Further discussion of Question 10 of Homework 9

On Thursday, I circulated a proposed "YES" answer to the "categorical" question, Question 10 of Homework 9, related to the axiom system in problem 13 on page 235 of PtMW, and asked everyone to look for possible counterarguments. Masashi has found one, and I think he is right. Here is his argument for a "NO" answer:

10. Is the axiom system *W* consistent? Is it categorical?
We know that *W* is consistent because we found a model for it, namely *P* = {1, 2, 3, 4}, *L* = {{1,2},{2,3},{3,4},{4,1},{1,3},{2,4}}, in the last homework.

Is it categorical? In my opinion, it is not categorical because there is a model quite different (so not isomorphic) from the above one: we can consider a flat plane as a model of the formal system *W*. Precisely, if we take points and straight lines[1] on a flat plane as the primitives of *W*, namely "points" and "lines", then all the axioms hold.
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So where did I go wrong? I think I misread Axiom 3 and misused it in my purported proof.

Axiom 3: If *p* and *q* are distinct points, then there is one and only one line of which *p* and *q* are both members.

My use of axiom 3 in my "proof", with the error in boldface:

> First let's show that there cannot be MORE than 4 points. Suppose there were (at least) 5 points, a,b,c,d,e. Axiom 3 requires that each pair **constitute** a line, and there cannot be any 3-point lines, also because of axiom 3. So we have 10 lines ab, ac, ad, ae, bc, bd, be, cd, ce, de. But now we get in trouble with axiom 5: Consider line ab and point d: there are TWO lines containing d that are disjoint from ab, namely cd and de. So there cannot be more than 4 points.

What was right in what I said was that if you do have a line consisting of just {a,b}, you cannot *also* have a 3-point line containing a and b. But all I really proved in my answer is that if lines have to consist of exactly 2 points, there can only be 4 points and 6 lines. But Axiom 3 doesn't require that lines consist of exactly 2 points.

In Masashi's model, every line has infinitely many points – this is classical Euclidean geometry, and those axioms are indeed perfectly compatible with classical Euclidean geometry.

Next question: Are there other finite models? Volodja and I don't know yet! This would be another challenging exercise. But we definitely have a negative answer to the categoricity question, with at least one finite model and at least one infinite one. **Aha, see next page!**

[1] Note that not straight *line segments*, but straight *lines* correspond to "lines" in *W* because of axiom A3. [BHP added: that is, each line extends forever in both directions. A line segment is a *part* of a line.]

There are infinitely many finite models. They even have a name, they are the *affine planes of order n.* Our initial model was the affine plane of order 2: it has 2^2 points, and $2^2 + 2$ lines; every line contains exactly 2 points, and every point is on exactly 3 lines.

In the affine plane of order 3, there are 3^2 points, $3^2 + 3$ lines; every line contains exactly 3 points, and every point is on exactly 4 lines.

You can find a picture of it here[2] (but caution: this site uses Java applets, and my computer froze once briefly):

http://home.wlu.edu/~mcraea/Finite_Geometry/NoneuclideanGeometry/Prob16FiniteHypGeom/Solution16.htm .

You can find a different axiomatization of affine geometry, along with the generalizations about "order n" that I just cited, here:

http://en.wikipedia.org/wiki/Finite_geometry

I copy the crucial part here:

> For an affine plane geometry, the axioms are as follows:
>
> 1. Given any two distinct points, there is exactly one line that includes both points.
> 2. The parallel postulate: Given a line *L* and a point *P* not on *L*, there exists exactly one line through *P* that is parallel to *L*.
> 3. There exists a set of four points, no three collinear (i.e. contained in a common line).
>
> The last axiom ensures that the geometry is not empty, while the first two specify the nature of the geometry. The simplest affine plane contains only four points; it is called the *affine plane of order* 2. Since no three are collinear, any pair of points determines a unique line, and so this plane contains six lines. It corresponds to a tetrahedron where non-intersecting edges are considered "parallel", or a square where not only opposite sides, but also diagonals are considered "parallel". More generally, a finite affine plane of order n has n^2 points and $n^2 + n$ lines; each line contains n points, and each point is on $n + 1$ lines.

An even more elegant axiomatization is found on Professor McRae's site, in the problems to which the link given above was to the solutions.

http://home.wlu.edu/~mcraea/Finite_Geometry/NoneuclideanGeometry/Prob16FiniteHypGeom/Problem16.htm

AFFINE PLANES

1. Two distinct points are contained in a unique line.
2. Given a line and a point, there is a unique line through the point that is parallel to the given line.
3. There exist three points that are not all contained in a line.

I believe that these two axiomatizations of affine planes are equivalent to each other and to the axioms on p. 235 in PtMW, although I'm not prepared to try to prove it.

[2] The author is Alan McRae of Washington and Lee University.