Formal Preliminaries, Part 1: Relations and Functions [1]

1. Basic Concepts of Set Theory

I assume that the reader is familiar with the following:

- Basic concept of a set
- Set abstraction notation, $\{ x : x \text{ is a boy} \}$
- The concepts of cardinality, subset, powerset, union, intersection, complementation
- Various key set theoretic identities $(X \cup (Y \cap Z) = ((X \cup Y) \cap (X \cup Z))$

For a review of these key concepts, the reader is referred to Partee *et al.* (1993), Chapter 1

2. Relations and Functions

2.1 Ordered Tuples, Products, Projections

(1) **Ordered Pair**

The defining property of an ordered pair is that 'order matters'. That is:

- If $<x,y> = <y,x>$, then $x = y$
- If $x \neq y$, then $<x,y> \neq <y,x>$

(2) **Ordered *n*-Tuple**

With the notion of an ordered pair, we can define the concept of an ordered triple:

$$<x, y, z> =_{def} < <x,y> , z >$$

Clearly, this will generalize to the definition of an arbitrary *n*-tuple:

$$< x_1 , \dots , x_n > =_{def} < <x_1, \dots x_{n-1} > , x_n >$$

(3) **Cartesian Product**

If we have two sets A, B, then A × B, the *cartesian product of A and B* is:

$$A \times B =_{def} \{ <x, y> : x \in A \text{ and } y \in B \}$$

Illustration: $\{ a, b \} \times \{ c, d \} = \{ <a,c>, <a,d>, <b,c>, <b,d> \}$
$\{ c, d, e \} \times \{ a, b \} = \{ <c,a>, <c,b>, <d,a>, <d,b>, <e,a>, <e,b> \}$

Note: For any set A, $A \times \emptyset = \emptyset$

[1] These notes are based upon material in the following required readings: Partee *et al.* (1993) Chapter 1, Chapter 2; Heim & Kratzer (1998) pp. 29-32; Stewart & Tall (1977) pp. 105-106.

(4) **n-Ary Cartesian Product**
Suppose that $A_1, \ldots, A_n$ is a series of n sets:

$A_1 \times \ldots \times A_n$ $=_{def}$ the set of all n-tuples $\langle a_1, \ldots, a_n \rangle$ such that $a_1 \in A_1, \ldots a_n \in A_n$
$=_{def}$ $\{ \langle a_1, \ldots, a_n \rangle : a_i \in A_i \}$

Illustration:
{ a, b } × { c, d } × {e , f } = {<ace>, <acf>, <ade>, <adf>, <bce>, <bcf> <bde>, <bdf>}

(5) **Cartesian Power**
- Let A be any set. $A^2 = A \times A$
- Let A be any set and n be any (natural) number. $A^n = A \times \ldots \times A$ (n times)

2.2 Relations, Domains, Ranges, Inverses

(6) **(Binary) Relation**
A set of ordered pairs is a *(binary) relation.*

Illustrations: { <x,y> : x is the mother of y }
{ <x,y> : x loves y }
{ <a,b>, <c,d>, <e,f>, … }

(7) **Domain and Range**
Let R be a relation.
a. The Domain of R: $\{ x : \langle x,y \rangle \in R \}$
b. The Range of R: $\{ y : \langle x,y \rangle \in R \}$

(8) **Inverse of Relation**
Let R be a relation. R^{-1} is the *inverse of R*, and is defined as follows:

$R^{-1} =_{def} \{ \langle y,x \rangle : \langle x,y \rangle \in R \}$

Illustration: Let R = { <a,b>, <c,d>, <e,f> }
R^{-1} = { <b,a>, <d,c>, <f,e> }

Note: $(R^{-1})^{-1} = R$

(9) **n-Ary Relation**
An *n*-ary relation R is a set of *n*-tuples.

Illustrations: { <x,y,z> : x is between y and z }	ternary relation
{ <x,y,z,s> : x bet y z dollars that s would lose }	quaternary relation
{ <a,b,c,d,e>, <f,g,h,i,j> <k,l,m,n,o> }	'5-ary' relation

Note: Given the definition in (2), an *n*-ary relation R is equivalent to a binary relation whose domain is a set of (*n*-1)-tuples, (*i.e.*, an (*n*-1)-ary relation).

2.3 Functions and Stuff

(10) **Function**

Let $R \subseteq A \times B$. R is a *function from A to B* if the following conditions hold:

a. The domain of R is A
b. If $\langle x,y\rangle \in R$ and $\langle x,z\rangle \in R$, then $y = z$.

(11) **Key Notations Relating to Functions**

a. $f: A \rightarrow B$ $=_{def}$ f is a function from A to B
b. $f(x)$ $=_{def}$ the unique y s.t. $\langle x,y\rangle \in f$
c. B^A $=_{def}$ the set of all functions from A to B.
d. $f: A \rightarrow B \rightarrow C$ $=_{def}$ f is a function from A to C^B

(12) **Surjection, Injection, Bijection**

Let $f: A \rightarrow B$.

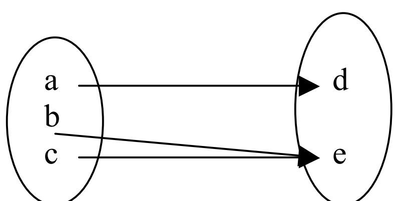
a. f is a surjection (onto) $=_{def}$ the range of $f = B$
$=_{def}$ for every $b \in B$, there is an $a \in A$ s.t. $f(a) = b$

b. f is an injection (one-to-one) $=_{def}$ if $f(a) = f(a')$, then $a = a'$
$=_{def}$ each $a \in A$ is mapped to a different $b \in B$

c. f is a bijection $=_{def}$ f is a surjection (onto) and an injection (one-to-one)

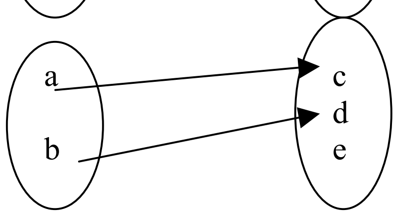
<u>Illustrations:</u>

Surjection:



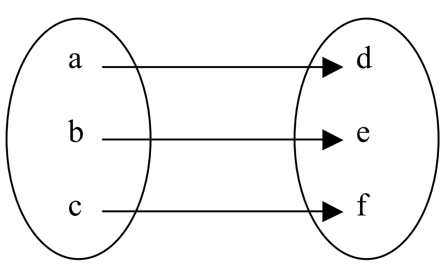
(not an injection)

Injection:



(not a surjection)

Bijection:



(13) **Inverses and Bijections**

a. Since a function f is a relation, we can speak of its inverse f^{-1}

b. The inverse of a function f isn't *necessarily* a function (consider surjections)

c. **If f: A → B is a bijection, then f^{-1} is a function. f^{-1} is also a bijection.**

Proof:

(i) *f^{-1} is a function.*
Suppose it weren't. Then there would be <x,y>, <x,z> ∈ f^{-1} where $y \neq z$. But this would entail f(z) = f(y) = x, and so f isn't an injection, and so f isn't a bijection, contrary to assumption.

(ii) *f^{-1} is an injection (one-to-one).*
Suppose it weren't. Then <x,y>, <z,y> ∈ f^{-1} where $x \neq z$. But this would entail that <y,x>, <y,z> ∈ f, where $x \neq z$, and so f isn't a function, contrary to assumption.

(iii) *f^{-1} is an surjection (onto).*
Suppose it weren't. Then there is an x ∈ A such that there is no y ∈ B such that $f^{-1}(y) = x$. But, then it follows that that there is an x ∈ A such that there is no y ∈ B such that f(x) = y, and so the domain of f isn't A, contrary to assumption.

(14) **n-Ary Function**

- Note that the domain of a function can be a set of n-tuples. Such a function will be dubbed an 'n-ary function'.

  $f: (A_1 \times \ldots \times A_n) \rightarrow B$

  Illustration: $\{ \langle\langle x,y \rangle, z \rangle : z = x + y \}$ binary function
  $\{ \langle\langle x,y,z \rangle, s \rangle : s = x + y + z \}$ ternary function

- Note that an n-ary function is a set of ordered pairs, the first member of which is an n-tuple.

- **Consequently, given the definition in (2), an n-ary function is equivalent to an (n+1)-ary relation**

  Illustration:
  $\{ \langle\langle x,y \rangle, z \rangle : z = x + y \}$ = $\{ \langle x,y,z \rangle : z = x + y \}$
  $\{ \langle\langle x,y,z \rangle, s \rangle : s = x + y + z \}$ = $\{ \langle x,y,z,s \rangle : s = x + y + z \}$

Note: Sometimes the term 'function' is restricted to *unary* functions. Functions of arity greater than 1 (*i.e.,* binary, ternary, etc.) are sometimes dubbed 'operations'.

2.4 Function Composition

(15) (**Basic) Function Composition**

Let f: A → B and g: B → C be unary functions. The *composition of f and g* is

g°f = { <x,z> : for some y, <x,y> ∈ f and <y,z> ∈ g }

Illustration: From Partee et al. (1993):

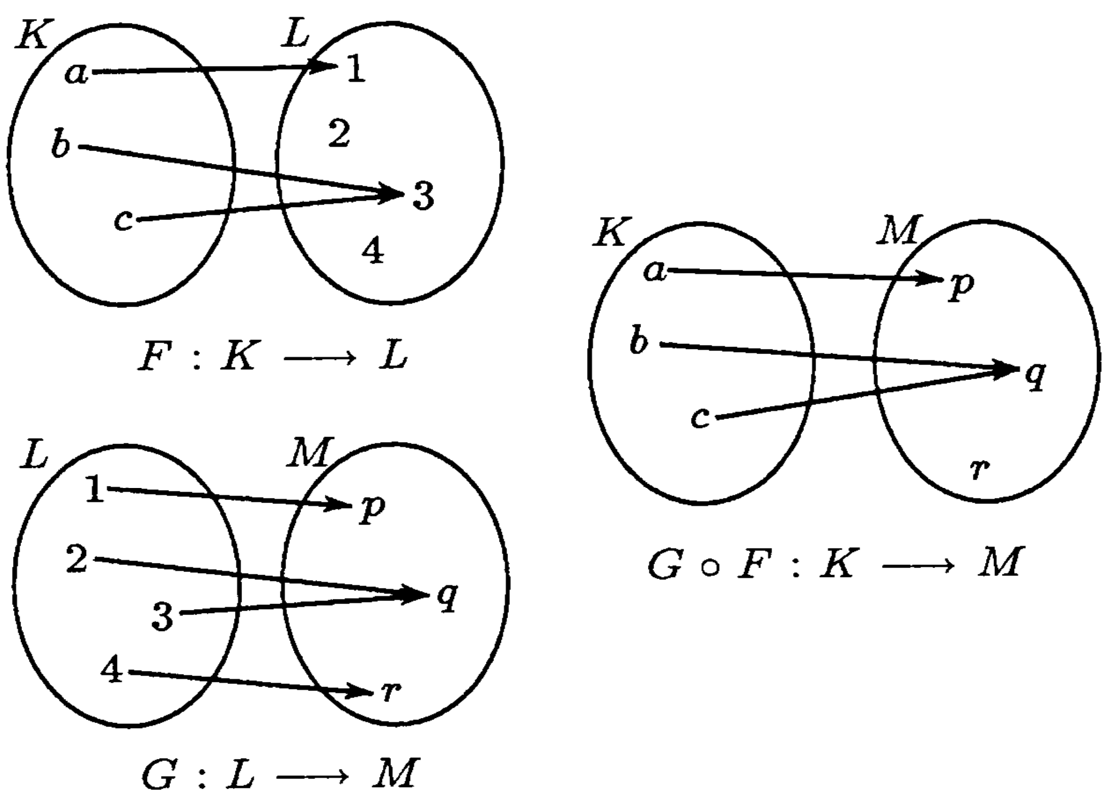


Figure 2–3: Composition of two functions F and G

Note: For all x, g°f(x) = g(f(x))

Note: Another way of writing 'the composition of f and g' is g<f>

Note: In the definitions above, the functions g and f are assumed to be unary… We can generalize these definitions to functions of any arity…

(16) **(Generalized) Function Composition**

Let g be an n-ary function, and let $f_1, \ldots, f_n$ be a series of n m-ary functions. The *composition of g and $f_1, \ldots, f_n$* is the m-ary function defined as follows:

$g\langle f_1, \ldots, f_n\rangle =_{def}$ the m-ary function such that for any m-ary sequence $a_1, \ldots a_m$
$g\langle f_1, \ldots, f_n\rangle(\langle a_1, \ldots a_m\rangle) =$
$g(f_1(\langle a_1, \ldots a_m\rangle), \ldots, f_n(\langle a_1, \ldots a_m\rangle))$

Illustration:

Let $g = \{ \langle\langle x,y\rangle,z\rangle : z = x + y \}$, $f = \{ \langle x,y\rangle : y = x - 1 \}$, $h = \{ \langle x,y\rangle : y = x + 2 \}$

Then: $g\langle f,h\rangle(2) = g(f(2),h(2)) = g(1,4) = 5$

$g\langle f,h\rangle = \{ \langle x,y\rangle : y = (x-1) + (x+2) \}$

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3. Characteristic Functions

(17) **Characteristic Functions and Characteristic Sets**

a. Characteristic Function f_A of a Set A
For every set $A \subseteq B$, there is a unique function f_A: B → {1, 0}, defined below:
For all $b \in B$, $f_A(b) = 1$ *iff* $b \in A$

b. Characteristic Set A_f of a Function f: B → {1,0}
For every function f: B → {1,0}, there is a unique set $A_f \subseteq B$, defined below:
For all $b \in B$, $b \in A_f$ *iff* $f(b) = 1$

Note:
- Given this regular correspondence between sets and (characteristic) functions, we will often shift freely between the two without comment.
- *Note, however, that such equivocation is not entirely innocent: sets are distinct set-theoretic objects from their characteristic functions*.

$\{a,b,c\} \neq \{ \langle a,1\rangle, \langle b,1\rangle, \langle c,1\rangle, \langle d,0\rangle, \langle e,0\rangle, \langle f,0\rangle \}$

(18) **Characteristic Functions of Relations**
Recalling the definition in (9) of an n-ary relation, the definition in (17) entails that every n-ary relation $R \subseteq A_1 \times \ldots \times A_n$ has its own characteristic function:

For all $\langle a_1, \ldots , a_n\rangle \in A_1 \times \ldots \times A_n$, $f_R(\langle a_1, \ldots , a_n\rangle) = 1$ *iff* $\langle a_1, \ldots , a_n\rangle \in R$

Note: For this reason, we will also often shift freely between n-ary relations and n-ary characteristic functions (even though they are, strictly speaking, different objects)

(19) **One Final Observation**

- The characteristic function of an n-ary relation is an n-ary function.
- But, recall from (14), that every n-ary function is equivalent to an (n+1)-ary relation
- **Thus, every n-ary relation corresponds to a unique (n+1)-ary relation**

Illustration: { <a,b>, <bc>, <c,d> } ≈ { <a,b,1>, <b,c,1>, <c,d,1>, <b,a,0>, <c,b,0>, <d,c,0>, … }

Note: Sometimes, Montague shifts freely between these two objects (so, watch out!)

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4. Currying Functions[2]

(20) **Currying of Binary Functions**

Let f be a binary function f: (A×B) → C. There's a unique function *curry*(f): A → B → C defined as follows:

For all <x,y> ∈ A × B, f(<x,y>) = c *iff* *curry*(f)(x)(y) = c

Illustration:
Suppose we have the following function f: { a, b } × { c, d } → {1, 0}

$$f: \begin{pmatrix} \langle a,c\rangle \rightarrow 1 \\ \langle a,d\rangle \rightarrow 0 \\ \langle b,c\rangle \rightarrow 0 \\ \langle b,d\rangle \rightarrow 1 \end{pmatrix} \qquad curry(f): \begin{pmatrix} a \rightarrow \begin{pmatrix} c \rightarrow 1 \\ d \rightarrow 0 \end{pmatrix} \\ \\ b \rightarrow \begin{pmatrix} c \rightarrow 0 \\ d \rightarrow 1 \end{pmatrix} \end{pmatrix}$$

Note: The lambda notation makes it quite easy to define *curry*(f):

curry(f) = [λx : [λy : f(<x,y>)]]

[2] Given that the original discoverer of this technique was Moses Schönfinkel, some have proposed that the term 'schönfinkeling' be used instead of 'currying' (as Haskell Curry rediscovered the technique later). However, to my knowledge, the term 'schönfinkeling' has not widely caught on in mathematics and computer science.

(21) **Currying n-Ary Functions**

- Note that in the definition in (20), the set A could itself be a set of pairs.
  - In such a case, f(<<x,y>,z>) = c *iff* *curry*(f)(<x,y>)(z) = c

- Thus, in such a case, *curry*(f) will itself be a binary function. Thus, we could easily also speak of the function *curry(curry(f))*.
  - In such a case f(<<x,y>,z>) = c *iff* *curry*(*curry*(f))(x)(y)(z) = c
  - This reasoning clearly generalizes to the following:

Let f be an n-ary function f: $A_1 \times \ldots \times A_n \rightarrow C$. There is a unique function CUR(f), CUR(f): $A_1 \rightarrow \ldots \rightarrow A_n \rightarrow C$ defined as follows:

For all $<a_1, \ldots, a_n> \in A_1 \times \ldots \times A_n$,
$f(<a_1, \ldots, a_n>) = c$ *iff* $CUR(f)(a_1)\ldots(a_n) = c$

Note: The lambda notation again makes it quite easy to define CUR(f)

$$CUR(f) = [\ \lambda x_1 : \ldots [\ \lambda x_n : f(<x_1, \ldots x_n>)\ ] \ldots]$$

(22) **Key Consequence**

Let R $R \subseteq A_1 \times \ldots \times A_n$ be an n-ary relation. From (18) and (21), it follows that there is a unique function CUR(f_R) such that:

$<a_1, \ldots, a_n> \in R$ *iff* $CUR(f_R)(a_1)\ldots(a_n) = 1$

Note: Again, the lambda notation makes it quite easy to define CUR(f_R)

$$CUR(f_R) = [\ \lambda x_1 : \ldots [\ \lambda x_n : f_R(<x_1, \ldots x_n>)\ ] \ldots]$$

Note:
Putting all of this together, we will often shift freely between the following (distinct) set-theoretic objects:

- Set of n-tuples (n-ary relations)
- Functions from n-tuples to {1,0} (characteristic functions of n-ary relations)
- Curried functions from n-tuples to {1,0}
- A^{BXC} and $(A^C)^B$

5. Indexing

(23) **Indexed Family**

- Throughout the notes above, I've made use of the following informal notation:

  $A \quad = \quad \{a_1, \ldots, a_n\}$

  'A is a set consisting of n different elements a_i , for all $0 < i \leq n$'

- Note that this informal notation implies the existence of a bijection f: $\{1, 2, ..n\} \rightarrow A$
  $f(i) = a_i$

- Note that there's no special reason why use have to use numbers as indices; it's just convenient. This sets up the following general definition.

Definition:
Let J and A be sets such that there is a bijection f: J → A. We can say that A is an *indexed family*, and that J is the *index set*.

Notation:
Suppose that A is an indexed family, whose index set is J. We can represent A as follows:

(i) $\{ a : \text{there is a } j \in J \text{ such that } f^{-1}(a) = j \}$

(ii) $\{ a_j \}_{j \in J}$

(24) **Indexing and Tuples**

- Throughout the notes above, we've also used numerical indices to represent *n*-tuples:

  $< a_1 , \ldots , a_n >$

- We can adapt our notation in (ii) above as means for compactly representing *n*-tuples. *Both of the following are equivalent to* $< a_1 , \ldots , a_n >$:

(i) $< a_i >_{i \in \{1, \ldots, n\}}$

(ii) $< a_i >_{i \leq n}$

(25) **Union and Intersection of Indexed Sets**
Let S be an indexed family of sets with index set A.

a. $\cup S$ = $\{ x : x \in S_a \text{ for some } a \in A \}$

b. $\cap S$ = $\{ x : x \in S_a \text{ for all } a \in A \}$

(26) **Alternate Notations**

a. If S is an indexed family of sets with index set A, then '$\cup$S' is sometimes written:

$$\bigcup_{\alpha \in A} S_\alpha$$

b. If S is an indexed family of sets with index set A, then '$\cap$S' is sometimes written:

$$\bigcap_{\alpha \in A} S_\alpha.$$

c. If S is an indexed family of sets with index set {1, …, n }, then '$\cup$S' is sometimes written:

$$\bigcup_{r=1}^{n} S_r$$

c. If S is an indexed family of sets with index set {1, …, n }, then '$\cap$S' is sometimes written:

$$\bigcap_{r=1}^{n} S_r$$