Proving the Soundness and Completeness of Propositional Logic: Some Highlights [1]

(1) **A Summary of What We've Done So Far for PL**

a. We've given a purely *syntactic* characterization of 'valid inference' in PL
$S \vdash \psi$

b. We've given a formal semantics for PL notation, and used it to provide a (proper) semantic definition of 'valid inference' in PL.
$S \vDash \psi$

(2) **The BIG Question:** Do these two characterizations of validity coincide?
$S \vdash \psi$ *iff*(?) $S \vDash \psi$

(3) **The Theorems We Wish to Prove**

Soundness of PL:
If ψ can be derived from S in our natural deduction system for PL, then S entails ψ
- If $S \vdash \psi$, then $S \vDash \psi$
- If $S \vdash \psi$, then if V is a valuation for S, $V(\psi) = T$

Completeness of PL:
If S entails ψ, then ψ can be derived from S in our natural deduction system.
- If $S \vDash \psi$, then $S \vdash \psi$
- If every valuation V of S is also a valuation of ψ, then $S \vdash \psi$

(4) **Some History of the Proofs**
- Soundness was basically proven rather early on (it's easy, but tedious)
- Before there were proper proofs, people were largely convinced that PL and FOL were 'complete'
  - After all, anything anyone ever wanted to prove could be proved!
- The first proper proof that FOL is complete was Gödel's PhD thesis (1929)
  - It's crazy complicated, and nobody teaches it anymore
- In his PhD thesis, Henkin (1949) hit upon a much simpler, and just plain cooler proof
  - This is the one everybody teaches to this day…

[1] These notes are based upon material in the following required readings: Gamut (1991), Chapter 4 pp. 148-155; Crossley *et al.* (1972), Chapter 2; Partee *et al.* (1993) Chapter 8 pp. 225-227.

For reasons of time, I won't give the complete proof of either soundness or completeness:
- However, I will hit the 'highlights' of both…
- As we'll see, the completeness proof is more involved, *and more interesting!...*

1. Proving the Soundness of Natural Deduction for Propositional Logic

(5) **Theorem to Prove: Soundness** If S ⊢ ψ, then S ⊨ ψ

(6) **Key Observation**
If S ⊢ ψ, then there is a finite subset S' ⊆ S such that there is a derivation consisting of ***n* lines** where each φ ∈ S' appear as 'Assumptions' and where ψ appears on **line *n*.**

- *Key Idea:*
We can use (strong) induction to prove the following, which would prove (5):

(7) **Restatement of Soundness (for Mathematical Induction)**
For every natural number n > 0, if S ⊢ ψ with a proof consisting of n lines, then S ⊨ ψ

I won't give the <u>*entire*</u> *inductive proof of (7), but I'll give you the main gist…*

(8) **Preliminary Observation**

The following are, technically speaking, proofs in our natural deduction system.

a. 1. p Assumption

b. 1. p Assumption
2. (q & r) Assumption
3. (s → t) Assumption

- Proof (1a) consists of one single line. It terminates right after we add 'p' as an assumption. Thus, {p} ⊢p (which intuitively should be the case)

- Proof (1b) has three lines. It terminates right after we add the third assumption '(s→t)'. Thus, {p, (q&r), (s→t)} ⊢(s→t) (which intuitively should be the case)

(9) **Proof of Soundness Theorem (7) By Strong Induction**

a. Base Step: $n = 1$
Suppose that $S \vdash \psi$ with a proof consisting of 1 line.

- Thus, for some finite subset $S' \subseteq S$, $S' \vdash \psi$ with a proof consisting of 1 line.

- Given our system, the proof in question must be a 'degenerate' case like (8a), where ψ is an Assumption.

- Consequently, $\psi \in S'$, and so $\psi \in S$. Consequently, $S \vDash \psi$

b. Induction Step:
Let $n \in \mathbb{N}$ be such that for all $m < n$, if $S \vdash \psi$ with a proof consisting of m lines, then $S \vDash \psi$.

- We'll now show that if $S \vdash \psi$ with a proof consisting of n lines, then $S \vDash \psi$.

- We'll show this by considering all the ways that a proof consisting of (n-1) lines can be extended to a proof consisting of n lines.

- Given the structure of our system, there are *12* cases to consider:

1. Adding an assumption ψ
2. Deriving ψ by Repetition
3. Deriving ψ by I&
4. Deriving ψ by E&
5. Deriving ψ by I∨
6. Deriving ψ by E∨
7. Deriving ψ by E→
8. Deriving ψ by I→
9. Deriving ψ by E~
10. Deriving ψ by I~
11. Deriving ψ by ~~
12. Deriving ψ by EFSQ

For reasons of time, I won't do all 12…
Just a few notable ones…

1. *Adding ψ as an Assumption*
Suppose that $S \vdash \psi$ with a proof consisting of n lines, where the final line has 'Assumption' as the justification. It follows that $\psi \in S$, and so $S \vDash \psi$

2. *Deriving ψ by Repetition*
Suppose that S ⊢ ψ with a proof consisting of n lines, where the final line has 'Repetition' as the justification.

- By definition of 'Repetition', S ⊢ ψ with a proof of length m < n.
- Therefore, by the Induction Assumption, S ⊨ ψ.

3. *Deriving ψ by I&*
Suppose that S ⊢ ψ with a proof consisting of n lines, where the final line has 'I&' as the justification.

- By definition of 'I&', ψ = (φ & χ), and S ⊢ φ with a proof of length m < n, and S ⊢ χ with a proof of length m < n
- Therefore, by the induction assumption, S ⊨ φ and S ⊨ χ
- Therefore, S ⊨ (φ & χ) (= ψ)

4. *Deriving ψ by E&*
(Can be shown via an argument parallel to the one for I&)

5. *Deriving ψ by I∨*
(easily shown via an argument similar to those above)

6. *Deriving ψ by E∨*
(easily shown via an argument similar to those above)

7. *Deriving ψ by E→*
Suppose that S ⊢ ψ with a proof consisting of n lines, where the final line has 'E→' as the justification.

- By definition of 'E→', it follows that S ⊢ (φ → ψ) with a proof of length m < n, and S ⊢ φ with a proof of length m' < n.
- Therefore, by the induction assumption, S ⊨ (φ → ψ) and S ⊨ φ.
- Therefore S ⊨ ψ.[2]

The other steps in the proof are basically parallel to this…

EXCEPT THAT: the steps for I→ and I~ rely upon a minor (trivial) lemma concerning 'conditional proofs'…

[2] After all, if there were a valuation V of S s.t. V(ψ) = 0, then since S ⊨ φ, this valuation would be s.t.V(φ → ψ) = 0, and so S wouldn't entail (φ → ψ).

(10) **Important Note**
As trivial as it is, this proof of the soundness of PL would not even get off the ground without a clear, mathematically precise definition of what 'entailment' for PL is…

- And this requires a clear, mathematically precise definition of what an 'interpretation' of PL is…

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2. Proving the Completeness of Natural Deduction for Propositional Logic

(11) **Theorem to Prove: Completeness** If $S \vDash \psi$, then $S \vdash \psi$.

The first crucial step to proving completeness is the 'Key Lemma' in (13).

- For reasons of time, I won't review the demonstration here.
- Interested readers are referred to Gamut (1991), p. 150

(12) **Key Preliminary Definition: Consistency**
Let S be a set of formulae in PL. S is *inconsistent* if $S \vdash \bot$. S is *consistent* if $S \nvdash \bot$

- *Note that '(in)consistency' here is a* *syntactic* *property.*

(13) **Key Lemma**
Let S be a set of formulae in PL. $S \cup \{\psi\}$ is inconsistent *iff* $S \vdash {\sim}\psi$

- *Note that (13) just states that* $S \cup \{\psi\} \vdash \bot$ *iff* $S \vdash {\sim}\psi$
- *If you consider our rules of I~ and E~, you can see that (13) pretty trivially holds…*

The second crucial step to proving completeness is seeing how the 'Consistency Theorem' in (14) would entail Completeness in (11)

(14) **The Consistency Theorem**
If S is a consistent set of formulae in PL, then there is a valuation V of S.

(15) **The Consistency Theorem Entails Completeness**

- Suppose that $S \vDash \psi$. It follows that $S \cup \{{\sim}\psi\}$ has ***no valuation***.
- Therefore, by contraposition of (14), it follows that $S \cup \{{\sim}\psi\}$ is ***inconsistent.***
- Therefore, by (13), it follows that $S \vdash {\sim\sim}\psi$, and so $S \vdash \psi$.

The third and most arduous step in the completeness is proof is proving (14).
And, the most arduous part of proving (14) is proving the lemma in (16)…

(16) **Lindenbaum's Lemma**
Let S be a consistent set of formulae in PL. There is a consistent set S* such that $S \subseteq S^*$ and S* has the following key 'closure properties'.[3]

For any formulae φ and ψ of PL:

a. $\varphi \in S^*$ *iff* $\sim\varphi \notin S^*$
b. $(\varphi \,\&\, \psi) \in S^*$ *iff* $\varphi \in S^*$ and $\psi \in S^*$
c. $(\varphi \vee \psi) \in S^*$ *iff* $\varphi \in S^*$ or $\psi \in S^*$
d. $(\varphi \rightarrow \psi) \in S^*$ *iff* $\varphi \notin S^*$ or $\psi \in S^*$

Note:
For those who are interested, proving (16) isn't intellectually all that difficult. It just takes time to correctly lay out the procedure for constructing S* from S….

Now that we have this huge set S, we're home free!*

(17) **The Cool Central Insight of Henkin's Proof**
You can take a set S* with the properties in (16), and *directly build a valuation for S* from the formulas in S* itself!*

(18) **Model Existence Lemma**
If a consistent set S* has the 'closure properties' in (16a-d), then S* has a valuation. Namely, it has the valuation defined as follows:

a. The Valuation for S*:
Let the valuation V be such that for every proposition letter α of PL, $V(\alpha) = 1$ *iff* $\alpha \in S^*$

b. Claim:
The valuation V defined in (18a) is a valuation for S*.

The final step in the completeness theorem is proving the claim in (18).

- The proof will be by induction on the complexity of formulae…

Again, I won't do the whole proof here, but I'll review some key illustrative steps…

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[3] This set S* is commonly referred to as a 'maximally consistent set'. Note, too, that since S* is consistent, $\perp \notin S^*$.

(19) **Proof of the Model Existence Lemma**

Claim:
Let V be the valuation defined in (18a). For any formula φ of PL, $\varphi \in S^*$ iff $V(\varphi) = 1$.

Proof (by Induction on Complexity of Formulae):

a. *Base Step: Proposition Letters*
Suppose that φ is a proposition letter. Then, by the very definition in (18a), it trivially follows that $V(\varphi) = 1$ iff $\varphi \in S^*$

b. *Induction Step:*
Suppose that φ is a (complex) formula of PL, and that for any of its immediate subformulae $\psi \in S^*$ iff $V(\psi) = 1$. We will now that that $\varphi \in S^*$ iff $V(\varphi) = 1$. There are four cases to consider: ~, &, v, →

1. $\varphi = \sim\psi$
   - $\sim\psi \in S^*$ *iff* (by closure property in (16a))
   - $\psi \notin S^*$ *iff* (by induction assumption)
   - $V(\psi) = 0$ *iff* (by definition of a valuation)
   - $V(\sim\psi) = 1$

2. $\varphi = (\psi \,\&\, \chi)$
   - $(\psi \,\&\, \chi) \in S^*$ *iff* (by closure property in (16b))
   - $\psi \in S^*$ and $\chi \in S^*$ *iff* (by induction assumption)
   - $V(\psi) = 1$ and $V(\chi) = 1$ *iff* (by definition of a valuation)
   - $V(\psi \,\&\, \chi) = 1$

3. $\varphi = (\psi \vee \chi)$ *Proof is parallel to those for 1. and 2.*

4. $\varphi = (\psi \rightarrow \chi)$ *Proof is parallel to those for 1. and 2.*

(20) **Putting It All Together**

a. Given the lemma in (16), we've shown that any consistent set S can be 'expanded' into a larger consistent set S* with the properties in (16a-d).

b. Given (18)-(19), we've shown that any such set S* with the properties in (16a-d) has a valuation V.

c. Since V is a valuation for S*, and $S \subseteq S^*$, it follows that V is a valuation for S.

d. **Thus, any consistent set of formulae has a valuation V. QED (14).**

(21) **Taking Stock of What We've Done**

- In our last set of notes, we developed a mathematically rigorous characterization of what it means for a formula of PL to be 'true under an interpertation' (valuation).

- We've just seen how these notions have allowed us to prove that our syntactic proof system for PL *is a perfect syntactic characterization of validity in PL*

- **For the first time in human history, we've shown that we can indeed give a perfect, purely syntactic characterization of what it means for an inference to be valid (in a specified language)**

  - That's a huge achievement…
  - And it's even more an achievement when we do it for FOL…