

## Propositional Logic: Formal Semantics and Valuations <sup>1</sup>

### 1. The Motivation for Developing a Formal Semantics for Propositional Logic

#### (1) A Central Goal of (Formal) Logic, Since Aristotle

To provide a *purely syntactic* characterization of valid argumentation.

#### (2) The Means of Achieving This in Formal Logic

- a. A precisely defined formal notation for representing certain aspects of the ‘logical structure’ of an assertion.
- b. A set of *syntactically defined* rules for deriving formulas in the notation from other formulas in the notation (e.g., our system of ‘Natural Deduction’).

#### (3) An Immediate Challenge for This Program

How do we know when we’ve succeeded? How do we know whether our formal system indeed produces *all and only* the inferences that are valid (in the notation)?

##### a. Entailment:

A set of formulae S **entails**  $\psi$ , if ‘whenever’ every  $\phi \in S$  is true, so is  $\psi$

• Notation:  $S \models \psi$

• Note: If S entails  $\psi$ , then the inference of  $\psi$  from S is valid

##### b. Soundness:

We’ve established the **soundness** of our system, if we can show that every derivation is / corresponds to a valid inference.

• Notation: If  $S \vdash \psi$ , then  $S \models \psi$

##### c. Completeness:

We’ve established the **completeness** of our system, if we can show that every valid inference (in the notation) corresponds to a derivation.

• Notation: If  $S \models \psi$ , then  $S \vdash \psi$ ,

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<sup>1</sup> These notes are based upon material in the following required readings: Gamut (1991), Chapter 2 pp. 44-54; Partee *et al.* (1993), Chapter 6 pp. 97-112.

(4) **The Main Obstacle to Proving Soundness and Completeness**

The notion of ‘entailment’ in (3a) is currently too vague and informal for precise (mathematical) argumentation.

- What, exactly, does it mean to say ‘*whenever all the formulae in S are true*’?

**To have a precise proof of soundness/completeness, we need rigorous explication of what it means for a formula of PL to be ‘true’.**

(5) **Towards a Formalization of ‘Truth’ for PL**

Let us begin by considering our *informal* semantics for PL, and how it can be used to determine a truth-value for a PL formula:

Example Sentence: (p & ~q)

Determining Its Truth (Informally):

a. *Step One:*

- A formula of PL isn’t true or false *in and of itself*, but only relative to an (informal) interpretation (i.e., ‘key’).
- So the first thing an informal interpretation does is it maps the primitive proposition letters to particular statements of English

p: ‘Seth lives in Northampton.’  
q: ‘Rajesh lives in Northampton.’

b. *Step Two:*

Next, to determine whether the whole formula is true, we first consider the truth of the English sentences that the proposition letters are mapped to:

‘Seth lives in Northampton’ is *true*  
‘Rajesh lives in Northampton’ is *true*

c. *Step Three:*

Finally, once we have the truth-values of the English translations, the meanings of the (English) logical connectives determine a truth-value for the whole formula.

~q = ‘It is not the case that Rajesh lives in Northampton’ = *false*  
(p & ~q) = *false*

It seems, then, that an informal interpretation of PL determines a truth-value for a PL formula by

- Mapping the primitive proposition letters to truth-values (indirectly, through English translations)
- Then, for complex formulae, the main connective determines the truth-value of the whole formula, based on the truth-values of the component formulae

Thus, it seems we can develop a more abstract, mathematically precise conception of ‘PL interpretation’ in the following way...

## 2. A Review of Propositional Logic (PL): Formal Semantics and Valuations<sup>2</sup>

### (6) Definition of a ‘Valuation’ for Propositional Logic

A valuation  $V$  is a function from the well-formed formulae of PL to the set of truth-values  $\{1,0\}$  ( $V: \text{WFF} \rightarrow \{0,1\}$ ) such that:

- a. If  $\varphi$  is a proposition letter, then  $V(\varphi) \in \{1,0\}$  (redundant, but helpful)
- b. If  $\varphi, \psi \in \text{WFF}$ , then
1.  $V(\sim\varphi) = 1$  iff  $V(\varphi) = 0$
  2.  $V((\varphi \& \psi)) = 1$  iff  $V(\varphi) = 1$  and  $V(\psi) = 1$
  3.  $V((\varphi \vee \psi)) = 1$  iff  $V(\varphi) = 1$  or  $V(\psi) = 1$  (inclusive ‘or’)
  4.  $V((\varphi \rightarrow \psi)) = 1$  iff  $V(\varphi) = 0$  or  $V(\psi) = 1$  (inclusive ‘or’)
- (c.  $V(\perp) = 0$  (This is need for the proof of soundness))

### (7) The Truth-Tables of the Logical Connectives

<p>a. <table style="display: inline-table; border-collapse: collapse; margin-right: 20px;"> <tr><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>\varphi</math></td><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>\sim\varphi</math></td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td></tr> </table></p>	$\varphi$	$\sim\varphi$	1	0	0	1	<p>b. <table style="display: inline-table; border-collapse: collapse; margin-right: 20px;"> <tr><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>\varphi</math></td><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>\psi</math></td><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>(\varphi \&amp; \psi)</math></td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td></tr> </table></p>	$\varphi$	$\psi$	$(\varphi \& \psi)$	1	1	1	1	0	0	0	1	0	0	0	0	<p>c. <table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>\varphi</math></td><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>\psi</math></td><td style="border-bottom: 1px solid black; padding: 2px 10px;"><math>(\varphi \vee \psi)</math></td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">1</td><td style="padding: 2px 10px;">1</td></tr> <tr><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td></tr> </table></p>	$\varphi$	$\psi$	$(\varphi \vee \psi)$	1	1	1	1	0	1	0	1	1	0	0	0
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<sup>2</sup> My discussion here will assume prior familiarity with the formal semantics of Propositional Logic, particularly truth-tables, validity, tautology, logical-equivalence, etc. Students are referred to Partee *et al.* (1993), Chapter 6 for crucial background.

(8) **Calculating Truth-Tables for Complex Formulae**

- In top row, list all the proposition letters in the formula (followed by double lines).
- Then, list all the sub-formulae, going from bottom-up (followed by single lines)
- Each row below corresponds to one of the possible valuations of the proposition letters in the sentence ( $2^n$  rows, where  $n$  = number of proposition letters)

Illustration:  $((p \ \& \ q) \rightarrow \sim r)$

<u>p</u>	<u>q</u>	<u>r</u>	<u><math>\sim r</math></u>	<u><math>(p \ \&amp; \ q)</math></u>	<u><math>((p \ \&amp; \ q) \rightarrow \sim r)</math></u>
1	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	0	1
1	0	0	1	0	1
0	1	1	0	0	1
0	1	0	1	0	1
0	0	1	0	0	1
0	0	0	1	0	1

*We can use this notion of a ‘valuation’ to provide a more precise characterization of our key semantic concepts!*

(9) **Tautology**

- a. *Informal Notion:* S is a tautology if S is ‘necessarily true’  
 b. *Formal Notion:*  $\varphi$  is a tautology if for every valuation  $V$ ,  $V(\varphi) = 1$

Illustration:  $(p \vee \sim p)$

<u>p</u>	<u><math>\sim p</math></u>	<u><math>(p \vee \sim p)</math></u>
1	0	1
0	1	1

(10) **Contradiction**

- a. *Informal Notion:* S is a contradiction if S is ‘necessarily false’  
 b. *Formal Notion:*  $\varphi$  is a contradiction if for every valuation  $V$ ,  $V(\varphi) = 0$

Illustration:  $(p \ \& \ \sim p)$

<u>p</u>	<u><math>\sim p</math></u>	<u><math>(p \ \&amp; \ \sim p)</math></u>
1	0	0
0	1	0

(11) **Contingent**

- a. *Informal Notion:* S is contingent if S is ‘possibly true and possibly false’  
 b. *Formal Notion:*  $\varphi$  is contingent if there is a valuation  $V$  such that  $V(\varphi) = 1$ , and a valuation  $V'$  such that  $V'(\varphi) = 0$

**Key Consequences**

- A formula  $\varphi$  of PL is a tautology *iff*  $\sim\varphi$  is a contradiction.
- A formula  $\varphi$  of PL is contingent *iff*  $\sim\varphi$  is contingent

(12) **Logical Equivalence**

a. *Informal Notion:* S and S' are logically equivalent if S is true whenever S' is.

b. *Formal Notion:*

$\varphi$  and  $\psi$  are logically equivalent if for every valuation V,  $V(\varphi) = V(\psi)$

Illustration:    p,  $\sim\sim$ p

p	$\sim$ p	$\sim\sim$ p
1	0	<b>1</b>
0	1	<b>0</b>

**Key Consequences**

- Formulae  $\varphi$  and  $\psi$  of PL are logically equivalent *iff*  $\sim\varphi$  and  $\sim\psi$  are logically equivalent.
- If  $\varphi$  and  $\psi$  are both tautologies, then  $\varphi$  and  $\psi$  are logically equivalent.
- If  $\varphi$  and  $\psi$  are both contradictions, then  $\varphi$  and  $\psi$  are logically equivalent.

*Given our definition of valuation in (6), our definition of 'logical equivalence' in (12b) captures a wide variety of intuitive equivalences, many of which we also derived in our proof system!*

(13) **DeMorgans Laws:  $(p \vee q)$  is Logically Equivalent to  $\sim(\sim p \ \& \ \sim q)$**

p	q	$(p \vee q)$	$\sim p$	$\sim q$	$(\sim p \ \& \ \sim q)$	$\sim(\sim p \ \& \ \sim q)$
1	1	<b>1</b>	0	0	0	<b>1</b>
1	0	<b>1</b>	0	1	0	<b>1</b>
0	1	<b>1</b>	1	0	0	<b>1</b>
0	0	<b>0</b>	1	1	1	<b>0</b>

(14) **Associativity of  $\&$ :  $((p \vee q) \vee r)$  is Logically Equivalent to  $(p \vee (q \vee r))$**

p	q	r	$(p \vee q)$	$((p \vee q) \vee r)$	$(q \vee r)$	$(p \vee (q \vee r))$
1	1	1	1	<b>1</b>	1	<b>1</b>
1	1	0	1	<b>1</b>	1	<b>1</b>
1	0	1	1	<b>1</b>	1	<b>1</b>
1	0	0	1	<b>1</b>	0	<b>1</b>
0	1	1	1	<b>1</b>	1	<b>1</b>
0	1	0	1	<b>1</b>	1	<b>1</b>
0	0	1	0	<b>1</b>	1	<b>1</b>
0	0	0	0	<b>0</b>	0	<b>0</b>

(15) **Key Result: Substitution of Logically Equivalent Formulae**

*Informal Statement:*

If  $\varphi$  and  $\psi$  are logically equivalent, then replacing  $\varphi$  with  $\psi$  will have no effect on the truth-value of a larger sentence.

*Formal Statement:*

Suppose that  $\varphi$  and  $\psi$  are logically equivalent, that  $\varphi$  is a subformula of  $\chi$ , and that  $\chi'$  is the formula just like  $\chi$ , except that all instances of  $\varphi$  are replaced with  $\psi$ . It follows that  $\chi$  and  $\chi'$  are logically equivalent.

Illustration:  $(p \rightarrow q)$  and  $(\sim\sim p \rightarrow q)$ :

$p$	$q$	$(p \rightarrow q)$	$\sim p$	$\sim\sim p$	$(\sim\sim p \rightarrow q)$
1	1	1	0	1	1
1	0	0	0	1	0
0	1	1	1	0	1
0	0	1	1	0	1

(16) **Key Result: Dropping Operators from Our System**

- Informally speaking, what (15) shows is that if  $\varphi$  and  $\psi$  are logically equivalent, then anything we can ‘express’ in PL with  $\varphi$ , we can also ‘express’ with  $\psi$
- Now, recall the following key equivalences (proofs left to the student):
  - (i)  $(\varphi \rightarrow \psi)$  is logically equivalent to  $(\sim\varphi \vee \psi)$
  - (ii)  $(\varphi \vee \psi)$  is logically equivalent to  $\sim(\sim\varphi \ \& \ \sim\psi)$
  - (iii)  $(\varphi \ \& \ \psi)$  is logically equivalent to  $\sim(\sim\varphi \vee \sim\psi)$
- It follows from (i) that anything we can ‘express’ in PL with ‘ $\rightarrow$ ’, we can also express with ‘ $\sim$ ’ and ‘ $\vee$ ’
- It follows from (ii) that anything we can ‘express’ in PL with ‘ $\vee$ ’, we can also express with ‘ $\sim$ ’ and ‘ $\&$ ’.
- It follows from (iii) that anything we can ‘express’ in PL with ‘ $\&$ ’ we can also express with ‘ $\sim$ ’ and ‘ $\&$ ’
- **Consequently, we can drop ‘ $\rightarrow$ ’ from our PL system (leaving ‘ $\sim$ ’, ‘ $\&$ ’ ‘ $\vee$ ’), and still have an equivalently ‘expressive’ system**
- **Consequently, we can drop ‘ $\vee$ ’ from our system (leaving ‘ $\&$ ’ and ‘ $\sim$ ’), and still have an equivalently ‘expressive’ system**
- **Consequently, we can drop ‘ $\&$ ’ from our system (leaving ‘ $\vee$ ’ and ‘ $\sim$ ’), and still have an equivalently ‘expressive’ system**

(17) **Key Result: Defining Operators in Terms of Other Operators**

Given the result in (16), we could view the operators ‘ $\rightarrow$ ’ and ‘ $\vee$ ’ (or ‘ $\rightarrow$ ’ and ‘ $\&$ ’) not as primitive operators, but as *special abbreviations* for more complex formulae

- $(\varphi \rightarrow \psi)$  is ‘shorthand’ for  $(\sim\varphi \vee \psi)$
- $(\varphi \vee \psi)$  is ‘shorthand’ for  $\sim(\sim\varphi \& \sim\psi)$

We’ll make use of this commonplace idea in (17) later on, as it will greatly simplify the definitions of the logical languages we use for the semantic analysis of English...

Finally, the notion of ‘valuation’ provides us with a more rigorous definition of ‘entailment’...

(18) **Preliminary Definition**

If  $S$  is a set of formulae, then  $V$  is a **valuation for  $S$**  if for every  $\varphi \in S$ ,  $V(\varphi) = 1$

(19) **Entailment**

a. *Informal Notation:*

A set of formulae  $S$  **entails**  $\psi$ , if ‘whenever’ every  $\varphi \in S$  is true, so is  $\psi$

b. *Formal Notion:*

A set of formulae  $S$  **entails**  $\psi$ , if every valuation  $V$  of  $S$  is such that  $V(\psi) = 1$

(20) **Key Consequence: Entailment and Tautology**

Let  $S$  be a finite set of formulae  $\{\varphi_1, \dots, \varphi_n\}$ .  $S$  entails  $\psi$  iff  $((\varphi_1 \& \dots \& \varphi_n) \rightarrow \psi)$  is a tautology.

*Proof:* Suppose that  $S$  entails  $\psi$ . Thus, for any valuation  $V$ , if  $V((\varphi_1 \& \dots \& \varphi_n)) = 1$  then  $V$  is a valuation for  $S$ , and so  $V(\psi) = 1$ . Thus, for any valuation  $V$ , if  $V((\varphi_1 \& \dots \& \varphi_n)) = 1$ , then  $V(\psi) \neq 0$ , and so  $V(((\varphi_1 \& \dots \& \varphi_n) \rightarrow \psi)) = 1$ .

Suppose that  $((\varphi_1 \& \dots \& \varphi_n) \rightarrow \psi)$  is a tautology. Now suppose that  $V$  is a valuation of  $S$ . It follows that  $V((\varphi_1 \& \dots \& \varphi_n)) = 1$ . Moreover, since  $((\varphi_1 \& \dots \& \varphi_n) \rightarrow \psi)$  is a tautology, it follows that  $V(((\varphi_1 \& \dots \& \varphi_n) \rightarrow \psi)) = 1$ . Consequently, it must be the case that  $V(\psi) \neq 0$ , and so  $V(\psi) = 1$ .

(21) **Major Consequence: Computability of Entailment for PL**

One can ‘effectively’ compute whether a finite set of PL formulae  $S$  entail a formula  $\psi$

- Just compute the truth-table for  $(S \rightarrow \psi)$  and check whether it’s a tautology!

Now that we have this notion of 'valuation', the key issues in (3) become much more tractable!

(22) **Soundness of PL**

- If  $S \vdash \psi$ , then  $S \models \psi$
- If  $S \vdash \psi$ , then if  $V$  is a valuation for  $S$ ,  $V(\psi) = T$

(23) **Completeness of PL**

- If  $S \models \psi$ , then  $S \vdash \psi$
- If every valuation  $V$  of  $S$  is also a valuation of  $\psi$ , then  $S \vdash \psi$

*In the next set of notes, we'll see how (22) and (23) can be rigorously proved!*