Problem Set on 'Translation and Natural Deduction': Answers and Notes

1. Notes on the Answers

In Section 2, I have copied some illustrative answers from the problem sets submitted to me. In this section, I provide some notes on the answers below as well as on the problems themselves.

(1) Translation from English to Propositional Logic

- For (1b), (1c), and (1f) several different acceptable answers were given.

- For (1b), some version of any of the following formulae would be acceptable, relative to the key stated below:

$(\sim r \rightarrow ((e \vee n) \rightarrow (d \,\&\, c)))$	r: It is summer	d: It is damp
$((\sim r \,\&\, (e \vee n)) \rightarrow (d \,\&\, c))$	e: It is evening	c: It is cold
$((e \vee n) \rightarrow (\sim r \rightarrow (d \,\&\, c)))$	n: It is night	

- For (1c), some version of either of the following would be acceptable, relative to the key below.

$((i \rightarrow \sim h) \rightarrow (y \rightarrow \sim j))$	i: I need you	y: You need me
$((i \,\&\, \sim h) \rightarrow (y \rightarrow \sim j))$	h: You help me	j: I help you

- For (1e), many different formulae would be acceptable, including the ones below. Note that all acceptable translations are logically equivalent to simply 'If Johnny is nice, then he will get a bicycle from Santa Clause'. _Consider why this is apt._

$(((w \,\&\, j) \rightarrow b) \,\&\, ((\sim w \,\&\, j) \rightarrow b))$	w: Johnny wants a bicycle
$((w \rightarrow (j \rightarrow b)) \,\&\, (\sim w \rightarrow (j \rightarrow b)))$	j: Johnny is nice
$(j \rightarrow (b \,\&\, (w \vee \sim w)))$	b: Johnny'll get a bicycle from Santa
$(j \rightarrow ((w \vee \sim w) \rightarrow b))$	

- Note that we did not introduce the biconditional symbol $\leftrightarrow$ in class. Therefore, it would not have been proper to use it in the answers to this exercise.

(2) Derivations in Our Natural Deduction System for Propositional Logic

- For (2a), two different acceptable derivations were given. I've included both in Section 2.

- For all other proofs, each problem set submitted to me had some variant of the proofs provided in Section 2.

(3) **Translations from English to First Order Logic**

- Note that the English sentences in (3b), (3d), (3e) and (3f) all involve quantification over *people*. Some of the translations I received for these sentences neglected to reflect that, and instead had simple quantification over *things*. Please see the answers in Section 2 for illustration.

- For sentence (3e), several folks encoded "own a car" via a simple predicate letter 'Cx', translated as 'x owns a car'. Though technically acceptable, this kind of encoding would fail to represent the quantificational structure of the English sentence. Please see the answers in Section 2 for a more complete encoding of (3e).

- For sentence (3f), several folks translated the sentence as something akin to the following:

  $\exists x\ (\ (\ Px\ \&\ Nx\ ) \rightarrow \forall y\ (\ Py \rightarrow Ayx\ )\ )$
  Px: x is a person Nx: x is noisy Axy: x is annoyed at y

  Note, however, that this would equate to something like 'there is a thing *x* such that if x is a person and is noisy, then everybody is annoyed at x.' Consequently, this formula is made true simply in virtue of the existence of my stapler, which is not a person, and so is not a noisy person (*think carefully why this is so*).

  As shown in Section 2, a more accurate encoding of (3f) would be the following:

  $\forall x\ (\ (\ Px\ \&\ Nx\ ) \rightarrow \forall y\ (\ Py \rightarrow Ayx\ )\ )$
  Px: x is a person Nx: x is noisy Axy: x is annoyed at y

  Note that what's curious about cases like (3f) is that it seems like an indefinite ('somebody') in the antecedent of a conditional functions semantically like a universal with scope over the conditional. For more information on this phenomenon, feel free to take a look at the following (particularly Section 4):

  http://people.umass.edu/scable/LING620-SP11/Handouts/Donkey-Anaphora1.pdf

(4) **Identifying Sentences of FOL**
(no comments)

(5) **Proof by Induction on the Complexity of Formula**

- Several folks attempted to prove the claim in (5) via *weak* induction. That is, their induction step looked something like the following:

  Let $n \in \mathbb{N}$ be such that if a formula φ has n logical operators, then φ does not begin with a variable. We will now show that if φ has (n+1) logical operators, then φ does not begin with a variable.

- Unfortunately, this induction assumption will not be sufficient to prove the general claim in (5).
  - For example, one needs to show in the induction step that if φ is *any arbitrary conjunction*, then φ will not begin with a variable.
  - However, the induction assumption above will not allow us to show this, since we cannot guarantee that any subformula of φ has n logical operators.
    - Indeed, the folks who used the weak induction assumption above usually showed only that if φ is of the form $(\psi \ \& \ \Phi\alpha_1\ldots\alpha_n)$, then φ does not begin with a variable.
    - Note, though, that this only holds for conjunctions of that form, and so we've not shown that *any arbitrary conjunction* has the desired property.

- It is for this reason that the proof of (5) requires an induction assumption that looks as follows, and so (5) must be proven by strong induction (see Section 2 for details).

  Let $n \in \mathbb{N}$ be such that for all $m < n$, if a formula φ has m logical operators, then φ does not begin with a variable. We will now show that if φ has n logical operators, then φ does not begin with a variable.

(6) **Derivations in Our Natural Deduction System for First Order Logic**
(no comments)

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2. Illustrative Answers from Submitted Problem Sets

(1) **Translation from English to Propositional Logic**

a. $p \to q$

Key:

p: God wills

q: Peace comes

b. $(\neg s \to ((e \vee n) \to (d \wedge c)))$

s 'it is summer'
e 'it is evening'
n 'it is night'
d 'it is damp'
c 'it is cold'

b. *If it isn't summer, then it is damp and cold, if it is evening or night.*

p = It is summer
q = It is evening
r = It is night
s = It is damp
t = It is cold

$((\neg p \wedge (q \vee r)) \rightarrow (s \wedge t))$

b. If it isn't summer, then it is damp and cold, if it is evening or night.

$((s \vee t) \rightarrow (\sim p \rightarrow (q \;\&\; r)))$

p: It is summer.

q: It is damp.

r: It is cold.

s: It is evening.

t: It is night.

c. $((n \wedge \neg h) \rightarrow (y \rightarrow \neg i))$

n 'I need you'
h 'you help me'
y 'you need me'
i 'I will help you'

c. *If you do not help me if I need you, I will not help you if you need me.*

p = You help me
q = I need you
r = I help you
s = You need me

$((q \rightarrow \neg p) \rightarrow (s \rightarrow \neg r))$

d. $p \rightarrow \neg q$

Key:

p: John comes

q: Peter comes

e. $((p \rightarrow \sim q) \;\&\; (\sim q \rightarrow p))$
p: we are going
q: it is raining

f. (((p & r) → q) & ((p & ~r) → q))
p: Johnny is nice
q: Johnny will get a bike from Santa
r: Johnny wants a bike

f. If Johnny is nice, then he will get a bicycle from Santa Claus, whether he wants one or not.
(p → (q ∧ (s ∨ ~s)))
p: Johnny is nice.
q: Johnny will get a bicycle from Santa Claus.
s: Johnny wants a bicycle.

f. $(j \rightarrow ((w \vee \neg w) \rightarrow b))$
j 'John is nice'
w 'John wants a bicycle'
b 'John will get a bicycle from Santa Clause'

(2) **Derivations in Our Natural Deduction System for Propositional Logic**

a. (p ∨ q) ⊢ ((p → q) → q)

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1.   (p ∨ q)                    Assumption
2.      | (p → q)               Assumption
3.      |    | q                Assumption
4.      |    | q                Rep., 3
5.      | (q → q)               I→
6.      | q                     E∨, 1, 2, 5
7.   ( (p → q) → q )            I→
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a. $(p \vee q) \vdash ((p \to q) \to q)$

1	$(p \vee q)$	Assumption
2	$\vert\ p$	Assumption
3	$\vert\ \vert\ (p \to q)$	Assumption
4	$\vert\ \vert\ q$	E→ 2,3
5	$\vert\ ((p \to q) \to q)$	I→
6	$(p \to ((p \to q) \to q))$	I→
7	$\vert\ q$	Assumption
8	$\vert\ \vert\ (p \to q)$	Assumption
9	$\vert\ \vert\ q$	Repetition
10	$\vert\ ((p \to q) \to q)$	I→
11	$(q \to ((p \to q) \to q))$	I→
12	$((p \to q) \to q)$	E∨ 1,6,11

b. <u>$(p \,\&\, {\sim}q) \vdash {\sim}(p \to q)$</u>

1	$(p \,\&\, {\sim}\, q)$	Assumption
2	${\sim}\, q$	E& 1
3	$\vert\ (p \to q)$	Assumption
4	$\vert\ p$	E& 1
5	$\vert\ q$	E→ 3, 4
6	$\vert\ \perp$	E∼ 2, 5
7	${\sim}\, (p \to q)$	I∼

c. $(p \rightarrow \neg q) \vdash (q \rightarrow \neg p)$

1.	$(p \rightarrow \neg q)$	Assumption
2.	$\mid\ q$	Assumption
3.	$\mid\ \mid\ p$	Assumption
4.	$\mid\ \mid\ \neg q$	E→ 1, 3
5.	$\mid\ \mid\ \bot$	E¬ 2, 4
6.	$\mid\ \neg p$	I¬
7.	$(q \rightarrow \neg p)$	I→ □

d. $(P \ \& \ (Q \vee R)) \vdash ((P \ \& \ Q) \vee (P \ \& \ R))$

1	$\mid\ (P\&(Q \vee R))$	Assumption
2	$\mid\ P$	∧E, 1
3	$\mid\ (Q \vee R)$	∧E, 1
4	$\mid\ \mid\ Q$	Assumption
5	$\mid\ \mid\ (P\&Q)$	∧I, 2, 4
6	$\mid\ \mid\ ((P\&Q) \vee (P\&R))$	∨I, 5
7	$\mid\ \mid\ (Q \rightarrow ((P\&Q) \vee (P\&R)))$	⇒I
8	$\mid\ \mid\ R$	Assumption
9	$\mid\ \mid\ (P\&R)$	∧I, 2, 8
10	$\mid\ \mid\ ((P\&Q) \vee (P\&R))$	∨I, 9
11	$\mid\ \mid\ (R \rightarrow ((P\&Q) \vee (P\&R)))$	⇒I
12	$\mid\ ((P\&Q) \vee (P\&R))$	∨E, 3, 7, 11

(2.5) $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$

1	$(p \wedge q) \vee (p \wedge r)$	Assumption
2	$\quad p \wedge q$	Assumption
3	$\quad p$	$\wedge$E, 2
4	$\quad q$	$\wedge$E, 2
5	$\quad q \vee r$	$\vee$I, 4
6	$\quad p \wedge (q \vee r)$	$\wedge$I, 3, 5
7	$(p \wedge q) \to p \wedge (q \vee r)$	$\Rightarrow$I
8	$\quad p \wedge r$	Assumption
9	$\quad p$	$\wedge$E, 8
10	$\quad r$	$\wedge$E, 8
11	$\quad q \vee r$	$\vee$I, 10
12	$\quad p \wedge (q \vee r)$	$\wedge$I, 9, 11
13	$(p \wedge r) \to p \wedge (q \vee r)$	$\Rightarrow$I
14	$p \wedge (q \vee r)$	$\vee$E, 1, 7, 13

(3) **Translations from English to First Order Logic**

a. ((Ljm & Lmj) & (Ujm & Umj))

Lwx: w loves x deeply
Uyz: y makes z unhappy
j: John
m: Mary

b. ~∀x((Px & Ax) → ~Hx)

Px: x is a person
Ay: y is ambitious
Hz: z is honest

c. $(\exists y(Py \& Glyj) \& \sim\exists x(Tx \& Glxt))$

Gxyz: x got y from z
Px: x is a present
Tz: z is a thing
l: Lyn
j: John
t: Peter

d. $\sim\exists x((Px \& Lxh) \& \sim Bxh)$

Px: x is a person
Lyz: y lives in z
Byz: y was born in z
h: Hadley

e. $\forall x((Px \wedge (Lxa \vee Nxa)) \rightarrow \exists y(Cy \wedge Oxy))$
Px: *x* is a person
Lxy: *x* lives in *y*
Nxy: *x* lives near *y*
Cx: *x* is a car
Oxy: *x* owns *y*
a: Amherst

(3.5) $\forall x((Px \wedge (Lxa \vee Lxc)) \rightarrow \exists y(Cy \wedge Oxy))$

a: Amherst.

c: close to Amhest.

Cx: x is a car.

Oxy: x owns y.

Lxy: x lives in y.

f. $\forall x((Px \& Nx) \rightarrow \forall y(Py \rightarrow Ayx))$

Px: x is a person
Ny: y is noisy
Awz: w is annoyed at z

(4) **Identifying Sentences of FOL**

a. No (x is free in Bx)

b. Yes

c. Yes

d. No (y is free in Cy)

e. Yes

f. No (x is free in Bx)

g. Yes

(5) **Proof by Induction on the Complexity of Formula**

<u>Claim</u>: No formula of FOL begins with a variable.

<u>Proof by Induction:</u>

a *Base Step: 0*
If φ contains no logical constants, then φ begins with an n-ary predicate letter, so φ does not begin with a variable.

b *Induction Step*
Let n be such that for all m < n, if φ contains m logical constants, then φ does not begin with a variable.

- Now suppose that φ contains n logical constants. There are six possible cases to consider:

i. φ is of the form $\sim\psi$, where ψ contains (n-1) logicals constants. By assumption, then, ψ does not begin with a variable. Thus, so does φ.

ii. φ is of the form $(\chi \& \psi)$, where χ contains m < n logical constants, and ψ contains j < n logical constants. By assumption, then, χ and ψ both do not begin with a variable. Thus, so does φ.

iii. φ is of the form $(\chi \vee \psi)$, where χ contains m < n logical constants, and ψ contains j < n logical constants. By assumption, then, χ and ψ both do not begin with a variable. Thus, so does φ.

iv. φ is of the form $(\chi \to \psi)$, where χ contains m < n logical constants, and ψ contains j < n logical constants. By assumption, then, χ and ψ both do not begin with a variable. Thus, so does φ.

v. φ is of the form $\forall v \psi$, where ψ contains (n-1) logicals constants. By assumption, then, ψ does not begin with a variable. Thus, so does φ.

vi. φ is of the form $\exists v \psi$, where ψ contains (n-1) logicals constants. By assumption, then, ψ does not begin with a variable. Thus, so does φ.

(6) **Derivations in Our Natural Deduction System for First Order Logic**

a. $\forall x(Px \wedge Bx) \vdash (\forall x Px \wedge \forall x Bx)$

1	$\forall x(Px \wedge Bx)$	Assumption
2	$(Pa \wedge Ba)$	E∀ 1
3	Pa	E∧ 2
4	Ba	E∧ 2
5	$\forall y Py$	I∀ 3
6	$\forall y By$	I∀ 4
7	$(\forall y Py \wedge \forall y By)$	I∧ 5,6

(5.2) $\forall x Px \wedge \forall x Bx \vdash \forall x(Px \wedge Bx)$

1	$\forall x Px \wedge \forall x Bx$	Assumption
2	$\forall x Px$	∧E, 1
3	Pa	∀E, 2
4	$\forall x Bx$	∧E, 1
5	Ba	∀E, 4
6	$Pa \wedge Ba$	∧I, 3, 5
7	$\forall x(Px \wedge Bx)$	∀I, 6

c. $\exists x(Px \wedge Bx) \vdash (\exists x Px \wedge \exists x Bx)$

1	$\exists x(Px \wedge Bx)$	Assumption
2	│ $(Pa \wedge Ba)$	Assumption
3	│ Pa	E$\wedge$ 2
4	│ $\exists x Px$	I$\exists$ 3
5	│ Ba	E$\wedge$ 2
6	│ $\exists x Bx$	I$\exists$ 5
7	│ $(\exists x Px \wedge \exists x Bx)$	I$\wedge$ 4,6
8	$((Pa \wedge Ba) \rightarrow (\exists x Px \wedge \exists x Bx))$	I$\rightarrow$
9	$(\exists x Px \wedge \exists x Bx)$	E$\exists$ 1,8

d. $\exists x \neg Px \vdash \neg \forall x Px$

1.	$\exists x \neg Px$	Assumption	
2.	│ $\neg Pa$	Assumption	
3.	│ │ $\forall x Px$	Assumption	
4.	│ │ Pa	E$\forall$ 3	
5.	│ │ $\bot$	E$\neg$ 2, 4	
6.	│ $\neg \forall x Px$	I$\neg$	
7.	$(\neg Pa \rightarrow \neg \forall x Px)$	I$\rightarrow$	
8.	$\neg \forall x Px$	E$\exists$ 1, 7	□

e. ~∀xPx ⊢ ∃x~Px

1.	~∀xPx	Assumption
2.	│ ~∃x~Px	Assumption
3.	│ │ ~Pa	Assumption
4.	│ │ ∃x~Px	I∃, 3
5.	│ │ ⊥	E~, 2, 4
6.	│ ~~Pa	I~
7.	│ Pa	~~, 6
8.	│ ∀xPx	I∀, 7
9.	│ ⊥	E~, 1, 8
10.	~~ ∃x~Px	I~
11.	∃x~Px	~~, 10