

The categoricity question below has been nagging at the back of my mind since we taught this course in 2001, and there weren't any totally convincing student answers in 2001 or 2004, so I've finally tried to write down a proof of an answer. Please check it closely and see if you can find any holes. If not, I'll post it on the website. -- Barbara

Homework 9, question 10:

Is the axiom system \mathcal{W} (of PtMW Ch 8, p235, question 13) consistent? *Is it categorical?

Consistent: Yes – in the original homework, we constructed a model with 4 points and 6 lines. Since the axioms have a model, they are consistent.

Categorical: This question is harder. If the axiom system is categorical, then all its models are isomorphic. We have seen one model with 4 points and 6 lines. Can there be any models that are different? (Different number of points, or 4 points but either a different number of lines or different 'structuring' of the lines, something other than a relabelling of the points.)

1. Easy part of the proof: All 4-point models are isomorphic.

Let's name the points in the original model a, b, c, d , and the 6 lines ab, ac, ad, bc, bd, cd . (The line I'm naming "ab" is $\{a, b\}$, etc.)

(i) There cannot be fewer lines, because for each pair of points, there must be exactly one line that contains both.

(ii) There cannot be any 'different' two-member lines, since we've used up every pair of points.

(iii) In order to have any more lines, there would have to be an empty line or one or more 1-point lines. In the original problem we proved that there cannot be an empty line.

So let's try assuming that there exists at least one one-point line, say $\{a\}$. But we immediately hit a contradiction with Axiom 5: cd is a line, a is a point not on cd , and there is supposed to be one and only one line containing a and disjoint from cd . We already had line ab as such a line; if there could be a line $\{a\}$, it would be a second such line. Therefore there are no one-point lines in a 4-point model.

Therefore we can conclude that all 4-point models are isomorphic.

2. Can there be any models with more or fewer than 4 points?

First let's show that there cannot be MORE than 4 points. Suppose there were (at least) 5 points, a, b, c, d, e . Axiom 3 requires that each pair constitute a line, and there cannot be any 3-point lines, also because of axiom 3. So we have 10 lines $ab, ac, ad, ae, bc, bd, be, cd, ce, de$. But now we get in trouble with axiom 5: Consider line ab and point d : there are TWO lines containing d that are disjoint from ab , namely cd and de . So there cannot be more than 4 points.

Axioms 2,3,4 together require that there be at least 3 points. So the only possibility left is that maybe there could be models with 3 points. Can there be?

Assume we have points a , b , and c , and hence at least the three lines ab , ac , bc . That already satisfies axioms 1-4, but not axiom 5: Consider line ab , and the point c which is not on it: there is no line containing c that is disjoint from ab .

What if we now add the three singleton lines, $\{a\}$, $\{b\}$, and $\{c\}$? In the 4-point model, that got us into trouble, but the reasons it did don't exist in a 3-point model.

So suppose we have points a, b, c and lines $ab, ac, bc, \{a\}, \{b\}, \{c\}$. That still satisfies Axioms 1-4, but what about Axiom 5? No. Consider line $\{a\}$ and a point not on it, for instance b . Now there are TWO lines containing b that are disjoint from $\{a\}$, namely bc and $\{b\}$.

Would it help if we had just one singleton line, say $\{a\}$? No: the consideration that led us to introduce the singletons isn't taken care of with just one. So consider again the line ab and the point c not on it: without the singleton $\{c\}$, there is no line containing c disjoint from ab .

Would it help if we had just two singleton lines, say $\{a\}$ and $\{c\}$? No, we have already violated Axiom 5 in the same way as with all three singletons: consider line $\{a\}$ and the point c not on it. Now there are again two lines containing c and disjoint from $\{a\}$: bc and $\{c\}$.

We conclude that there are no models that are not isomorphic to the original model, and hence that the system is categorical.

Note that along the way we've shown that Axiom 5 is independent of the other 4, since we've found models that satisfy Axioms 1-4 but fail on Axiom 5.