
- If Φ is a binary predicate letter, then $I(\Phi) \subseteq D^2 = D \times D$ (a set of pairs from D)
- If Φ is a ternary predicate letter, then $I(\Phi) \subseteq D^3 = D \times D \times D$ (a set of triples from D), *etc.*

(8) Illustration:

The following is a model of FOL $\langle \{ \text{Angelika, Seth, Rajesh} \}, I \rangle$, where I consists of at least the following mappings:

$I(a) = \text{Angelika}$ $I(b) = \text{Seth}$ $I(c) = \text{Rajesh}$

$I(F) = \emptyset$ $I(P) = \{ \langle \text{Angelika, Rajesh} \rangle, \langle \text{Angelika, Seth} \rangle, \langle \text{Rajesh, Seth} \rangle \}$

² My discussion here will assume prior familiarity with the formal semantics of First Order Logic. Students are referred to Partee *et al.* (1993), Chapter 7 for crucial background.

With this notion of ‘model’ in place, we can introduce the key formal semantic notion below:

(9) **Valuation of FOL, Relative to a Model (To Be Revised)**

Let \mathcal{M} be a model $\langle D, I \rangle$. Then the ‘valuation based on \mathcal{M} ’ ($V_{\mathcal{M}}$) is a function whose domain is the set of FOL sentences (without quantifiers), whose range is $\{0,1\}$, and which satisfies the conditions below:

- (i) If $\varphi = \Phi\alpha_1 \dots \alpha_n$, then $V_{\mathcal{M}}(\varphi) = 1$ iff $\langle I(\alpha_1) \dots I(\alpha_n) \rangle \in I(\Phi)$
- (ii) If $\varphi = \sim\psi$, then $V_{\mathcal{M}}(\varphi) = 1$ iff $V_{\mathcal{M}}(\psi) = 0$
- (iii) If $\varphi = (\psi \ \& \ \chi)$, then $V_{\mathcal{M}}(\varphi) = 1$ iff $V_{\mathcal{M}}(\psi) = 1$ and $V_{\mathcal{M}}(\chi) = 1$
- (iv) If $\varphi = (\psi \ \vee \ \chi)$, then $V_{\mathcal{M}}(\varphi) = 1$ iff $V_{\mathcal{M}}(\psi) = 1$ or $V_{\mathcal{M}}(\chi) = 1$
- (v) If $\varphi = (\psi \ \rightarrow \ \chi)$, then $V_{\mathcal{M}}(\varphi) = 1$ iff $V_{\mathcal{M}}(\psi) = 0$ or $V_{\mathcal{M}}(\chi) = 1$

(10) **Illustration**

Let \mathcal{M} be the model (partially) defined in (8). From (9), it follows that

- $V_{\mathcal{M}}(\text{Fa} \ \& \ \sim\text{Pbc}) = 1$ iff (by 9iii)
- $V_{\mathcal{M}}(\text{Fa}) = 1$ and $V_{\mathcal{M}}(\sim\text{Pbc}) = 1$ iff (by 9ii)
- $V_{\mathcal{M}}(\text{Fa}) = 1$ and $V_{\mathcal{M}}(\text{Pbc}) = 0$ iff (by 9i)
- $I(a) \in I(F)$ and $\langle I(b), I(c) \rangle \notin I(P)$ iff (by definition of I in (8))
- $\text{Angelika} \in \emptyset$ and $\langle \text{Seth}, \text{Rajesh} \rangle \notin \{ \langle \text{Angelika}, \text{Rajesh} \rangle, \langle \text{Angelika}, \text{Seth} \rangle, \langle \text{Rajesh}, \text{Seth} \rangle \}$

Thus, we can calculate that $V_{\mathcal{M}}(\text{Fa} \ \& \ \sim\text{Pbc}) = 0$

In this way, the definition of ‘model’ in (8) and ‘valuation’ in (9) mirrors the way our earlier ‘informal interpretation’ (key) maps the formula ‘(Fa & ~Pbc)’ to the truth-value *false*

PROBLEM: We forgot about quantifiers! How do we add them into the picture?

(11) **Key Problem: The Semantics of Formulae with Free Variables**

Consider an FOL sentence like ' $\exists xPx$ '.

- This is a complex formula made up of the quantifier ' $\exists x$ ' and the atomic formula ' Px '
- Moreover, we're going to want this formula to end up being entailed by ' Pa '.
- **Thus, we're going to want the truth-value of this formula to somehow be determined by *some kind of semantic value* that the formula ' Px ' has (and for that value to 'connected' with the truth of Pa in some fashion...)**
- ***BUT HOW IN GOD'S NAME DO WE DO THAT???***

(12) **A Naïve Intuition, to Get Us Started**

- a. In English, "Something is red" is true *iff* there is a particular thing x such that we could 'point to it' and truthfully say 'THAT is red'.
- b. In English, "Everything is red" is true *iff* for any thing in the world x , you could 'point to it', and truthfully say 'THAT is red'.

(13) **Building From That Naïve Intuition**

- a. $\exists x\varphi$ is true *iff* there is a thing α such that φ is true when x 'picks out' α
- b. $\forall x\varphi$ is true *iff* for any thing α , φ is true when x 'picks out' α

But what do we mean by 'when x picks out α '?!?

Well, this will be familiar to the semanticists...

(14) **Variable Assignment**

Let \mathcal{M} be a model $\langle D, I \rangle$. Then g is a *variable assignment (based on \mathcal{M})* if:

- (i) g is a function whose domain is the set of all the variables in FOL,
- (ii) and whose range is D

Note: g needn't be an injection; it could map two different variables to the same $\alpha \in D$!

(15) **Notation**

Let \mathcal{M} be a model $\langle D, I \rangle$, and let g be a variable assignment based on \mathcal{M} , and let $\alpha \in D$, and let v be a variable of FOL.

$g(v/\alpha)$ is the variable assignment exactly like g except (at most) that it maps v to α

Note: The following follow from the definition in (15)

- $g(x/a)(y/b)$ is just like g except (at most) that it maps x to a and y to b .
- $g(x/a)(x/b) = g(x/b)$

(16) **The Term ‘Term’**

t is a ‘term (of FOL)’ iff t is either a variable or an individual constant

(17) **Some More Notation**

Let t be a term, \mathcal{M} be a model $\langle D, I \rangle$ and g be a variable assignment based on \mathcal{M} . Then *the interpretation of t relative to \mathcal{M} and g* $[[t]]^{M,g}$ is defined as follows:

- If t is an individual constant, then $[[t]]^{M,g} = I(t)$
- If t is a variable, then $[[t]]^{M,g} = g(t)$

With these notions in place, we can revise our notion of ‘valuation’ in (9) so that it now can map every formula (without quantifiers) to truth values, even those with free variables!

(18) **Valuation of FOL, Relative to a Model and a Variable Assignment (To Be Revised)**

Let \mathcal{M} be a model $\langle D, I \rangle$ and g be a variable assignment (based on \mathcal{M}) Then the ‘valuation based on \mathcal{M} and g ’ ($V_{M,g}$) is a function whose domain is the set of FOL formulae (without quantifiers), whose range is $\{0,1\}$, and which satisfies the conditions below:

- If $\varphi = \Phi t_1 \dots t_n$, then $V_{M,g}(\varphi) = 1$ iff $\langle [[t_1]]^{M,g}, \dots, [[t_n]]^{M,g} \rangle \in I(\Phi)$
- If $\varphi = \sim\psi$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 0$
- If $\varphi = (\psi \ \& \ \chi)$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 1$ and $V_{M,g}(\chi) = 1$
- If $\varphi = (\psi \ \vee \ \chi)$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 1$ or $V_{M,g}(\chi) = 1$
- If $\varphi = (\psi \ \rightarrow \ \chi)$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 0$ or $V_{M,g}(\chi) = 1$

(19) **Illustration**

Let \mathcal{M} be the model (partially) defined in (8) and let g be the following variable assignment: $\langle x, \text{Rajesh} \rangle, \langle y, \text{Seth} \rangle, \langle z, \text{Angelika} \rangle$. From (18), it follows that

- $V_{M,g}(\sim Pxc) = 1$ iff (by 18ii)
- $V_{M,g}(Pxc) = 0$ iff (by 18i)
- $\langle [[x]]^{M,g}, [[c]]^{M,g} \rangle \notin I(P)$ iff (by definition in (17))
- $\langle g(x), I(c) \rangle \notin I(P)$ iff (by definition of g and I)
- $\langle \text{Rajesh}, \text{Rajesh} \rangle \notin \{ \langle \text{Angelika}, \text{Rajesh} \rangle, \langle \text{Angelika}, \text{Seth} \rangle, \langle \text{Rajesh}, \text{Seth} \rangle \}$

Thus, we can calculate that $V_{M,g}(\sim Pxc) = 1$

Our notion of ‘interpretation (valuation) with respect to a variable assignment’ explicates what we mean by ‘when a variable picks out α ’

- ‘ Φ is true when x ‘picks out’ α ’ \approx $V_{M,g}(\Phi) = 1$ if $g(x) = \alpha$

With this in mind we can extend our definition in (18) to include quantifiers!

(20) Extending (18) to Include Quantifiers

a. What We Ultimately Want: Two equations whose left hand side looks like:

- (vi) If $\varphi = \exists v\psi$, then $V_{M,g}(\varphi) = 1$ iff ...
- (vii) If $\varphi = \forall v\psi$, then $V_{M,g}(\varphi) = 1$ iff ...

b. Recall Our Naïve Intuition:

- (i) $\exists x\psi$ is true iff there is a thing α such that ψ is true when x ‘picks out’ α
- (ii) $\forall x\psi$ is true iff for any thing α , ψ is true when x ‘picks out’ α

c. Spelling Out the Naïve Intuition With Variable Assignments

- (vi) If $\varphi = \exists v\psi$, then $V_{M,g}(\varphi) = 1$ iff **there is an $a \in D$ such that $V_{M,g(v/a)}(\psi) = 1$**
- (vii) If $\varphi = \forall v\psi$, then $V_{M,g}(\varphi) = 1$ iff **for every $a \in D$, $V_{M,g(v/a)}(\psi) = 1$**

- In (vi) and (vii), we interpret ψ relative to the variable assignment $g(v/a)$; thus, we interpret ψ with ‘ v picking out a ’

(21) Valuation of FOL, Relative to a Model and a Variable Assignment (Final Version)

Let \mathcal{M} be a model $\langle D, I \rangle$ and g be a variable assignment (based on \mathcal{M}). Then the ‘valuation based on M and g ’ ($V_{M,g}$) is a function whose domain is the set of FOL formulae, whose range is $\{0,1\}$, and which satisfies the conditions below:

- (i) If $\varphi = \Phi t_1 \dots t_n$, then $V_{M,g}(\varphi) = 1$ iff $\langle [[t_1]]^{M,g}, \dots, [[t_n]]^{M,g} \rangle \in I(\Phi)$
- (ii) If $\varphi = \sim\psi$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 0$
- (iii) If $\varphi = (\psi \ \& \ \chi)$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 1$ and $V_{M,g}(\chi) = 1$
- (iv) If $\varphi = (\psi \ \vee \ \chi)$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 1$ or $V_{M,g}(\chi) = 1$
- (v) If $\varphi = (\psi \ \rightarrow \ \chi)$, then $V_{M,g}(\varphi) = 1$ iff $V_{M,g}(\psi) = 0$ or $V_{M,g}(\chi) = 1$
- (vi) **If $\varphi = \exists v\psi$, then $V_{M,g}(\varphi) = 1$ iff there is an $a \in D$ such that $V_{M,g(v/a)}(\psi) = 1$**
- (vii) **If $\varphi = \forall v\psi$, then $V_{M,g}(\varphi) = 1$ iff for every $a \in D$, $V_{M,g(v/a)}(\psi) = 1$**

(22) **Illustration**³

Let \mathcal{M} be a model $\langle \{\text{Dave, Joe}\}, I \rangle$, where $I(L) = \{ \langle \text{Dave, Joe} \rangle, \langle \text{Joe, Dave} \rangle \}$. Let g be a variable assignment such that $g(x) = \text{Dave}$ and $g(y) = \text{Joe}$.

a. Interpreting ‘ $\exists y Lxy$ ’ with respect to \mathcal{M} and g

1. $V_{M,g}(\exists y Lxy) = 1$ *iff* (by (21vi))
2. There is an $a \in D$ s.t. $V_{M,g(y/a)}(Lxy) = 1$ *iff* (by (21i))
3. There is an $a \in D$ s.t. $\langle [[x]]^{M,g(y/a)}, [[y]]^{M,g(y/a)} \rangle \in I(L)$ *iff*
4. There is an $a \in D$ s.t. $\langle g(y/a)(x), g(y/a)(y) \rangle \in I(L)$
iff (by def. of g and I)
5. There is an $a \in D$ s.t. $\langle \text{Dave}, a \rangle \in \{ \langle \text{Dave, Joe} \rangle, \langle \text{Joe, Dave} \rangle \}$

Thus, we can calculate that $V_{M,g}(\exists y Lxy) = 1$

b. Interpreting ‘ $\forall x \exists y Lxy$ ’ with respect to \mathcal{M} and g

1. $V_{M,g}(\forall x \exists y Lxy) = 1$ *iff* (by (21vii))
2. For every $a \in D$, $V_{M,g(x/a)}(\exists y Lxy) = 1$ *iff* (by (21vi))
3. For every $a \in D$, there is an $a' \in D$ s.t. $V_{M,g(x/a)(y/a')} (Lxy) = 1$
iff (by (21i))
4. For every $a \in D$, there is an $a' \in D$ s.t.
 $\langle [[x]]^{M,g(x/a)(y/a')}, [[y]]^{M,g(x/a)(y/a')} \rangle \in I(L)$ *iff*
5. For every $a \in D$, there is an $a' \in D$ s.t.
 $\langle g(x/a)(y/a')(x), g(x/a)(y/a')(y) \rangle \in I(L)$ *iff* (by def. of g and I)
6. For every $a \in D$ ($\{\text{Dave, Joe}\}$), there is an $a' \in D$ ($\{\text{Dave, Joe}\}$) s.t.
 $\langle a, a' \rangle \in \{ \langle \text{Dave, Joe} \rangle, \langle \text{Joe, Dave} \rangle \}$

Thus, we can calculate that $V_{M,g}(\forall x \exists y Lxy) = 1$

- We’ve found a viable way of using *models* to assign truth-values to formulae of FOL
- **So, we’ve also found a way of using models to assign truth-values to FOL sentences**
 - *So, models seem like an excellent characterization of ‘interpretation’ for FOL*

³ For another helpful illustration of the key definitions in (21), see Partee *et al.* (1993), Chapter 13 pp. 326-327.

3. Some Important Consequences and Notations

(23) Key Fact: Sentences of FOL and Variable Assignments

Let \mathcal{M} be any model and g, h be variable assignments based on \mathcal{M} . If φ is a sentence of FOL, then $V_{M,g}(\varphi) = V_{M,h}(\varphi)$

- That is, the truth-value of a *sentence* relative to a model and a variable assignment doesn't depend upon the variable assignment at all (since there are no free variables)
 - Notice how in (22b), we don't ever actually calculate the value of g for any variable.
 - Thus, even if we replaced g with some other variable assignment h in (22b), we'd still get the same result!...
- Consequently, if a sentence φ is true with respect to a model \mathcal{M} and a variable assignment g , **then it's true with respect to \mathcal{M} and any variable assignment h**

So, even though we need to refer to a variable assignment to calculate the truth-value of a sentence, the sentence's truth-value doesn't depend upon the assignment we pick...

For this reason, we can introduce the following terminology:

(24) Key Terminology: A Model of an FOL Sentence (Set of Sentences)

Let \mathcal{M} be a model, φ be a sentence of FOL, and S be a set of FOL sentences.

- φ is true in \mathcal{M} iff for any variable assignment g , $V_{M,g}(\varphi) = 1$
- \mathcal{M} is a model for φ iff φ is true in \mathcal{M}
- \mathcal{M} is a model for S iff for all $\varphi \in S$, \mathcal{M} is a model for φ

Also, if we want to – and some do – we can eliminate direct reference to $V_{M,g}$ in our semantics for FOL, by extending the notation ' $[[\cdot]]$ '^{M,g}, defined in (17)

(25) **Interpretation With Respect to a Model**

Let \mathcal{M} be a model $\langle D, I \rangle$ and g be a variable assignment based on \mathcal{M} . Then the interpretation (a.k.a. denotation) of an ‘expression’ of FOL relative to \mathcal{M} and g $[[\cdot]]^{\mathcal{M},g}$ is defined as follows:

- a. If v is a variable, then $[[v]]^{\mathcal{M},g} = g(v)$
- b. If α is an individual constant, then $[[\alpha]]^{\mathcal{M},g} = I(\alpha)$
- c. If Φ is a predicate letter, then $[[\Phi]]^{\mathcal{M},g} = I(\Phi)$
- d. If $\varphi = \Phi t_1 \dots t_n$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff $\langle [[t_1]]^{\mathcal{M},g}, \dots, [[t_n]]^{\mathcal{M},g} \rangle \in [[\Phi]]^{\mathcal{M},g}$
- e. If $\varphi = \sim\psi$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff $[[\psi]]^{\mathcal{M},g} = 0$
- f. If $\varphi = (\psi \ \& \ \chi)$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff $[[\psi]]^{\mathcal{M},g} = 1$ and $[[\chi]]^{\mathcal{M},g} = 1$
- g. If $\varphi = (\psi \vee \chi)$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff $[[\psi]]^{\mathcal{M},g} = 1$ or $[[\chi]]^{\mathcal{M},g} = 1$
- h. If $\varphi = (\psi \rightarrow \chi)$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff $[[\psi]]^{\mathcal{M},g} = 0$ or $[[\chi]]^{\mathcal{M},g} = 1$
- i. If $\varphi = \exists v\psi$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff there is an $a \in D$ such that $[[\psi]]^{\mathcal{M},g(v/a)} = 1$
- j. If $\varphi = \forall v\psi$, then $[[\varphi]]^{\mathcal{M},g} = 1$ iff for all $a \in D$, $[[\psi]]^{\mathcal{M},g(v/a)} = 1$

(26) **Notation**

Let φ be a sentence of FOL and \mathcal{M} be a model. The interpretation (a.k.a. denotation) of φ with respect to \mathcal{M} , $[[\varphi]]^{\mathcal{M}}$, is $[[\varphi]]^{\mathcal{M},g}$ for an arbitrary variable assignment g .

Consequences:

- φ is true in \mathcal{M} iff $[[\varphi]]^{\mathcal{M}} = 1$ iff \mathcal{M} is a model for φ
- \mathcal{M} is a model for S iff for all $\varphi \in S$, $[[\varphi]]^{\mathcal{M}} = 1$

4. Using Models to Define Semantic Concepts for FOL

With our formal characterization of ‘truth’ for FOL, we can give precise definitions to the concepts of ‘tautology’, ‘contradiction’, ‘logical equivalence’ and ‘entailment’....

These definitions are going to be restricted to sentences of FOL...

(27) **Tautology**

Let φ be a sentence of FOL. φ is a tautology *iff* for every model \mathcal{M} $[[\varphi]]^{\mathcal{M}} = 1$

Illustration: $\forall x(Px \vee \sim Px)$

Proof That It's a Tautology

Let \mathcal{M} be any model $\langle D, I \rangle$ and g be any variable assignment (based on \mathcal{M}).

- Suppose that $[[\forall x(Px \vee \sim Px)]]^{M,g} = 0$
- Then it's not the case that for all $a \in D$, $[[Px \vee \sim Px]]^{M,g(x/a)} = 1$
- Then there is some $a \in D$ such that $[[Px \vee \sim Px]]^{M,g(x/a)} = 0$.
- Then there is some $a \in D$ such that $[[Px]]^{M,g(x/a)} = 0$ and $[[\sim Px]]^{M,g(x/a)} = 0$
- Then there is some $a \in D$ such that $[[Px]]^{M,g(x/a)} = 0$ and $[[Px]]^{M,g(x/a)} = 1$
- Then there is some $a \in D$ such that $g(x/a)(x) \notin I(P)$ and $g(x/a)(x) \in I(P)$
- Then there is some $a \in D$ such that $a \notin I(P)$ and $a \in I(P)$ **CONTRADICTION**

Important Note:

It's common for logicians to refer to FOL sentences true in every model as *universally valid*.

- The term 'tautology' is often restricted to FOL formulae that can be obtained by taking a tautology of PL and replacing the propositional letters with FOL sentences.

(28) **Contradiction**

Let φ be a sentence of FOL. φ is a contradiction *iff* for every model \mathcal{M} $[[\varphi]]^{\mathcal{M}} = 0$

Illustration: $\exists x(Px \ \& \ \sim Px)$

Proof That It's a Contradiction

Let \mathcal{M} be any model $\langle D, I \rangle$ and g be any variable assignment (based on \mathcal{M}).

- Suppose that $[[\exists x(Px \ \& \ \sim Px)]]^{M,g} = 1$
- Then there is some $a \in D$ such that $[[Px \ \& \ \sim Px]]^{M,g(x/a)} = 1$.
- Then there is some $a \in D$ such that $[[Px]]^{M,g(x/a)} = 1$ and $[[\sim Px]]^{M,g(x/a)} = 1$
- Then there is some $a \in D$ such that $[[Px]]^{M,g(x/a)} = 1$ and $[[Px]]^{M,g(x/a)} = 0$
- Then there is some $a \in D$ such that $g(x/a)(x) \in I(P)$ and $g(x/a)(x) \notin I(P)$
- Then there is some $a \in D$ such that $a \in I(P)$ and $a \notin I(P)$ **CONTRADICTION**

(29) **Contingent**

Let φ be a sentence of FOL. φ is contingent *iff* there are models \mathcal{M} and \mathcal{M}' such that $[[\varphi]]^{\mathcal{M}} = 1$ and $[[\varphi]]^{\mathcal{M}'} = 0$.

Illustration: $\exists x(Px \ \& \ Qx)$ [proof trivial]

(30) **Consequences**

- φ is a tautology (universally valid) iff $\sim\varphi$ is a contradiction
- φ is contingent iff $\sim\varphi$ is contingent

(31) **Key Observation**

- Note that in showing that a given sentence of FOL is a tautology or contradiction, we need to use some (modest) mathematical ingenuity.
 - Unlike PL, we don't just 'mechanically calculate' it out (via truth-tables)
- **This is no accident. A very deep and interesting theorem of computability theory is that there can be no purely mechanical, algorithmic procedure for calculating whether an FOL sentence is a tautology or not.**
- Similarly, there is no purely mechanical procedure for calculating whether or not one FOL sentence 'entails' another (defined below) or whether two sentences of FOL are 'logically equivalent' (defined below).
 - Unlike PL, 'you gotta use your noodle...'

(32) **Logical Equivalence**

Let φ, ψ be sentences of FOL. φ and ψ are logically equivalent if for every model \mathcal{M}
 $[[\varphi]]^{\mathcal{M}} = [[\psi]]^{\mathcal{M}}$

Illustration: $\forall x \sim Px$ and $\sim \exists x Px$

Proof of Logical Equivalence:

Let \mathcal{M} be any model $\langle D, I \rangle$ and g be any variable assignment (based on \mathcal{M}).

- $[[\forall x \sim Px]]^{M,g} = 1$ iff
- For all $a \in D$, $[[\sim Px]]^{M,g(x/a)} = 1$ iff
- For all $a \in D$, $[[Px]]^{M,g(x/a)} = 0$ iff
- For all $a \in D$, $a \notin I(P)$ iff
- **It is not the case that there is an $a \in D$ s.t. $a \in I(P)$** iff
- It is not the case that there is an $a \in D$ s.t. $[[Px]]^{M,g(x/a)} = 1$ iff
- It is not the case that $[[\exists x Px]]^{M,g} = 1$ iff
- $[[\exists x Px]]^{M,g} = 0$ iff
- $[[\sim \exists x Px]]^{M,g} = 1$ iff

(33) **Consequences**

- Any two tautologies (universal validities) are logically equivalent
- Any two contradictions are logically equivalent.
- If φ, ψ are logically equivalent, then so are $\sim\varphi, \sim\psi$

(34) **Consequence**

The following two formulae are logically equivalent: $\exists xPx$ and $\sim\forall x\sim Px$

- $\forall x\sim Px$ and $\sim\exists xPx$ are logically equivalent (proof in (32))
- $\sim\forall x\sim Px$ and $\sim\sim\exists xPx$ are logically equivalent (consequences in (33))
- $\sim\forall x\sim Px$ and $\exists xPx$ are logically equivalent (double negation)

(35) **Key Result: Substitution of Logically Equivalent Formulae**

Informal Statement:

If φ and ψ are logically equivalent, then replacing φ with ψ will have no effect on the truth-value of a larger sentence.

Formal Statement:

Suppose that φ and ψ are logically equivalent, that φ is a subformula of χ , and that χ' is the formula just like χ , except that all instances of φ are replaced with ψ . It follows that χ and χ' are logically equivalent.

(36) **Key Result: Dropping Operators (Quantifiers) from Our System**

- Informally speaking, what (35) says is that if φ and ψ are logically equivalent, then anything we can 'express' in FOL with φ , we can also 'express' with ψ
- Now, note the logical equivalence we just proved in (34):
 $\exists xPx$ is logically equivalent to $\sim\forall x\sim Px$
- From (34), it follows that anything we can 'express' in FOL with ' \exists ' we can express with ' \forall ' and ' \sim '.
- **Consequently, we could drop ' \exists ' from our FOL system, leaving just ' \forall ' and ' \sim ', and still have an equally 'expressive' system**
- **In addition, we could simply view ' \exists ' as a *special abbreviation* for more complex formulae:**
 $\exists v\psi$ is 'shorthand' for $\sim\forall x\sim\psi$
- (We'll make use of these ideas later, where they will come in handy for the definitions of logical languages we use for the semantic analysis of English...)

(37) **Entailment**

Let φ, ψ be a sentences of FOL. φ entails ψ if every model \mathcal{M} such that $[[\varphi]]^{\mathcal{M}} = 1$ is also such that $[[\psi]]^{\mathcal{M}} = 1$

Illustration: $(\forall xPx \vee \forall xQx)$ entails $\forall x(Px \vee Qx)$

Proof of Entailment:

Let \mathcal{M} be any model $\langle D, I \rangle$ such that $[[\forall xPx \vee \forall xQx]]^{\mathcal{M}} = 1$. Let g be any variable assignment based on \mathcal{M} .

- Thus, $[[\forall xPx \vee \forall xQx]]^{\mathcal{M},g} = 1$
- Thus, either (i) $[[\forall xPx]]^{\mathcal{M},g} = 1$ or (ii) $[[\forall xQx]]^{\mathcal{M},g} = 1$
- Suppose (i). Then for all $a \in D$, $[[Px]]^{\mathcal{M},g(x/a)} = 1$
 - Therefore, for all $a \in D$, either $[[Px]]^{\mathcal{M},g(x/a)} = 1$ or $[[Qx]]^{\mathcal{M},g(x/a)} = 1$
 - Therefore, for all $a \in D$, either $[[Px \vee Qx]]^{\mathcal{M},g(x/a)} = 1$
 - Therefore, $[[\forall x(Px \vee Qx)]]^{\mathcal{M}} = 1$
- Suppose (ii). Then for all $a \in D$, $[[Qx]]^{\mathcal{M},g(x/a)} = 1$
 - Therefore, for all $a \in D$, either $[[Px]]^{\mathcal{M},g(x/a)} = 1$ or $[[Qx]]^{\mathcal{M},g(x/a)} = 1$
 - Therefore, for all $a \in D$, either $[[Px \vee Qx]]^{\mathcal{M},g(x/a)} = 1$
 - Therefore, $[[\forall x(Px \vee Qx)]]^{\mathcal{M}} = 1$
- Thus, $[[\forall x(Px \vee Qx)]]^{\mathcal{M},g} = 1$

Now that we have this notion of 'interpretation with respect to a model', it becomes possible to prove rigorously that our proof system for FOL is sound and complete!

(38) **Soundness of PL**

- If $S \vdash \psi$, then $S \models \psi$
- If $S \vdash \psi$, then if \mathcal{M} is a model for S , then $[[\psi]]^{\mathcal{M}} = 1$

(39) **Completeness of PL**

- If $S \models \psi$, then $S \vdash \psi$
- If every model \mathcal{M} for S is also a model for ψ , then $S \vdash \psi$