Problem Set on Propositional Logic and First Order Logic, Formal Semantics: Answers and Notes

1. Notes on the Answers

In Section 2, I have copied some illustrative answers from the problem sets submitted to me. In this section, I provide some notes on the answers below as well as on the problems themselves.

(1) **Computing Entailment in Propositional Logic**
(no comments)

(2) **Computing Logical Equivalence in Propositional Logic**
(no comments)

(3) **Defining Operators in Propositional Logic**

- For this problem, I wanted you to give me two formulae containing only the 'nor'-operator (↓), one of which is equivalent to $\sim\varphi$ and the other of which is equivalent to $(\varphi \vee \psi)$

- **I also wanted you to demonstrate the equivalence of those formulae using a truth-table.** Since I didn't explicitly ask for that, though, I don't hold it against you if you didn't provide such truth-tables ;)

(4) **Computing Truth of Formulas Relative to a Model**

- For this problem, I wanted you to actually show me the calculations for each formulae; an answer of simply '1' or '0' would be insufficient.

- In addition, the calculation for each formula should begin with the following statement: "Let g be a (any / an arbitrary) variable assignment based on $\mathcal{M}$ ". This crucial assumption was left implicit in many people's calculations.

- Finally, I also wanted you to provide actual concrete justifications for the determined truth-value at the end of each calculation. For example:

  - For (4a), you could note that where x = a, y = e, and z = f, the formula is true.

  - For (4b), you should note that no letter precedes itself in alphabetical order. Thus, the conditional antecedent will be false for all $x \in L$, and so the conditional will be true for all such $x \in L$, rendering the quantificational formula true.

  - For (4c), you should note that where x = z, the antecedent of the conditional is true, but the consequent is false. Thus, the entire conditional is false when x = z, and so the quantificational formula is false.

(5) **Proving Logical Equivalence for First Order Logic Sentences**

- For this problem, the proofs for each of the pairs in (5) should begin with the following statement: "Let $\mathcal{M}$ be any model, and let g be any variable assignment based on $\mathcal{M}$." Again, this crucial assumption was left implicit in many people's proofs.

- It's interesting to note how the crucial steps in the proofs rest upon the validity *in our meta-language* (English) of the key equivalence we're seeking to prove for FOL.

    - For example, in (5a), the proof rests upon the equivalence in English of the following statements:
        - For all y, for all x, …
        - For all x, for all y, …

    - Similarly, in (5b), the proof rests upon the equivalence in English of the following statements:
        - There is an x such that there is a y such that…
        - There is a y such that there is an x such that…

(6) **Proving Entailment for First Order Logic Sentences**

- For this problem, the proofs of each of the entailments in (6) should begin with the following statement (where φ is the formula occurring to the left of ⊨): "Let $\mathcal{M}$ be a model such that $[[\varphi]]^{M} = 1$. Let g be any variable assignment based on $\mathcal{M}$." Again, this crucial assumption was left implicit in many people's proofs.

- As with (5), it is interesting to note how the crucial steps in the proofs for (6b,c) again rest upon the validity *in our meta-language* (English) of the key validity we're seeking to prove for FOL.

    - For example, for (6b), the proof rests upon the validity in English of the following inference:
        - There is an α such that X and Y…
        - Therefore, there is an α such that X and there is an α such that Y.

    - Similarly, for (6c), the proof rests upon the validity in English of the following inference:
        - There is an α such that for all β…
        - Therefore, for all β, there is an α such that…

2. Illustrative Answers from Submitted Problem Sets

(1) **Computing Entailment in Propositional Logic**
Please use truth-tables to establish whether the following claims are accurate.

a. $\{(\neg p \vee \neg q)\} \vDash \neg(p \vee q)$

p	q	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$(p \vee q)$	$\neg(p \vee q)$
1	1	0	0	0	1	0
1	0	0	1	1	1	0
0	1	1	0	1	1	0
0	0	1	1	1	0	1

Since $\neg(p \vee q)$ is not true in every instance in which $(\neg p \vee \neg q)$ is true, the entailment does not hold.

b. $\{ (p \vee q), (\sim p \rightarrow \sim q)\} \nvDash q$

Counterexample: V(p)=1 and V(q)=0

p	q	**(p∨q)**	~p	~q	**(~p→~q)**	**q**
1	1	**1**	0	0	**1**	**1**
1	0	**<u>1</u>**	0	1	**<u>1</u>**	**<u>0</u>**
0	1	1	1	0	0	1
0	0	0	1	1	1	0

c. $\{((p \rightarrow q) \rightarrow r)\} \vDash (p \rightarrow (q \rightarrow r))$

	p	q	r	(p → q)	(q → r)	((p → q) → r)	(p → (q → r))
V_1	1	1	1	1	1	**1**	**1**
V_2	1	0	1	0	1	**1**	**1**
	1	1	0	1	0	0	0
V_3	1	0	0	0	1	**1**	**1**
V_4	0	1	1	1	1	**1**	**1**
V_5	0	0	1	1	1	**1**	**1**
	0	1	0	1	0	0	1
	0	0	0	1	1	0	1

- The claim is accurate.

(2) **Computing Logical Equivalence in Propositional Logic**
Please use truth-tables to show that the following pairs are logically equivalent.

a. (ϕ & Ψ) is logically equivalent to ~(~ϕ v ~Ψ) because,

V_1(ϕ & Ψ) = V_1 ~(~ϕ v ~Ψ) =1;
V_2(ϕ & Ψ) = V_2 ~(~ϕ v ~Ψ) =0;
V_3(ϕ & Ψ) = V_3 ~(~ϕ v ~Ψ) =0;
V_4(ϕ & Ψ) = V_4 ~(~ϕ v ~Ψ) =0.

	ϕ	Ψ	(ϕ & Ψ)	~ϕ	~Ψ	(~ϕ v ~Ψ)	~(~ϕ v ~Ψ)
V_1	1	1	1	0	0	0	1
V_2	1	0	**0**	0	1	1	**0**
V_3	0	1	**0**	1	0	1	**0**
V_4	0	0	**0**	1	1	1	**0**

b. $(\varphi \to \psi) = (\sim\psi \to \sim\varphi)$ (Logically equivalent)

φ	ψ	$\sim\varphi$	$\sim\psi$	$(\varphi \to \psi)$	$(\sim\psi \to \sim\varphi)$
1	1	0	0	**1**	**1**
0	1	1	0	**1**	**1**
1	0	0	1	**0**	**0**
0	0	1	1	**1**	**1**

c. $\phi \wedge (\psi \vee \chi) \equiv (\phi \wedge \psi) \vee (\phi \wedge \chi)$

ϕ	ψ	χ	$\psi \vee \chi$	$\phi \wedge (\psi \vee \chi)$	$\phi \wedge \psi$	$\phi \wedge \chi$	$(\phi \wedge \psi) \vee (\phi \wedge \chi)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

(3) **Defining Operators in Propositional Logic**
Please show how the operators '~' and 'v' can be defined using the 'nor'-operator (↓) defined below.

a. In order to define $\neg$ in terms of $\downarrow$, we show that $\neg\phi \equiv (\phi \downarrow \phi)$

ϕ	$\neg\phi$	$\phi \downarrow \phi$
1	0	0
0	1	1

b. $(\phi \vee \psi) \equiv ((\phi \downarrow \psi) \downarrow (\phi \downarrow \psi))$

ϕ	ψ	$\phi \vee \psi$	$\phi \downarrow \psi$	$(\phi \downarrow \psi) \downarrow (\phi \downarrow \psi)$
1	1	1	0	1
1	0	1	0	1
0	1	1	0	1
0	0	0	1	0

(4) **Computing Truth of Formulas Relative to a Model**

Let g be a variable assignment based upon the model $\mathcal{M}$.

a. 1. $[\![\exists x \exists y \exists z(Rxy \wedge (Ay \wedge (Rxz \wedge \neg Az)))]\!]^{M,g} = 1$ *iff* (by 21vi)

2. There is an $a \in L$ such that $[\![\exists y \exists z(Rxy \wedge (Ay \wedge (Rxz \wedge \neg Az)))]\!]^{M,g(x/a)} = 1$ *iff* (by 21vi)

3. There is an $a \in L$ such that there is a $b \in L$ such that
$[\![\exists z(Rxy \wedge (Ay \wedge (Rxz \wedge \neg Az)))]\!]^{M,g(x/a)(y/b)} = 1$ *iff* (by 21vi)

4. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$[\![Rxy \wedge (Ay \wedge (Rxz \wedge \neg Az))]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ *iff* (by 21iii)

5. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$[\![Rxy]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Ay \wedge (Rxz \wedge \neg Az)]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ *iff* (by 21iii)

6. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$[\![Rxy]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Ay]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Rxz \wedge \neg Az]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ *iff* (by 21iii)

7. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$[\![Rxy]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Ay]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Rxz]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![\neg Az]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ *iff* (by 21ii)

8. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$[\![Rxy]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Ay]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Rxz]\!]^{M,g(x/a)(y/b)(z/c)} = 1$ and
$[\![Az]\!]^{M,g(x/a)(y/b)(z/c)} = 0$ *iff* (by 21i)

9. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$\langle [\![x]\!]^{M,g(x/a)(y/b)(z/c)}, [\![y]\!]^{M,g(x/a)(y/b)(z/c)} \rangle \in I(R)$ and
$[\![y]\!]^{M,g(x/a)(y/b)(z/c)} \in I(A)$ and
$\langle [\![x]\!]^{M,g(x/a)(y/b)(z/c)}, [\![z]\!]^{M,g(x/a)(y/b)(z/c)} \rangle \in I(R)$ and
$[\![z]\!]^{M,g(x/a)(y/b)(z/c)} \notin I(A)$ *iff*

10. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$\langle g(x/a)(y/b)(z/c)(x), g(x/a)(y/b)(z/c)(y) \rangle \in I(R)$ and
$g(x/a)(y/b)(z/c)(y) \in I(A)$ and
$\langle g(x/a)(y/b)(z/c)(x), g(x/a)(y/b)(z/c)(z) \rangle \in I(R)$ and
$g(x/a)(y/b)(z/c)(z) \notin I(A)$ *iff*

11. There is an $a \in L$ such that there is a $b \in L$ such that there is a $c \in L$
$\langle a, b \rangle \in I(R)$ and
$b \in I(A)$ and
$\langle a, c \rangle \in I(R)$ and
$c \notin I(A)$

Thus, we can calculate that $[\![\exists x \exists y \exists z(Rxy \wedge (Ay \wedge (Rxz \wedge \neg Az)))]\!]^{M,g} = 1$. For example, $x =$ 'a', $y =$ 'e', and $z =$ 'f'.

b.

$V_{\mathcal{M},g}(\forall x(Rxx \rightarrow \neg Ax)) = 1$ *iff* (by (21vii))
for every $\alpha \in L$, $V_{\mathcal{M},g(x/\alpha)}(Rxx \rightarrow \neg Ax) = 1$ *iff* (by (21v))
for every $\alpha \in L$, $V_{\mathcal{M},g(x/\alpha)}(Rxx) = 0$ or $V_{\mathcal{M},g(x/\alpha)}(\neg Ax) = 1$ *iff* (by (21ii))
for every $\alpha \in L$, $V_{\mathcal{M},g(x/\alpha)}(Rxx) = 0$ or $V_{\mathcal{M},g(x/\alpha)}(Ax) = 0$ *iff* (by (21i))
for every $\alpha \in L$, $\langle [\![x]\!]^{\mathcal{M},g(x/\alpha)}, [\![x]\!]^{\mathcal{M},g(x/\alpha)} \rangle \notin I(R)$ or $[\![x]\!]^{\mathcal{M},g(x/\alpha)} \notin I(A)$ *iff*
for every $\alpha \in L$, $\langle g(x/\alpha)(x), g(x/\alpha)(x) \rangle \notin I(R)$ or $g(x/\alpha)(x) \notin I(A)$ *iff* (by def.)
for every $\alpha \in L$, $\langle \alpha, \alpha \rangle \notin I(R)$ or $\alpha \notin I(A)$

Again, this is true, because for every $\alpha \in L, \langle \alpha, \alpha \rangle \notin I(R)$; that is to say, no letter precedes itself in alphabetical order.

c. $\forall x(\neg Ax \rightarrow \exists yRxy)$

1. $[\![\forall x(\neg Ax \rightarrow \exists yRxy)]\!]^{M,g} = 1$ *iff* (by (25j))
2. for all $a \in L$, $[\![\neg Ax \rightarrow \exists yRxy]\!]^{M,g(x/a)} = 1$ *iff* (by (25h))
3. for all $a \in L$, $[\![\neg Ax]\!]^{M,g(x/a)} = 0$ or $[\![\exists yRxy]\!]^{M,g(x/a)} = 1$ *iff* (by (25e))
4. for all $a \in L$, $[\![Ax]\!]^{M,g(x/a)} = 1$ or $[\![\exists yRxy]\!]^{M,g(x/a)} = 1$ *iff* (by (25i))
5. for all $a \in L$, $[\![Ax]\!]^{M,g(x/a)} = 1$ or there is an $a' \in L$, s.t. $[\![Rxy]\!]^{M,g(x/a)(y/a'')} = 1$ *iff* (by (25d))
6. for all $a \in L$, $\langle [\![x]\!]^{M,g(x/a)} \rangle \in [\![A]\!]^{M,g(x/a)}$ or there is an $a' \in L$, s.t. $\langle [\![x]\!]^{M,g(x/a)(y/a')}, [\![y]\!]^{M,g(x/a)(y/a')} \rangle \in [\![R]\!]^{M,g(x/a)(y/a')}$ *iff* (by (25a,c))
7. for all $a \in L$, $\langle g(x/a)(x) \rangle \in I(A)$ or there is an $a' \in L$, s.t. $\langle g(x/a)(y/a')(x), g(x/a)(y/a')(y) \rangle \in I(R)$ *iff* (by def. of g and I)
8. for all $a \in L$, $\langle a \rangle \in \{x \mid x \text{ is a vowel}\}$ or there is an $a' \in L$, s.t. $\langle a, a' \rangle \in \{\langle x, y \rangle \mid x \text{ precedes } y\}$

$\rightsquigarrow$ *false in M*

Note:
The conclusion that the formula is false in $\mathcal{M}$ is based on the existence of 'z', which is neither a vowel, nor is followed by any letter alphabetically.

(5) **Proving Logical Equivalence for First Order Logic Sentences**
Please show that the following pairs of formulae are logically equivalent.

a. $\forall x \forall y Pxy \equiv \forall y \forall x Pxy$

Let $\mathcal{M}$ be any model $\langle D, I \rangle$, and g be any variable assignment based on $\mathcal{M}$.

1. $[\![\forall x \forall y Pxy]\!]^{M,g} = 1$ *iff*
2. For all $a \in D$, $[\![\forall y Pxy]\!]^{M,g^{(x/a)}} = 1$ *iff*
3. For all $a \in D$ and for all $b \in D$, $[\![Pxy]\!]^{M,g^{(x/a)(y/b)}} = 1$ *iff*
4. For all $a \in D$ and for all $b \in D$, $\langle a, b \rangle \in I(P)$ *iff*
5. For all $b \in D$ and for all $a \in D$, $\langle a, b \rangle \in I(P)$ *iff*
6. For all $b \in D$ and for all $a \in D$, $[\![Pxy]\!]^{M,g^{(x/a)(y/b)}} = 1$ *iff*
7. For all $b \in D$, $[\![\forall x Pxy]\!]^{M,g^{(y/b)}} = 1$ *iff*
8. $[\![\forall y \forall x Pxy]\!]^{M,g} = 1$

b. Let M be any model $\langle D, I\rangle$ and g be any variable assignment based on M.

1. $[\![\exists x \exists y Pxy]\!]^{M,g} = 1$ *iff*
2. There is an $a \in D$ such that $[\![\exists y Pxy]\!]^{M,g(x/a)} = 1$ *iff*
3. There is an $a \in D$ such that there is a $b \in D$ such that $[\![Pxy]\!]^{M,g(x/a)(y/b)} = 1$ *iff*
4. There is an $a \in D$ such that there is a $b \in D$ such that $\langle a, b\rangle \in I(P)$ *iff*
5. There is a $b \in D$ such that there is an $a \in D$ such that $\langle a, b\rangle \in I(P)$ *iff*
6. There is a $b \in D$ such that there is an $a \in D$ such that $[\![Pxy]\!]^{M,g(y/b)(x/a)} = 1$ *iff*
7. There is a $b \in D$ such that $[\![\exists x Pxy]\!]^{M,g(y/b)} = 1$ *iff*
8. $[\![\exists y \exists x Pxy]\!]^{M,g} = 1$

(6) **Proving Entailment for First Order Logic Sentences**
Please show that the following entailment relations hold.

a. $\underline{\forall xPx \models \exists xPx}$

$\forall xPx$ entails $\exists xPx$ if every model $\mathcal{M}$ such that $[\![\forall xPx]\!]^{M} = 1$ is also such that $[\![\exists xPx]\!]^{M} = 1$

- Let $\mathcal{M}$ be any model $\langle D, I\rangle$ such that $[\![\forall xPx]\!]^{M} = 1$. Let g be any variable assignment based on $\mathcal{M}$.
- Thus, $[\![\forall xPx]\!]^{M,g} = 1$ (by 21vii)
- Therefore, for all $a \in D$, $[\![Px]\!]^{M,g^{(x/a)}} = 1$ (by 21i)
- Therefore, for all $a \in D$, $[\![x]\!]^{M,g^{(x/a)}} \in I(P)$
- Therefore, for all $a \in D$, $g^{(x/a)}(x) \in I(P)$
- Therefore, for all $a \in D$, $a \in I(P)$
- Therefore, there exists an $a \in D$ such that $a \in I(P)$
- Therefore, there exists an $a \in D$ such that $g^{(x/a)}(x) \in I(P)$
- Therefore, there exists an $a \in D$ such that $[\![x]\!]^{M,g^{(x/a)}} \in I(P)$ (by 21i)
- Therefore, there exists an $a \in D$ such that $[\![Px]\!]^{M,g^{(x/a)}} = 1$ (by 21vi)
- Therefore, $[\![\exists xPx]\!]^{M,g} = 1$
- Thus, $\underline{\forall xPx \models \exists xPx}$

b. $\exists x(px \wedge Qx) \vDash (\exists xPx \wedge \exists xQx)$

Let M be any model $\langle D, I\rangle$ such that $[\![\forall xPx]\!]^{M} = 1$. Let g be any variable assignment based on M.

$[\![\exists x(Px \wedge Qx)]\!]^{M,g} = 1$

$\rightsquigarrow$ there is an $a \in D$, s.t. $[\![Px \wedge Qx]\!]^{M,g(x/a)} = 1$

$\rightsquigarrow$ there is an $a \in D$, s.t. $[\![Px]\!]^{M,g(x/a)} = 1$ and $[\![Qx]\!]^{M,g(x/a)} = 1$

$\rightsquigarrow$ there is an $a \in D$, s.t. $[\![Px]\!]^{M,g(x/a)} = 1$ and there is an $a' \in D$, s.t. $[\![Qx]\!]^{M,g(x/a)} = 1$

$\rightsquigarrow$ $[\![\exists xPx]\!]^{M,g(x/a)} = 1$ and $[\![\exists xPx]\!]^{M,g(x/a)} = 1$

$\rightsquigarrow$ $[\![(\exists xPx \wedge \exists xQx)]\!]^{M,g} = 1$ □

c. ∃y∀xPxy ⊨ ∀x∃yPxy

Let $\mathcal{M}$ be any model <D,I> such that $[[\exists y\forall xPxy]]^{\mathcal{M}} = 1$. Let g be any variable assignment based on $\mathcal{M}$.

$[[\exists y\forall xPxy]]^{\mathcal{M},g} = 1$
Then there is an $a \in D$, $[[\forall xPxy]]^{\mathcal{M},g(y/a)} = 1$
Then there is an $a \in D$, such that for all $b \in D$, $[[Pxy]]^{\mathcal{M},g(y/a)(x/b)} = 1$
Then for all $b \in D$, there is an $a \in D$, such that , $[[Pxy]]^{\mathcal{M},g(y/a)(x/b)} = 1$
Then for all $b \in D$, $[[\exists yPxy]]^{\mathcal{M},g(x/b)} = 1$
Therefore $[[\forall x\exists yPxy]]^{\mathcal{M},g} = 1$