I Exercises from PtMW

Chapter 6

(3) Let p, q and r be true and let s be false. Find the truth value of the following statements.

(a) \( ((p \& q) \& s) \) \( False \)

(b) \( (p \& (q \& s)) \) \( False \)

(e) \( ((p \& q) \iff (r \& \neg s)) \) \( True \)

(4) Construct truth tables for the following statements.

(a) \( (p \lor \neg q) \)

<table>
<thead>
<tr>
<th>P</th>
<th>q</th>
<th>\neg q</th>
<th>( p \lor \neg q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(b) \( \neg (\neg p \& q) \)

<table>
<thead>
<tr>
<th>P</th>
<th>q</th>
<th>\neg p</th>
<th>\neg p &amp; q</th>
<th>\neg (\neg p &amp; q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
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<td>0</td>
<td>1</td>
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</tbody>
</table>
(a) and (b) are logically equivalent: they have the same truth value for any possible assignment of truth values to their atomic parts (as can be seen when comparing the last column of the corresponding truth tables)

Chapter 13

(1) (a) Give the derivation tree for ((¬p & q) ∨ p)

\[
\begin{align*}
\lor & \quad ((\neg p & q) \lor p) \\
\land & \quad (\neg p & q) \quad p \\
\neg & \quad \neg p \quad q \\
& \quad p
\end{align*}
\]

(b) Give the corresponding semantic interpretation of the tree of (1a) assuming that both atomic statements are true

\[
\begin{align*}
\lor & \quad 1 \\
\land & \quad 3 \\
\neg & \quad 0 \quad 1 \\
& \quad 1
\end{align*}
\]

II Show that the algebra Form is not a Boolean algebra

An algebra in a signature \( \Omega_{BA} = \{0,1,\neg,\lor,\land\} \) is a Boolean algebra iff the following properties hold for its operations: associative laws, commutative laws, distributive laws, top and bottom laws and complementation laws. None of these properties hold for the operations of the algebra Form\(^1\).

Consider, for instance, the commutative law: The operations \( \lor \) and \( \land \) of the algebra Form do not satisfy it, because ‘p \( \land q \)’ is not the same formula as ‘q \( \land p \)’ and ‘p \( \lor q \)’ is not the same formula as ‘q \( \lor p \)’. (To say that they are would be equivalent to saying that Mary is sleeping and Peter is reading is the same English sentence as Peter is reading and Mary is sleeping).

\(^1\) I think that this is true even for the associative law. Even though brackets can be dropped for convenience, ‘(p \( \lor q \) \( \lor r \)’ is not, strictly speaking, the same formula as ‘p \( \lor (q \lor r) \)’. Is this right? At any rate, this doesn’t affect the argument above, since, to show that the algebra Form is not a Boolean algebra is enough to show that there is one operation that does not satisfy one of the properties.
Or consider of the one the top and bottom laws: \( a \land 0 = 0 \). The operation \( \land \) does not satisfy this property, since ‘\( p \land 0 \)’ and ‘0’ are different formulas of the set \( Form \).