

HOMEWORK 11

4. Prove by induction that the power set of a set with n members, has 2^n members, for any finite positive integer n .

(1) Show: $|\wp(\emptyset)| = 2^0$

Show: $|\wp(\emptyset)| = 1$

By definition, the power set of a given set A is the set of all subsets of A . Since the only subset of the empty set is itself, $\wp(\emptyset) = \{\emptyset\}$, which has one member.

(2)

1. Assume: $(|\wp(A_k)| = 2^k)$

2. Show: $(|\wp(A_{k+1})| = 2^{k+1})$

(where the subscripts indicate the number of members of the set. That is, X_n is a member with n members)

$A_{k+1} = A_k \cup \{x\}$, where x is some arbitrary element not in A_k . $\wp(A_{k+1})$. Let $\wp(A_k) = B_1, B_2, \dots, B_{2^k}$. $\wp(A_{k+1})$ has all these elements, plus $B_1 \cup \{x\}, B_2 \cup \{x\}, \dots, B_{2^k} \cup \{x\}$. That gives us an additional 2^k . So $|\wp(A_{k+1})| = 2(2^k) = 2^{k+1}$

(3) $\forall X_n (|\wp(X_n)| = 2^n)$ (by Math. Induction)

Right. Good.

6. (the horses exercise)

The problem is in the second step (“now assume that all sets of n horses contain only horses of the same color”). The assumption made is a universal statement, while we are supposed to make an assumption about an arbitrary element. If we do so, the argument does not go through. Let us assume that all the horses in a given set A are of the same color. Nothing guarantees that if we add a horse, this new horse will be of the same color as all the horses in A .

=====

Instructors’ comment: No, that’s not it. In fact, in order to formulate the proposition “All horses are the same color” as a proposition about natural numbers n , we do have to put in that universal quantifier. The thing we’re trying to prove becomes “Every set of n horses, for any n , is a set all of whose members are the same color.” Or, abbreviating, “For all n , every set of n horses is a uniform-color set.”

So the *form* of the proof of the induction set is correct. The problem lies in the proof of that step itself. The proof is supposed to work to get you from k to $k+1$ for any arbitrary natural number k . But the trouble was that the way the proof was done carried a covert presupposition that we are dealing with $k+1 > 2$, i.e. $k > 1$. But if you consider the possible case where $k = 1$ and $k+1 = 2$, and try to do what it says in that proof, you'll discover that the proof fails for this case.

In this case, you are looking at a set with two horses and removing each of the two horses one at a time. And it's true that in each case you're left with a (1-member) uniform-color set. Only this time there isn't any horse who stayed in the set all along to whom you could compare the color of the horse remaining behind when the other horse was removed.

Moral: What fools us in this proof is that usually when we think about an "arbitrary number k ", we forget that that means that in our reasoning we mustn't use any properties of numbers that are not shared by all numbers. (Here we covertly used the property "being greater than 1" for k . And as a result it didn't work for $k = 1$.) We tend to subconsciously equate a 'stereotypical' number with an 'arbitrary number'.