Seth Cable *Proseminar on Semantic Theory*
Fall 2013 *Ling 720*

Problem Set on the 'Formal Preliminaries':
Answers and Notes

1. Notes on the Answers

In the pages that follow, I have copied some illustrative answers from the problem sets submitted to me. In this section, I provide some notes on the answers below as well as on the problems themselves.

(1) Basic Comprehension Questions on Relations and Functions

- Nothing needs to be said for (1a), (1b) and (1d). See the answers copied below for details.

- Regarding (1c), another way of showing that the two functions are equivalent is by showing that any pair $<x,y>$ is a member of $(g°f)^{-1}$ *iff* it is a member of $(f^{-1})°(g^{-1})$:

  Let $<x,y>$ be any pair. $<x,y> \in (g°f)^{-1}$ *iff*
  $<y,x> \in (g°f)$ *iff*
  there is a z such that $<y,z> \in f$ and $<z,x> \in g$ *iff*
  there is a z such that $<z,y> \in f^{-1}$ and $<x,z> \in g^{-1}$ *iff*
  there is a z such that $<x,z> \in g^{-1}$ and $<z,y> \in f^{-1}$ *iff*
  $<x,y> \in (f^{-1})°(g^{-1})$

- Why did I ask you to assume that f and g are bijections? Really, it was only because in class I had defined '°' only for functions. Thus, for this problem to well-stated, both f^{-1} and g^{-1} would need to be functions, and so f and g would need to be bijections.

  - However, Partee *et al.* (1993) begin by defining '°' for all relations. If you're interested, confirm that the proofs above and below also establish that if F and G are any binary relations, $(G°F)^{-1} = (F^{-1})°(G^{-1})$.

(2) Proving that Sets are Countable

- For (2a), it should be noted that the integer powers of 10 include 0.01 (= 10^{-2}), 0.0001 (=10^{-4}) and so on. I recognize that my representation of the set as {10, 100, 1000, … } was misleading in this regard, and so I still accept your answer even if you assume the set to be the *positive* integer powers of ten.

- For (2a), some folks simply gave the function $f(n) = 10^n$ and asserted it to be a bijection (from ℕ to the integer powers of ten) *without actually explaining* *<u>why</u>* *it is a bijection.* Although the explanation is trivial, a full answer does require it.

- Besides the function $f(n) = 10^n$, there's also another bijection between these two sets, our old friend 'log' (a.k.a. $\log_{10}$, a.k.a. Log). See the answers copied below for the details.

- For (2b), many people answered the problem by showing that the set of all *strings of English characters* A is countable. Note, however, that the problem asked you to show that the set of all English *sentences* E is countable. To properly finish the proof, you need only note the following:

(i) $|A| = \aleph_0$ (by aforementioned proof)
(ii) $E \subseteq A$ (the English sentences are a subset of all possible strings)
(iii) $|E| \geq \aleph_0$ (the set of English sentences is infinite)
(iv) **Therefore,** $|E| \leq |A| = \aleph_0$ (by (i) and (ii) together)
(v) **Therefore,** $|E| = \aleph_0$ (by (iii) and (iv) together)

- As detailed in the answers copied below, there are three elegant ways of showing that the set of all strings of English characters A is countable:
  - First, you can use 'alphabetical order' to order each set of strings of length n. Then, one can order all of these finite ordered sets, by beginning with the strings of length 1, followed by the strings of length 2, and so on. As detailed below, this easily gives you a bijection from A to $\mathbb{N}$.
  - Secondly, one can view each string of English characters as an 'encoding' of some numeral within a base-27 system.
  - Finally, one can use the 'consequence of the fundamental theorem of arithmetic' mentioned on the problem set to map each string to a unique natural number. *Note that this only provides an injection to* $\mathbb{N}$, and so we conclude only that $|A| \leq \aleph_0$. However, since we know that A is infinite, $|A| \geq \aleph_0$, and so we can conclude that $|A| = \aleph_0$.

(3) **Proofs by Induction**

- For the base step of the induction proof (n = 2), most people simply wrote 'DeMorgans', which I accepted. However, it would be even better to *prove the equivalence yourself*. The following would suffice to show that for any x, $x \in (A \cup B)'$ *iff* $x \in A' \cap B'$

$x \in (A \cup B)'$ *iff*
$x \notin A \cup B$ *iff*
$x \notin A$ and $x \notin B$ *iff*
$x \in A'$ and $x \in B'$ *iff*
$x \in A' \cap B'$

2. Illustrative Answers from Submitted Problem Sets

(1) Basic Comprehension Questions on Relations and Functions

a Let $A = \{b,c\}$ and $B = \{2,3\}$. State whether the following are true or false:

- i $(A \times B) \cap (B \times A) = \phi$ TRUE
- ii $\langle c,c \rangle \subseteq A^2$ FALSE
- iii $\{\langle b,3\rangle, \langle 2,a\rangle\} \subseteq (A \times B) \cup (B \times A)$ FALSE
- iv $\phi \subseteq (A \times A)$ TRUE

b Let $A = \{b,c\}$ and $B = \{2,3\}$. Let $R = \{\langle b,b\rangle, \langle b,2\rangle, \langle c,2\rangle, \langle c,3\rangle\}$

- i What is the range and domain of R?
  - Domain$=\{b,c\}$
  - Range$=\{b,2,3\}$
- ii What is R^{-1}?
  - $R^{-1}=\{\langle b,b\rangle, \langle 2,b\rangle, \langle 2,c\rangle, \langle 3,c\rangle\}$

c Let f: $A \rightarrow B$ and g: $B \rightarrow C$ both be *bijections*. Show that $(g \circ f)^{-1} = (f^{-1}) \circ (g^{-1})$

$$
\begin{aligned}
(g \circ f) &= \{\langle x,z\rangle: \text{ for some } y, \langle x,y\rangle \in f \text{ and } \langle y,z\rangle \in g\} \\
(g \circ f)^{-1} &= \{\langle z,x\rangle: \text{ for some } y, \langle x,y\rangle \in f \text{ and } \langle y,z\rangle \in g\} \\
&= \{\langle z,x\rangle: \text{ for some } y, \langle z,y\rangle \in g^{-1} \text{ and } \langle y,x\rangle \in f^{-1}\} \\
&= (f^{-1}) \circ (g^{-1})
\end{aligned}
$$

d Let $R = \{ \langle x,y\rangle : x, y \in \{ 1, 2 \} \text{ and } x < y \}$.

- i Represent R as a set of pairs: $R=\{\langle 1,2\rangle\}$
- ii Represent the characteristic function of R as a set of pairs: $\{\langle\langle 1,1\rangle,0\rangle, \langle\langle 1,2\rangle,1\rangle, \langle\langle 2,1\rangle,0\rangle, \langle\langle 2,2\rangle,0\rangle\}$

iii Represent the characteristic function of R as a ternary relation: {<1,1,0>, <1,2,1>, <2,1,0>, <2,2,0>}

iv Represent the curried characteristic function of R as a matrix:

$$\begin{bmatrix} 1 & \rightarrow & \begin{bmatrix} 1 & \rightarrow 0 \\ 2 & \rightarrow 1 \end{bmatrix} \\ & & \\ 2 & \rightarrow & \begin{bmatrix} 1 & \rightarrow 0 \\ 2 & \rightarrow 0 \end{bmatrix} \end{bmatrix}$$

(2) **Proving that Sets are Countable**

a. Show that the set of integer powers of ten is countable.

Let A be the set of integer powers of ten. $A = \{f(x) \mid f(x) = 10^x \text{ and } x \in \mathbb{Z}\}$.
f is a bijection from $\mathbb{Z}$ to the set of powers of ten.

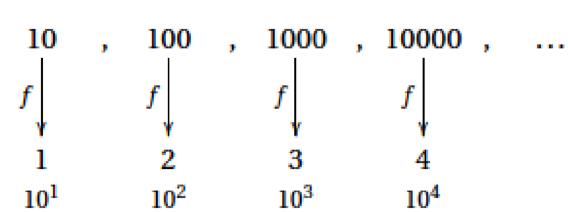
- It's an injection: every x is mapped to a different power of ten.
- It's a surjection: every power of ten is equal to 10^x for some $x \in \mathbb{Z}$.

As assumed in class, $|C| = |B|$ if and only if there is an bijection f: $C \rightarrow B$, therefore $|A| = |\mathbb{Z}|$. The set of powers of ten is countable (countably infinite given that we know that $|\mathbb{Z}| = \aleph_0$).

a. Take the function f in (1). f is an injection from the set of integer powers of ten to $\mathbb{N} - \{0\}$ because no two integer powers of ten will be mapped to the same number in $\mathbb{N} - \{0\}$. f is a surjection from the set of integer powers of ten to $\mathbb{N} - \{0\}$ because, since there is an infinite amount of integer powers of ten, every number in $\mathbb{N} - \{0\}$ will map to some integer power of ten. This is schematised in (2). Therefore, f is a bijection from the set of integer powers of ten to $\mathbb{N} - \{0\}$ and, consequently, the set of integer powers of ten is countably infinite.

(1) $f(x) = \log x$

(2)



b. The elements in the set of all English sentences can be deterministically ordered in the following way: The ordering of the alphabetical elements is $\langle a, b, c, \ldots, z, \llcorner\!\lrcorner \rangle$. The sentences can then be arranged in a sequence of columns, where column n contains all the expressions of length n. The number of elements in each column is finite, corresponding to 27^n for a column containing strings of length n. Within each column, the elements will be ordered based on the alphabet above. Thus, the first column will contain $\langle\langle a\rangle, \langle b\rangle, \ldots, \langle z\rangle, \langle \llcorner\!\lrcorner\rangle\rangle$, the second column will contain $\langle\langle a, a\rangle, \langle a, b\rangle, \ldots \langle a, \llcorner\!\lrcorner\rangle, \langle b, a\rangle, \langle b, b\rangle, \ldots \langle \llcorner\!\lrcorner, z\rangle, \langle \llcorner\!\lrcorner, \llcorner\!\lrcorner\rangle\rangle$ and so on for columns containing strings of arbitrary length. Even though the length of English sentences is unbounded, any given string is finite in length and will hence show up in the arrangment. This arrangment can then be mapped to the natural numbers by assigning '0' to the first element in the first column, '1' to the second element in the first column, ending up with '26' to the last element in the first column, then assigning '27' to the first element of the second column and so on. Conversely, every natural number corresponds to exactly one place in the arrangement and hence exactly one string. There is, then, a isomorphism between the elements in the arrangment and the natural numbers. Because we know that the set of natural numbers is countably infinite, we can conclude that the set of all English sentences is as well.

Seth's Comment: Again, the argument above only shows that the set of all *strings* is countably infinite. To finish the proof, we need some brief explanation of why it also follows that the subset consisting of all the *sentences* of English is countably infinite.

b. The set of English sentences is infinite, as evidenced by the following sequence:

The cat runs.

The red cat runs.

The big red cat runs.

The big fat red cat runs.

⋮

The big fat … red cat runs.

Any number of adjectives can be included; this includes arbitrarily many iterations of the same adjective. It now suffices to show that the set of English sentences is countable.

Given an alphabet $\Sigma := \{a, b, c \ldots, y, z, \llcorner\!\lrcorner\}$, note that $|\Sigma| = 27$.

Consider now the set S of all n-length strings $\sigma_1\sigma_2\sigma_3\ldots\sigma_{n-1}\sigma_n$, $n \in \mathbb{N}, \sigma_i \in \Sigma$.

Given an n-length string $\sigma_{n-1}\sigma_{n-2}\sigma_{n-3}\ldots\sigma_1\sigma_0 \in S$, map this string to a unique $m \in \mathbb{N}$ as follows:

$$m = \sum_{i=0}^{n-1} 27^i \sigma_i,$$

where each σ_i is assigned a numerical value as follows: $\llcorner\!\lrcorner = 0, a = 1, b = 2, \ldots, y = 25, z = 26$.

Note that this is equivalent to converting natural numbers from base 27 to base 10. Assuming that no string begins with $\llcorner\!\lrcorner$, then, we have a bijection from the natural numbers to $\mathbb{N} \setminus \{0\}$: each string has a unique natural number associated with it, and each unique natural number can be represented in a unique way by a string of $\sigma_i \in \Sigma$.

As such, S is countably infinite, and since $T \subset S$, T must also be countable.

b. To prove that the set of English sentences (E = the set of English sentences) is countably infinite, we can start by proving the idea that each member of E can be mapped to a unique number.

- We have an alphabet of 26 letters plus a space, of which English sentences consist:

  Alphabet = {a, b, c, d, e, ..., w, x, y, z, _}

- To assign each string of English sentences a unique number, the Consequence of Fundamental Theorem of Arithmetic is adopted:

  $n = p_1^{a1} \times p_2^{a2} \times \ldots \times p_k^{ak}$, where each p_i is prime, and each $a_i \in \mathbb{N}$

- We can arbitrarily assign a different natural number to each member in the alphabet as follows:

a	⟶	1
b	⟶	2
c	⟶	3
⋮		
y	⟶	25
z	⟶	26
_	⟶	27

Now the Consequence of Fundamental Theorem of Arithmetic is set as the positions in which the alphabet occurs in all English sentences:

$$n = \underset{}{\overset{\_}{p}}{}_1^{a1} \times \overset{\_}{p}{}_2^{a2} \times \overset{\_}{\ldots} \times \overset{\_}{p}{}_k^{ak}$$

→ where the letters occur

The numbers that are arbitrarily assigned to the letters represent the powers of the prime numbers. Therefore, if the letter *a* occurs in the first position in an English sentence, the power of the first prime number would be 1. And if *b* occurs in the first position, the power of the first prime number would be 2. Since all the letters of any given sentence occur in different positions in the sentence and signal different powers of different prime numbers related to the positions, no two sentences would have the same prime number sequence; that is, each English sentence would be assigned a unique number through this mechanism.

Seth's Comment: As mentioned above, this only shows that there is an *injection* from the set of English sentences to the natural numbers. However, since we know that the set of English sentences are infinite it thereby follows that that set is countably infinite.

(3) **Proofs by Induction**

Construct a proof by induction for the following general equivalence:
$(X_1 \cup \ldots \cup X_n)' = X_1' \cap \ldots \cap X_n'$

<u>Claim:</u>
For all n $\in \mathbb{N}$ such that $2 \leq n$, $(X_1 \cup \ldots \cup X_n)' = X_1' \cap \ldots \cap X_n'$

<u>Proof by Induction:</u>

a *Base Step: n = 2*
$(X_1 \cup X_2)' = X_1' \cap X_2'$ *DeMorgan's Laws*

b *Induction Step*
Let n $\in \mathbb{N}$ be such that: $(X_1 \cup \ldots \cup X_n)' = X_1' \cap \ldots \cap X_n'$

- By the associativity of intersection:
  $(X_1 \cup \ldots \cup X_{n+1})' = ((X_1 \cup \ldots \cup X_n) \cup X_{n+1})'$
- By the *base step*:
  $((X_1 \cup \ldots \cup X_n) \cup X_{n+1})' = (X_1 \cup \ldots \cup X_n)' \cap X_{n+1}'$
- By the induction assumption for n:
  $(X_1 \cup \ldots \cup X_n)' \cap X_{n+1}' = (X_1' \cap \ldots \cap X_n') \cap X_{n+1}'$
- By the associativity of intersection:
  $(X_1' \cap \ldots \cap X_n') \cap X_{n+1}' = X_1' \cap \ldots \cap X_n' \cap X_{n+1}'$
- Thus, $(X_1 \cup \ldots \cup X_{n+1})' = X_1' \cap \ldots \cap X_n' \cap X_{n+1}'$

Therefore, by weak induction, it follows that for all n $\in \mathbb{N}$ such that $2 \leq n$:
$(\mathbf{X}_1 \cup \ldots \cup \mathbf{X}_n)' = \mathbf{X}_1' \cap \ldots \cap \mathbf{X}_n'$