Problem Set on Syntactic Operations, Semantic Operations, and Homomorphisms

(1) **Exercise on the Syntactic Operations for PL**
Please show how the following formulae of PL can be constructed via applications of the syntactic operations Not, And, Or, and If.

For example, if the formula were (p → ~(q & r)), the correct answer would be If(p, Not(And(q, r))).

a. ~(p & (p → (q ∨ r)))

b. (((p ∨ r) → q) ∨ s)

c. (((p → q) → (~q → ~p)) & ((p → q) → (~p ∨ q)))

(2) **Exercise on the Syntactic Operations for FOL**
Please show how the following formulae of FOL can be constructed via applications of the syntactic operations Concat, Not, And, Or, If, Ext, and All.

For example, if the formula were (∃x((Px)a) & (Rb)), the correct answer would be And(Ext(x, Concat(Concat(P, x), a)), Concat(R, b)).

a. ∀x((Px) → ∃y((Lx)y))

b. ∃z(((Ra)z) ∨ ~((Ra)z))

c. ((Ab) ∨ ~∃x(Ax))

(3) **Exercise on (Montagovian) Syntactic Rules**
Let Merge be an operation that takes two strings φ and ψ and forms the string 'φ ψ'. Using this operation Merge, please show how each of the PS rules below can be abstractly characterized as triple <Op, <Cat_1, … Cat_n>, Cat>, where Op is an *n*-ary syntactic operation, and each of Cat_i and Cat is some syntactic category label.

a. DP → D NP

b. PP → P NP

c. NP → A NP

d. S → NP VP

e. How could we capture the following PS rule using this 'triplet' notation? Please do not in any way alter the notation:

VP → V (NP)

(4) **Our New Definition of a Model**
Let $\mathcal{M}$ be a model <D,I> as defined in the handout "First Order Logic: Formal Semantics and Models", where D = { a, b, c }, and I consists of at least the mappings below:

$I(P) = \{ x : x \in D \text{ and } x \text{ is a vowel} \}$

$I(Q) = \{ \langle x,y \rangle : x,y \in D \text{ and } x \text{ precedes } y \text{ in the alphabet} \}$

a. Please convert $\mathcal{M}$ into a model as defined in (18) on the handout "An Algebraic Perspective on Propositional Logic." That is, state what D and I should be under the new definition in (18).

b. Please use the new definition of 'valuation' in (19) on the handout "An Algebraic Perspective on Propositional Logic" to show how the converted model $\mathcal{M}$ assigns truth-values to the following formulae.

(i) $\exists x(Px)$
(ii) $\forall x \exists y((Qy)x)$

(5) **Exercise on 'Interpretations' for PL**
Let <{0,1}, Neg, Conj, Disj, Imp, V > be an interpretation for PL, as defined in (29) of "An Algebraic Perspective on Propositional Logic." Moreover, assume that V is such that V(p) = 1, V(q) = 0, and V(r) = 1. Please use the assumption that V is a homomorphism to calculate truth-values of the following formulae.

For example, if the formula were (p → q), then the calculation would be:

(i)	V(p → q)	=	(by definition of PL)
(ii)	V(If(p,q))	=	(by homomorphism property of V)
(iii)	Imp(V(p),V(q))	=	(by definition of V)
(iv)	Imp(1, 0)	=	(by definition of Imp)
(v)	0		

a. $\sim(p \;\&\; (p \rightarrow (q \vee r)))$
b. $(((p \vee \sim r) \rightarrow q) \vee p)$
c. $(((p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)) \;\&\; ((p \rightarrow q) \rightarrow (\sim p \vee q)))$

(6) **Exercise on Homomorphisms and Compositions**
Let **A** $< A, f_1, \dots f_n >$ and **B** $< B, g_1, \dots g_n >$ and **C** $< C, h_1, \dots h_n >$ all be algebras. Assume that $k: A \rightarrow B$ is a homomorphism from **A** to **B**, and that $j: B \rightarrow C$ is a homomorphism from **B** to **C**. Please show that $j°k$ is a homomorphism from **A** to **C**.

<u>Note:</u> To properly show this, you need to show the following:

(i) That $j°k$ is a function from A to C [trivial]
(ii) That the operations $f_1, \dots f_n$ can be put into a one-to-one correspondence with $h_1, \dots h_n$, such that for all $i \in \mathbb{N}$, f_i is the same arity as h_i [trivial]
(iii) That for all $i \in \mathbb{N}$, $j°k(\, f_i(a_1, \dots, a_m)\,) \;=\; h_i(\, j°k(a_1), \dots, j°k(a_m)\,)$ [important part]