

**Problem Set on the ‘Formal Preliminaries’<sup>1</sup>**

**(1) Basic Comprehension Questions on Relations and Functions**

a. Let  $A = \{b,c\}$  and  $B = \{2,3\}$ . State whether the following are true or false:

- (i)  $(A \times B) \cap (B \times A) = \emptyset$
- (ii)  $\langle c,c \rangle \subseteq A^2$
- (iii)  $\{\langle b,3 \rangle, \langle 2,a \rangle\} \subseteq (A \times B) \cup (B \times A)$
- (iv)  $\emptyset \subseteq (A \times A)$

b. Let  $A = \{b,c\}$  and  $B = \{2,3\}$ . Let  $R = \{\langle b,b \rangle, \langle b,2 \rangle, \langle c,2 \rangle, \langle c,3 \rangle\}$

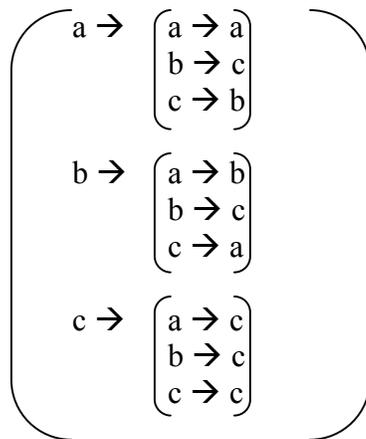
- (i) What is the range and domain of  $R$ ?
- (ii) What is  $R^{-1}$ ?

c. Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  both be *bijections*. Show that  $(g \circ f)^{-1} = (f^{-1}) \circ (g^{-1})$

d. Let  $R = \{\langle x,y \rangle : x, y \in \{1, 2\} \text{ and } x < y\}$ .

- (i) Represent  $R$  as a set of pairs.
- (ii) Represent the characteristic function of  $R$  as a set of pairs<sup>2</sup>
- (iii) Represent the characteristic function of  $R$  as a ternary relation
- (iv) Represent the curried characteristic function of  $R$  as a matrix

*Reminder: By ‘matrix’ I mean a diagram like the following:*



<sup>1</sup> Most of these exercises are taken from Partee *et al.* (1993), Chapters 2 and 4.

<sup>2</sup> Assume that the domain of the characteristic function of  $R$  is  $\{1,2\} \times \{1,2\}$ .

(2) **Proving that Sets are Countable**

- a. Show that the set of integer powers of ten  $\{10, 100, 1000, 10000, \dots\}$  is countable.
- b. Suppose that the following is true of English:
- (i) There is a finite alphabet for writing sentences, consisting of 26 letters and a space (forget punctuation marks for now)
  - (ii) Every sentence of English is a finite string in the alphabet in (i)
  - (iii) There is no upper bound on the length of sentences of English. That is, for any sentence  $S$  of English, there is always a longer sentence  $S'$ .

Show that the set of English sentences is countably infinite.

Some Hints:

- To solve this, you simply have to show that there is a defined way of ordering the sentences of English into an infinite ‘list’ (as we did for the rationals greater than 0).
- One way of solving this makes use of ‘alphabetical order’.
- Another, more round-about way of solving this makes use of the following key consequence of the ‘fundamental theorem of arithmetic’

Consequence of Fundamental Theorem of Arithmetic:

For every positive integer  $n > 1$ , there is *exactly one* way of representing  $n$  as a product of powers of primes:

$$n = p_1^{a_1} \times p_2^{a_2} \times \dots \times p_k^{a_k} \quad \text{where each } p_i \text{ is prime, each } a_i \in \mathbb{N}$$

(3) **Proofs by Induction**

Construct a proof by induction for the following general equivalence:

$$(X_1 \cup \dots \cup X_n)' = X_1' \cap \dots \cap X_n'$$

Hint:

You can appeal to the following general equivalences:

- (i)  $X'' = X$
- (ii)  $X_1 \cap \dots \cap X_n = (X_1 \cap \dots \cap X_{(n-1)}) \cap X_n$
- (iii)  $X_1 \cup \dots \cup X_n = (X_1 \cup \dots \cup X_{(n-1)}) \cup X_n$